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Box Dimensions of a Class of Self-conformal Sets

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Abstract: In this paper, we construct a class of self-conformal sets based on self-similar sets, and obtain the formula for their box dimensions supported by "finite" type of self-similar sets.

Keywords: self-conformal set; box dimension; finite type

1 Introduction

Let $\{\phi_i\}_{i=1}^q$ be an iterated function system of contractive similitudes on $\mathbb{R}$, defined by

$$\phi_i(x) = a_i x + c_i, \quad 1 \le i \le q, \tag{1}$$

and $\{\phi_i^{'}\}_{i=1}^q$ be a contractive self-conformal IFS on $\mathbb{R}$, defined by

$$\phi_i^{'}(x) = a_i x + b_i x^2 + c_i, \quad 1 \le i \le q, \tag{2}$$

where for all i, $0 < a_i < 1$, $0 < b_i < 1$, $0 < a_i + b_i < 1$, $c_i \in \mathbb{R}$. Let F be self-similar set (or attractor) defined by the IFS $\{\phi_i\}_{i=1}^q$, and $F^{'}$ be the self-conformal set defined by the IFS $\{\phi_i^{'}\}_{i=1}^q$. It is well known that if the contractive self-similar IFS $\{\phi_i\}_{i=1}^q$ satisfies the open set condition (OSC), then $\dim_B(F) = s$, where s is the unique solution of

$$\sum_{i=1}^{q} a_i^s = 1.$$

It has been the subject of several studies[1-5]. However, in the absence of the OSC the images of F under the ϕ_i have overlaps and the above dimension formula fails in general. In this case it is much harder to compute dimension of F, Ngai and Wang[6,7] introduced the notion of finite type and described a scheme for the computation of dimension when the finite type occurs. The scheme based on the notion of finite type can be outlined as follows.

Let $\sum_q = \{1, 2, ..., q\}$ and $\sum_q^* = \bigcup_{n\ge 0} \sum_q^n$ be the set of all finite words in $\sum_q$, where $\sum_q^n$ is the set of all words of length n, with $\sum_q^0$ containing only the empty word $\emptyset$. For $\mathbf{j} \in \sum_q^n$ let $|\,\mathbf{j}| = n$ denote the length of $\mathbf{j}$. For $\mathbf{i} \in \sum_q^m$ and $\mathbf{j} \in \sum_q^n$ let $\mathbf{ij} \in \sum_q^{m+n}$ be the concatenation of $\mathbf{i}$ and $\mathbf{j}$, and call $\mathbf{i}$ an initial segment of $\mathbf{ij}$.

Let $\mathbf{j} = (j_1, j_2, \cdots, j_m) \in \sum_q^m$, then

$$\phi_{\mathbf{j}} = \phi_{j_1} \circ \cdots \circ \phi_{j_m}, a_{\mathbf{j}} = a_{j_1} \cdots a_{j_m}.$$

Now let $\rho = \min_i \{a_i\}$. For all $k \ge 0$ define

$\Lambda_k = \{\mathbf{j} \in \sum_q^* \,|a_{\mathbf{j}} \le \rho^k$ but $a_{\mathbf{i}} > \rho^k$ if $\mathbf{i}$ is a proper initial segment of $\mathbf{j}\}$.

Intuitively, all $\phi_{\mathbf{j}}$ for $\mathbf{j} \in \Lambda_k$ have comparable contraction ratios, which are in the order of ρ^k.

