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Boundary Control of the Viscous Generalized Camassa-Holm Equation

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Abstract: This paper investigates the boundary control of the viscous generalized Camassa-Holm equation on $[0,1]$. The existence of its solution is proved in a short time interval under the boundary condition. Meanwhile, we prove the global exponential stability of the solution in four different spaces, i.e. in L^2,H^1,H^2,H^3.

Keywords: the viscous generalized Camassa-Holm equation; boundary control; global exponential stability

1 Introduction

In [1], Camassa and Holm used Hamiltonian methods to derive a new completely integrable dispersive wave equation for water waves by retaining two terms that are usually neglected in the small amplitude shallow water limit

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.1}$$

where u is the fluid velocity in the x direction (or equivalently the height of the water's free surface above a flat bottom), k is a constant related to the critical shallow water wave speed. They showed that for all$k$, (1.1) is integrable, and for$k = 0$, (1.1) has traveling solutions of the form$ce^{-|x-ct|}$. After that, there has been much research and observations about this equation. Zhengrong Liu and Tifei Qian et al [3] investigate generalized Camassa-Holm (denote GCH) equation, or called modified Camassa-Holm (or denote mCH) equation,

$$u_t + 2ku_x - u_{xxt} + au^m u_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.2}$$

where $a > 0$,$k \in R$, $m \in N$and $m$is called the strength of the nonlinearity. They discussed the peakons and their bifurcation in (1.2) and they introduced GCH equation from the mathematical point of view. In [2-4], Lixin Tian et al studied its peakons and traveling wave solutions.

In the other hand, more people have paid attentions to boundary control of different equations. Byrnes et al studied local stability of Burgers equation, Vanly et al further consummated this result, but still local. Miroslav Kristic studied global stability of Burgers equation [5]. Biler Rassel and Zhang studied KdVB equation under periodical boundary condition [6]-[8]. Liu and Kristic studied the stability of KdVB equation in a limited area [9]. Wei-jiu Liu et al did some studies on K-S equation [10, 11]. Yiping Meng studied boundary control of b-family equation [12]. In this paper, we study the existence of the boundary control of the viscous generalized Camassa-Holm equation with the following boundary

$$\begin{cases} u_t - u_{xxt} - \varepsilon\,(u - u_{xx})_{xx} + ku_x + (u^3)_x = 2u_x u_{xx} + uu_{xxx} \\ u\,(x,0) = u_0\,(x) \\ u\,(0,t) = u\,(1,t) = u_x\,(0,t) = u_x\,(1,t) = u_{xx}\,(0,t) = u_{xx}\,(1,t) = 0 \end{cases}, \tag{1.3}$$

where $t > 0, x \in \Omega, \Omega = [0,1], u \in H = L^2\,(\Omega)$.

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We will discuss the global well-posedness of (1.3). It will be shown that for given $T > 0$,$u_0 \in H^3$, (1.3) admits a unique solution $u \in C\left((0,T), H^3(0,1)\right) \cap C'\left((0,T), H^2(0,1)\right)$.

2 Main theorem

Denote $A = -\Delta$, where Δ is the Laplace operator. Let $v = u + Au$. Denote bi-linear item $B(u,v) = uv_x$ and $C(u) = ku_x$. Eq.(1.3) can be denoted as

$$\begin{cases} v_t + \varepsilon Av + 2B(v,u) + B(u,v) + C(u) + (u^3)_x - (\frac{3u^2}{2})_x = 0 \\ u_0 = u(x,0) \\ u(0,t) = u(1,t) = u_{xx}(0,t) = 0 \\ u_{xxx}(1,t) = u_x(1,t) = u_x(0,t) \end{cases}, \tag{2.1}$$

where A is a self-adjoint positive operator with compact inverse.

Theorem 1 *With $u_0 \in H^3$, Eq.4 has a global solution*

$$u \in C\left((0,\infty), H^3(0,1)\right) \cap C'\left((0,\infty), H^2(0,1)\right)$$

and satisfies the following L^2, H^1, H^2, H^3 stability:

(1) $\|u\|_{L^2}^2+\|u\|_{H^1}^2 \le (\|u_0\|_{L^2}^2+\|u_0\|_{H^1}^2)\exp(-2\varepsilon\lambda_1)t$, λ_1 *is a Poincare coefficient.* *(2.2)*