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Let $\mathcal{V} = \bigcup_{k\geq 0}\mathcal{V}_k$, where $\mathcal{V}_0 = \{(I,0)\}$ and $\mathcal{V}_k = \{(\phi_{\mathbf{i}},k) : \mathbf{i} \in \Lambda_k\}$ for all integer $k \geq 1$. A directed graph $\mathcal{G}$ is the constructed with $\mathcal{V}$ as the set of vertices and $\mathbf{j} \in \sum_{\mathbf{q}}^*$ as the directed edge connecting from $(\phi_{\mathbf{i}},k)$ to $(\phi_{\mathbf{ij}},k+1)$ (this implies that $\mathbf{i} \in \Lambda_k, (\phi_{\mathbf{i}},k) \in \mathcal{V}_k$, and $\mathbf{ij} \in \Lambda_{k+1}$, $(\phi_{\mathbf{ij}},k+1) \in \mathcal{V}_{k+1}$). The vertex $(\phi_{\mathbf{ij}},k+1)$ is called an offspring of the vertex $(\phi_{\mathbf{i}},k)$, and conversely, the vertex $(\phi_{\mathbf{i}},k)$ is called the parent of the vertex $(\phi_{\mathbf{ij}},k+1)$. Thus each vertex in $\mathcal{V}_k$ has at least one offspring in $\mathcal{V}_{k+1}$, and each vertex in $\mathcal{V}_{k+1}$ has at least one parent in $\mathcal{V}_k$. The reduced graph $\mathcal{G}_k$ is obtained from $\mathcal{G}$ by first removing all but the smallest (in the lexicographical order) directed edge going to a vertex. The set of the k-th order vertices of the reduced graph $\mathcal{G}_k$ is denoted by $\mathcal{V}_{R,k}$ and $\mathcal{V}_R = \bigcup_{k\geq 0}\mathcal{V}_{R,k}$.

Fix any nonempty bounded open set $\Omega \subset \mathbb{R}$ which is invariant under $\{\phi_i\}_{i=1}^q$,i.e., $\bigcup_{i=1}^q \phi_i(\Omega) \subset \Omega$. Two vertices $\mathbf{v}, \mathbf{v}' \in \mathcal{V}_k$(allowing $\mathbf{v} = \mathbf{v}'$) are neighbors (with respect to Ω) if $\phi_{\mathbf{v}}(\Omega) \cap \phi_{\mathbf{v}'}(\Omega) \neq \emptyset$ (where for $\mathbf{v} = (\phi_{\mathbf{i}},k) \in \mathcal{V}_k$ we use the convenient notation $\phi_{\mathbf{v}} := \phi_{\mathbf{i}}$). The set of vertices

$$\Omega(\mathbf{v}) = \{\mathbf{v}' : \mathbf{v}' \text{is a neighbor of } \mathbf{v}\}$$

is called the neighborhood of $\mathbf{v}$ (with respect to Ω). Two vertices $\mathbf{v} \in \mathcal{V}_k$ and $\mathbf{v}' \in \mathcal{V}_k'$ are equivalent, denoted by $\mathbf{v} \sim \mathbf{v}'$, if, for $\tau := \phi_{\mathbf{v}'} \circ \phi_{\mathbf{v}}^{-1} : \mathbb{R} \to \mathbb{R}$, the following conditions are satisfied[7]:

(i) $\{\phi_{\mathbf{u}'} : \mathbf{u}' \in \Omega(\mathbf{v}')\} = \{\tau \circ \phi_{\mathbf{u}} : \mathbf{u} \in \Omega(\mathbf{v})\}$;

(ii) for $\mathbf{u} \in \Omega(\mathbf{v})$ and $\mathbf{u}' \in \Omega(\mathbf{v}')$ such that $\phi_{\mathbf{u}'} = \tau \circ \phi_{\mathbf{u}}$, and for any positive integer $l \geq 1$, an index $\mathbf{i} \in \sum_q^*$ satisfies $(\phi_{\mathbf{u}} \circ \phi_{\mathbf{i}}, k+l) \in \mathcal{V}_{k+l}$ if and only if it satisfies $(\phi_{\mathbf{u}'} \circ \phi_{\mathbf{i}}, k'+l) \in \mathcal{V}_{k'+l}$.

The IFS $\{\phi_i\}_{i=1}^q$ is said to be of finite type if there are finitely many distinct neighborhood types.

In this case we can define the incidence matrix $S = [s_{ij}]$ for IFS. Suppose that there are N neighborhood types. Choose any vertex $\mathbf{v}$ has neighborhood type i. Its offspring in some reduced graph will have various neighborhood types. The entry s_{ij} denotes the number of offspring that have neighborhood type j.