(2) $\|u\|_{H^1}^2+\|u\|_{H^2}^2 \le \left(\|u_0\|_{H^1}^2+\|u_0\|_{H^2}^2\right)\exp(b_1-2\varepsilon\lambda_2)t$, *(2.3)*

where $\varepsilon \in \left(\frac{c}{2\lambda_2}, +\infty\right)$ *and* b_1 *is a nonnegative constant,* λ_2 *is a Poincare coefficient.*

(3) $\|u\|_{H^2}^2+\|u\|_{H^3}^2 \le \left(\|u_0\|_{H^2}^2+\|u_0\|_{H^3}^2\right)\exp\left(2b_2-2\varepsilon\lambda_3\right)t$, *(2.4)*

where $\varepsilon \in (\frac{b_2}{\lambda_3}, +\infty)$ *and* b_2 *is a nonnegative constant,* λ_3 *is a Poincare coefficient.*

3 Proof of Theorem

First, we prove the global existence of the solution.

We use the Galerkin procedure to prove global existence.

Let $\{\phi_j\}_{j=1}^{\infty}$ be an orthonormal basis of Hconsisting of eigenfunctions of the operator A. For $m \in N$ define the discrete ansatz space by $H_m = \{\phi_1,\phi_2,\cdots,\phi_m\}$, and P_m is the orthogonal projection from H to H_m. Performing the Galerkin Procedure for Eq.(2.1), then (2.1) is the ordinary differential system,

$$\begin{cases} v_{m,t} + \varepsilon Av_m + 2P_mB(v_m,u_m) + P_mB(u_m,v_m) + P_mC(u_m) + (u_m^3)_x - (\frac{3u_m^2}{2})_x = 0 \\ u_m(x,0) = p_m u_{m,0}(x) \end{cases}, \tag{3.1}$$

where $v_m = u_m - u_{m,xx}$. Since the nonlinear term is quadratic in u_m, then by the classical theory of ordinary differential equation, the system (3.1) has a unique solution for a short interval of the time $(0,T_m)$. Our purpose is to show the global existence of the solution under given boundary control.

Taking inner product in (3.1) to be u_m onΩ, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|u_m\|_{L^2}^2+\|u_m\|_{H^1}^2\right)+\varepsilon\left(\|u_m\|_{H^1}^2+\|Au_m\|_{L^2}^2\right)=0. \tag{3.2}$$

By Poincare inequality and Young's inequality, we can construct

$$\frac{d}{dt}\left(\|u_m\|_{L^2}^2+\|u_m\|_{H^1}^2\right)+2\varepsilon\lambda_1\left(\|u_m\|_{L^2}^2+\|u_m\|_{H^1}^2\right)\le 0,$$

where λ_1 is a Poincare coefficient. Then, we can get

$$\|u_m\|_{L^2}^2+\|u_m\|_{H^1}^2 \le \left(\|u_m(0)\|_{L^2}^2+\|u_m(0)\|_{H^1}^2\right)\exp(-2\varepsilon\lambda_1 t) \le \|u_m(0)\|_{L^2}^2+\|u_m(0)\|_{H^1}^2 \triangleq r_1, \tag{3.3}$$

$\forall t \in [0,T]$, where r_1 is positive constant.

Let r be a positive constant, integrating (3.2) in the interval$[t, t+r] \subset [0,T]$, we have

$$\int_t^{t+r} \left(\|u_m(s)\|_{H^1}^2 + \|Au_m(s)\|_{L^2}^2 \right) ds \le \frac{r_1}{\varepsilon}. \tag{3.4}$$

Taking inner product in (3.1) to be Au_m onΩ, we obtain

$$\begin{array}{l} \frac{1}{2}\frac{d}{dt}\left(\|u_m\|_{H^1}^2 + \|Au_m\|_{L^2}^2 \right) + \varepsilon \left(\|Au_m\|_{L^2}^2 + \|Au_m\|_{H^1}^2 \right) + P_m\left(B\left(u_m, v_m\right), Au_m\right) \\ +2P_m\left(B\left(v_m, u_m\right), Au_m\right) + P_m\left(C\left(u_m\right), Au_m\right) + ((u_m^3)_x, Au_m) - ((\frac{3u_m^2}{2})_x, Au_m) = 0 \end{array}. \tag{3.5}$$

By computing, we have

$$((u_m^3)_x, Au_m) = 3\int_0^1 u_m (u_{mx})^3 dx, ((\frac{3u_m^2}{2})_x, Au_m) = \frac{3}{2}\int_0^1 (u_{mx})^3 dx$$

$$\left| P_m\left(B\left(u_m, v_m\right), Au_m\right) + 2P_m\left(B\left(v_m, u_m\right), Au_m\right) + P_m\left(C\left(u_m\right), Au_m\right) \right|$$

$$\le 3\left| P_m\left(B\left(u_m, u_m\right), Au_m\right)\right| + 2\left| P_m\left(B\left(Au_m, u_m\right), Au_m\right)\right| + \left| P_m\left(B\left(u_m, Au_m\right), Au_m\right)\right|.$$

According to Agmon inequality when $n=1$, then we have

$$\left| P_m\left(B\left(u_m, u_m\right), Au_m\right)\right| \le \|u_{m,x}\|_{L^\infty(\Omega)} \|u_m\|_{H^1}^2 \le C_1 \|u_m\|_{H^1}^{\frac{5}{2}} \|Au_m\|_{L^2}^{\frac{1}{2}},$$

$$\left| P_m\left(B\left(Au_m, u_m\right), Au_m\right)\right| \le \|u_{m,x}\|_{L^\infty(\Omega)} \|Au_m\|_{L^2}^2 \le C_2 \|u_m\|_{H^1}^{\frac{1}{2}} \|Au_m\|_{L^2}^{\frac{5}{2}},$$

$$\left| P_m\left(B\left(u_m, Au_m\right), Au_m\right)\right| \le \|u_{m,x}\|_{L^\infty(\Omega)} \|Au_m\|_{L^2}^2 \le C_3 \|u_m\|_{H^1}^{\frac{1}{2}} \|Au_m\|_{L^2}^{\frac{5}{2}},$$

$$\begin{array}{l} \left| \int_0^1 u_m (u_{mx})^3 dx \right| \le \|u_m\|_{L^\infty(\Omega)} \|u_{m,x}\|_{L^\infty(\Omega)} \|u_m\|_{H^1}^2 \\ \le (C_4 \|u_m\|_{L^2}^{\frac{1}{2}} \|u_m\|_{H^1}^{\frac{1}{2}})(C_5 \|u_m\|_{H^1}^{\frac{1}{2}} \|Au_m\|_{L^2}^{\frac{1}{2}}) \|u_m\|_{H^1}^2 \\ \le C_4 C_5 C_6^{\frac{1}{2}} C_7^{\frac{1}{2}} \|Au_m\|_{L^2}^{\frac{1}{2}} \|u_m\|_{H^1}^{\frac{5}{2}} \le C_8 \|Au_m\|_{L^2}^{\frac{1}{2}} \|u_m\|_{H^1}^{\frac{5}{2}} \end{array},$$

where $C_8 = C_4 C_5 C_6^{\frac{1}{2}} C_7^{\frac{1}{2}}$ and$\|u_m\|_{L^2} \le C_6, \|u_m\|_{H^1} \le C_7$.

$$\left| \int_0^1 (u_{m,x})^3 dx \right| \le \|u_{m,x}\|_{L^\infty(\Omega)} \|u_m\|_{H^1}^2 \le C_9 \|u_m\|_{H^1}^{\frac{5}{2}} \|Au_m\|_{L^2}^{\frac{1}{2}}.$$

From above inequalities, we can obtain

$$\left| P_m\left(B\left(u_m, v_m\right), Au_m\right) + 2P_m\left(B\left(v_m, u_m\right), Au_m\right) + P_m\left(C\left(u_m\right), Au_m\right) + ((u_m^3)_x, Au_m) - ((\frac{3u_m^2}{2})_x, Au_m) \right|$$

$$\le (3C_1 + 3C_8 + \frac{3}{2}C_9) \|u_m\|_{H^1}^{\frac{5}{2}} \|Au_m\|_{L^2}^{\frac{1}{2}} + (2C_2 + C_3) \|u_m\|_{H^1}^{\frac{1}{2}} \|Au_m\|_{L^2}^{\frac{5}{2}}$$

$$\le C_{10} \|u_m\|_{H^1}^{\frac{1}{2}} \|Au_m\|_{L^2}^{\frac{1}{2}} \left(\|u_m\|_{H^1}^2 + \|Au_m\|_{L^2}^2 \right),$$

where $C_{10} = \max\left\{3C_1 + 3C_8 + \frac{3}{2}C_9, 2C_2 + C_3\right\}$.