Theorem 1 $(See[6])$ *Let $\{\phi_i\}_{i=1}^q$ be an iterated function system defined as in (1). Suppose that the iterated function system is of finite type with respect to a bounded invariant open set Ω, and let S be the corresponding incidence matrix. Then the attractor F of the iterated function system satisfies*

$$\dim_H(F) = \dim_B(F) = \frac{\log\lambda}{-\log\rho},$$

where $\rho = \min_i\{a_i\}$ and $\lambda = \lambda(S)$ is the spectral radius of S.

In this paper, for the contractive self-conformal IFS $\{\phi_i'\}_{i=1}^q$, we define

$$\Lambda_k' = \{\mathbf{j} \in \Sigma_q'^* \mid a_{\mathbf{j}} \leq \rho^k \text{ but } a_{\mathbf{i}} > \rho^k \text{ if } \mathbf{i} \text{ is a proper initial segment of } \mathbf{j}\},$$

where $\sum_q'^*$ defined as $\sum_q^*$. We denote $\mathcal{V}_0' = \{(I,0)\}$ and $\mathcal{V}_k' = \{(\phi_{\mathbf{i}}',k) : \mathbf{i} \in \Lambda_k'\}$ for all $k \geq 1$. This leads to our main result.

Condition 2 *For $\{\phi_i'\}_{i=1}^q$ satisfies: let E, U be any subsets in $\mathbb{R}$ with diam$(U) \leq K_1\rho^k$ and diam$(E) \leq K_2$. Then there exists an $M = M(K_1,K_2) > 0$, such that for all $k \geq 0$, $\sharp\{\mathbf{v} \in \mathcal{V}_k' | U \cap \phi_{\mathbf{v}}'(E) \neq \emptyset\} \leq M$.*

Theorem 3 *Let $\{\phi_i\}_{i=1}^q$,$\{\phi_i'\}_{i=1}^q$ be iterated function systems defined as $(1),(2)$. Suppose that $\{\phi_i\}_{i=1}^q$ is of finite type with respect to a bounded invariant open set Ω. $\{\phi_i'\}_{i=1}^q$ satisfies Condition 2 and $|\mathcal{V}_k'| = |\mathcal{V}_k|$. Then the self-conformal set F' satisfies*

$$\dim_B(F') = \frac{\log\lambda}{-\log\rho},$$

where $\rho = \min_i\{a_i\}$, S be the corresponding incidence matrix for $\{\phi_i\}_{i=1}^q$, and $\lambda = \lambda(S)$ is the spectral radius of S.

Corollary 4 *If $\{\phi_i\}_{i=1}^q$ and $\{\phi_i'\}_{i=1}^q$ are both of no complete overlap, we will obtain*

$$\dim_B(F') = \frac{\log\lambda}{-\log\rho}.$$

where $\rho = \min_i\{a_i\}$ and $\lambda = \lambda(S)$ is the spectral radius of S.

2 The proof of the results

Lemma 5 *Let $|\phi_{\mathbf{i}}(\Omega)|$ denote the diameter of $\phi_{\mathbf{i}}(\Omega)$, then there exists a positive real number $P > 1$ such that for any $\mathbf{i} \in \sum_q^*$,*

$$1 < \frac{|\phi'_{\mathbf{i}}(\Omega)|}{|\phi_{\mathbf{i}}(\Omega)|} < P,$$

where Ω is a bounded invariant open set of $\{\phi_i\}_{i=1}^q$ and $\{\phi'_i\}_{i=1}^q$.

Proof. Let

$$f(n) = \prod_{i=1}^{n} \frac{1}{1 + AB^i|\Omega|},$$

where

$$A = \max_{1\le i\le q}\{\frac{b_i}{a_i}\},\ B = \max_{1\le i\le q}\{a_i + b_i\} < 1.$$

Obviously, $0 < f(n) < 1, (n = 1, 2, \cdots)$ and $\{f(n)\}$ is monotone decreasing. So we have

$$0 \le \sigma < 1,\ \text{where}\ \sigma := \prod_{i=1}^{\infty} \frac{1}{1 + AB^i|\Omega|}.$$

Let

$$g(n) = \prod_{i=1}^{n} \frac{1}{1 + \frac{1}{i^2}} > 0,$$

$\{g(n)\}$ is also monotone decreasing. Note that

$$\prod_{i=1}^{\infty} \frac{1}{1 + \frac{1}{i^2}} = \prod_{i=1}^{\infty} \frac{i^2}{1 + i^2} \ge \frac{1}{2}\prod_{i=2}^{\infty} \frac{i^2 - 1}{i^2 + 1} = \frac{\pi}{2}\cosh(\pi) > 0.$$

There exists a positive integer N large enough, we have

$$\prod_{i=N}^{\infty} \frac{1}{1 + AB^i|\Omega|} > \prod_{i=N}^{\infty} \frac{1}{1 + \frac{1}{i^2}}.$$

By (3), we have

$$\prod_{i=N}^{\infty} \frac{1}{1 + \frac{1}{i^2}} > 0,\ \text{so}\ \sigma > 0.$$

Thus we can put

$$P = [\frac{1}{\sigma}] + 1.$$

For any positive integer number k, and any $\mathbf{i} \in \Sigma_q^k$, we have

$$\frac{|\phi_{\mathbf{i}}(\Omega)|}{|\phi'_{\mathbf{i}}(\Omega)|} = \prod_{j=1}^{k} \frac{a_{i_j}}{a_{i_j} + b_{i_j}|\phi_{i_1} \circ \cdots \phi_{i_{j-1}}(\Omega)|} = \prod_{j=1}^{k} \frac{1}{1 + \frac{b_{i_j}}{a_{i_j}}|\phi_{i_1} \circ \cdots \phi_{i_{j-1}}(\Omega)|}.$$

From the definition of $\{\phi'_i\}_{i=1}^q$, we can obtain that

$$\prod_{i=1}^{\infty} \frac{1}{1 + AB^i|\Omega|} < \frac{|\phi_{\mathbf{i}}(\Omega)|}{|\phi'_{\mathbf{i}}(\Omega)|} < 1.$$

Therefore

$$1 < \frac{|\phi'_{\mathbf{i}}(\Omega)|}{|\phi_{\mathbf{i}}(\Omega)|} < P,\ \text{for any}\ \mathbf{i} \in \Sigma_q^*. \tag{3}$$

■

Lemma 6 *Let* $\{\phi_i\}_{i=1}^q$, $\{\phi_i^{'}\}_{i=1}^q$ *be iterated function systems defined as* (1), (2). *Suppose that* $\{\phi_i\}_{i=1}^q$ *is of finite type with respect to a bounded invariant open set* Ω. $\{\phi_i^{'}\}_{i=1}^q$ *satisfies Condition 2. Then the self-conformal set* $F^{'}$ *satisfies*

$$\liminf_{k\to\infty}\frac{\log|\mathcal{V}_k^{'}|}{-k\log\rho}\le\underline{\dim}_B(F^{'})\le\overline{\dim}_B(F^{'})\le\limsup_{k\to\infty}\frac{\log|\mathcal{V}_k^{'}|}{-k\log\rho}.$$

Proof. By Lemma 5, let $N(cP\rho^k)$ be the minimal number of balls of radius $cP\rho^k$ needed to cover $F^{'}$. Then for any $c_1, c_2 > 0$ there exist positive constants $K^+(c_1, c_2)$ and $K^-(c_1, c_2)$ such that

$$K^-(c_1,c_2)N(c_1P\rho^k)\le N(c_2P\rho^k)\le K^+(c_1,c_2)N(c_1P\rho^k).$$

Observe that $F^{'}=\bigcup_{\mathbf{j}\in\Lambda_k^{'}}\phi_{\mathbf{j}}(F^{'})=\bigcup_{\mathbf{v}\in\mathcal{V}_k^{'}}\phi_{\mathbf{v}}(F^{'})$, and there exists a $c_0>0$ such that each $\phi_{\mathbf{v}}(F^{'})$ can be covered by a ball of radius $c_0P\rho^k$. So we have