Then, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|u_m\|_{H^1}^2 + \|Au_m\|_{L^2}^2 \right) + \varepsilon \left(\|Au_m\|_{L^2}^2 + \|Au_m\|_{H^1}^2 \right)$$

$$\le C_{10} \|u_m\|_{H^1}^{\frac{1}{2}} \|Au_m\|_{L^2}^{\frac{1}{2}} \left(\|u_m\|_{H^1}^2 + \|Au_m\|_{L^2}^2 \right) + \frac{1}{2}\left(\|u_m\|_{H^1}^2 + \|Au_m\|_{L^2}^2 \right). \tag{3.6}$$

Applying Young's inequality and Poincare inequality, we have

$$\frac{d}{dt}\left(\|u_m\|_{H^1}^2 + \|Au_m\|_{L^2}^2 \right) + 2\varepsilon\lambda_2 \left(\|u_m\|_{H^1}^2 + \|Au_m\|_{L^2}^2 \right) \le 2\left(\varepsilon\lambda_2 - \frac{1}{2} \right) \left(\|u_m\|_{H^1}^2 + \|Au_m\|_{L^2}^2 \right)$$

$$+\left(\|u_m\|_{H^1}^2+\|Au_m\|_{L^2}^2\right)+\frac{C_{10}^2}{2\varepsilon\lambda_2-1}\|u_m\|_{H^1}\|Au_m\|_{L^2}\left(\|u_m\|_{H^1}^2+\|Au_m\|_{L^2}^2\right).$$

$\frac{d}{dt}\left(\|u_m\|_{H^1}^2+\|Au_m\|_{L^2}^2\right)\le C_{11}\left(\|u_m\|_{H^1}^2+\|Au_m\|_{L^2}^2\right)^2,$

where $C_{11}=\frac{C_{10}^2}{2(2\varepsilon\lambda_2-1)}$ is a constant,

λ_2 is a Poincare coefficient and $\varepsilon>\max\left\{\frac{1}{2\lambda_2},\frac{1}{2}\right\}$.

Let $y=\|u_m\|_{H^1}^2+\|Au_m\|_{L^2}^2$,$g=C_{11}\left(\|u_m\|_{H^1}^2+\|Au_m\|_{L^2}^2\right)$. Then,

we can derive $\int_t^{t+r}y(s)ds\le\frac{r_2}{\varepsilon},\int_t^{t+r}g(s)ds\le C_{11}\frac{r_2}{\varepsilon}$, for $r>0$.

According to uniform Gronwall inequality, we have

$$\|u_m\|_{H^1}^2+\|Au_m\|_{L^2}^2\le\left(\frac{r_2}{\varepsilon r}\right)\exp\left(C_{11}\frac{r_2}{\varepsilon}\right)\triangleq r_3.$$

Integrating (3.6) in the interval $[t,t+r]$, we can get

$$\varepsilon\int_t^{t+r}\left(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2\right)ds\le\frac{C_{10}r_2r_3}{4\varepsilon}+\frac{C_{10}r_2}{2\varepsilon}+\frac{r_3}{2}+\frac{r_2}{2\varepsilon}\triangleq r_4. \tag{3.7}$$

Taking inner product in (3.1) to be A^2u_m on Ω, we have

$$\begin{array}{l}\frac{1}{2}\frac{d}{dt}\left(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2\right)+\varepsilon\left(\|Au_m\|_{H^1}^2+\left\|A^2u_m\right\|_{L^2}^2\right)+P_m\left(B\left(u_m,v_m\right),A^2u_m\right)\\+2P_m\left(B\left(v_m,u_m\right),A^2u_m\right)+P_m\left(C\left(u_m\right),A^2u_m\right)+((u_m^3)_x,A^2u_m)-((\frac{3u_m^2}{2})_x,A^2u_m)=0\end{array}.$$

By computing, we have

$$((u_m^3)_x,A^2u_m)=-3(\int_0^1 2u_mu_{m,x}^2u_{m,xxx}+u_m^2u_{m,xx}u_{m,xxx}dx)=15\int_0^1 u_mu_{m,x}u_{m,xx}^2dx,$$

$$((\frac{3u_m^2}{2})_x,A^2u_m)=3(-\int_0^1 u_{m,x}^2u_{m,xxx}+u_mu_{m,xx}u_{m,xxx}dx)=\frac{15}{2}\int_0^1 u_{m,x}u_{m,xx}^2dx.$$