$$|\mathcal{V}_k^{'}|\ge N(c_0P\rho^k).$$

Now let $B_1,\cdots,B_{N(\delta)}$ be balls of radius $\delta>0$ that cover $F^{'}$. We may uniquely write $\delta=cP\rho^k$ for some k and $\rho<c\le1$. By Condition 2, the cardinality of $\{v\in\mathcal{V}_k^{'}|B_j\cap\phi_v^{'}(F^{'})\ne\emptyset\}$ is bounded by some fixed $M>0$ for all $1\le j\le N(\delta)$. Therefore $|\mathcal{V}_k^{'}|\le MN(\delta)$. On other hand,

$$|\mathcal{V}_k^{'}|\ge N(c_0P\rho^k)\ge K^-(c,c_0)N(cP\rho^k)=K^-(c,c_0)N(\delta). \tag{4}$$

Therefore

$$\underline{\dim}_B(F^{'})=\liminf_{\delta\to0}\frac{\log N(\delta)}{-\log\delta}\ge\liminf_{k\to\infty}\frac{\log(|\mathcal{V}_k^{'}|/M)}{-\log(cP\rho^k)}=\liminf_{k\to\infty}\frac{\log|\mathcal{V}_k^{'}|}{-k\log\rho}.$$

Similarly, applying (4) we obtain

$$\overline{\dim}_B(F^{'})=\limsup_{\delta\to0}\frac{\log N(\delta)}{-\log\delta}\le\limsup_{k\to\infty}\frac{\log(|\mathcal{V}_k^{'}|/K^-(c,c_0))}{-\log(cP\rho^k)}=\limsup_{k\to\infty}\frac{\log|\mathcal{V}_k^{'}|}{-k\log\rho}.$$

So

$$\liminf_{k\to\infty}\frac{\log|\mathcal{V}_k^{'}|}{-k\log\rho}\le\underline{\dim}_B(F^{'})\le\overline{\dim}_B(F^{'})\le\limsup_{k\to\infty}\frac{\log|\mathcal{V}_k^{'}|}{-k\log\rho}.$$

■

Proof of Theorem 3.

Proof. We can evaluate $|\mathcal{V}_k|$ using the incidence matrix S of the finite type IFS $\{\phi_i\}_{i=1}^q$ with respect to Ω. By[6]

$$|\mathcal{V}_k|=e_1^TS^k\xi,$$

where $\xi=[1,1,...,1]^T$ and $e_1=[1,0,...,0]^T$ are vectors in $\mathbb{R}^N$. Then

$$\lim_{k\to\infty}(e_1^TS^k\xi)^{1/k}=\lambda.$$

Now let $\varepsilon>0$ be arbitrary. Then there exists K large enough, for all $k\ge K$,

$$(\lambda-\varepsilon)^k<e_1^TS^k\xi<(\lambda+\varepsilon)^k.$$

Using the fact that $|\mathcal{V}_k|=e_1^TS^k\xi$, $|\mathcal{V}_k^{'}|=|\mathcal{V}_k|$, and applying Lemma 6, we get

$$\frac{\log(\lambda-\varepsilon)}{-\log\rho}\le\liminf_{k\to\infty}\frac{\log|\mathcal{V}_k|}{-k\log\rho}\le\underline{\dim}_B(F^{'})\le\overline{\dim}_B(F^{'})\le\limsup_{k\to\infty}\frac{\log|\mathcal{V}_k|}{-k\log\rho}\le\frac{\log(\lambda+\varepsilon)}{-\log\rho}.$$

Letting $\varepsilon\to0$ yields

$$\dim_B(F^{'})=\frac{\log\lambda}{-\log\rho}.$$

■

Proof of Corollary 4.

Proof. When $\{\phi_i\}_{i=1}^q$,$\{\phi_i'\}_{i=1}^q$ are also of no complete overlap, it is easy to obtain

$$|\mathcal{V}_k'| = |\mathcal{V}_k|,$$

so

$$\dim_B(F') = \frac{\log\lambda}{-\log\rho}.$$

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