According to Agmon inequality when $n=1$, then we have

$$\left|P_m\left(B\left(u_m,u_m\right),A^2u_m\right)\right|\le\|u_{m,x}\|_{L^\infty(\Omega)}\|Au_m\|_{L^2}^2$$

$$\le C_{12}\|u_m\|_{H^1}^{\frac{1}{2}}\|Au_m\|_{L^2}^{\frac{1}{2}}(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2),$$

$$\left|P_m\left(B\left(Au_m,u_m\right),A^2u_m\right)\right|\le\|u_{m,x}\|_{L^\infty(\Omega)}\|Au_m\|_{H^1}^2\le C_{13}\|u_m\|_{H^1}^{\frac{1}{2}}\|Au_m\|_{L^2}^{\frac{1}{2}}(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2),$$

$$\left|P_m\left(B\left(u_m,Au_m\right),A^2u_m\right)\right|\le\|u_{m,x}\|_{L^\infty(\Omega)}\|Au_m\|_{H^1}^2\le C_{14}\|u_m\|_{H^1}^{\frac{1}{2}}\|Au_m\|_{L^2}^{\frac{1}{2}}(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2),$$

$$\left|\int_0^1 u_mu_{m,x}u_{m,xx}^2dx\right|\le\|u_m\|_{L^\infty(\Omega)}\|u_{m,x}\|_{L^\infty(\Omega)}\|Au_m\|_{L^2}^2$$

$$\le(C_{15}\|u_m\|_{L^2}^{\frac{1}{2}}\|u_m\|_{H^1}^{\frac{1}{2}})(C_{16}\|Au_m\|_{H^1}^{\frac{1}{2}}\|Au_m\|_{L^2}^{\frac{1}{2}})(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2)$$

$$\le C_{17}\|u_m\|_{H^1}^{\frac{1}{2}}\|Au_m\|_{L^2}^{\frac{1}{2}}(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2),\text{where}C_{17}=C_{15}C_{16}C_6^{\frac{1}{2}}C_7^{\frac{1}{2}},$$

$$\left|\int_0^1 u_{m,x}u_{m,xx}^2dx\right|\le\|u_{m,x}\|_{L^\infty(\Omega)}\|Au_m\|_{L^2}^2\le C_{18}\|u_m\|_{H^1}^{\frac{1}{2}}\|Au_m\|_{L^2}^{\frac{1}{2}}(\|Au_m\|_{L^2}^2+\|Au_m\|_{H^1}^2).$$

From above inequalities, we can obtain

$$\begin{array}{l}\left|P_m\left(B\left(u_m,v_m\right),A^2u_m\right)+2P_m\left(B\left(v_m,u_m\right),A^2u_m\right)+P_m\left(C\left(u_m\right),A^2u_m\right)\right.\\\left.+((u_m^3)_x,A^2u_m)-((\frac{3u_m^2}{2})_x,A^2u_m)\right|\end{array}$$

$$\leq (3C_{12} + 2C_{13} + +C_{14} + C_{15} + 15C_{17} + \frac{15}{2}C_{18}) \|u_m\|_{H^1}^{\frac{1}{2}} \|Au_m\|_{L^2}^{\frac{1}{2}} (\|Au_m\|_{L^2}^2 + \|Au_m\|_{H^1}^2)$$

$$\leq C_{19} \|u_m\|_{H^1}^{\frac{1}{2}} \|Au_m\|_{L^2}^{\frac{1}{2}} (\|Au_m\|_{L^2}^2 + \|Au_m\|_{H^1}^2).$$

Then, we have

$$\begin{array}{l} \frac{1}{2}\frac{d}{dt}\left(\|Au_m\|_{L^2}^2 + \|Au_m\|_{H^1}^2\right) + \varepsilon\left(\|Au_m\|_{H^1}^2 + \left\|A^2u_m\right\|_{L^2}^2\right) \\ \leq C_{19}\|u_m\|_{H^1}^{\frac{1}{2}}\|Au_m\|_{L^2}^{\frac{1}{2}}\left(\|Au_m\|_{L^2}^2 + \|Au_m\|_{H^1}^2\right) + \frac{1}{2}\left(\|Au_m\|_{H^1}^2 + \left\|A^2u_m\right\|_{L^2}^2\right) \end{array} \tag{3.8}$$

Applying Young's inequality and Poincare inequality, we can obtain

$$\frac{d}{dt}\left(\|Au_m\|_{L^2}^2 + \|Au_m\|_{H^1}^2\right) \leq C_{20}\|u_m\|_{H^1}\|Au_m\|_{L^2}\left(\|Au_m\|_{L^2}^2 + \|Au_m\|_{H^1}^2\right), \tag{3.9}$$

where $C_{20} = \frac{C_{19}^2}{\lambda_3(2\varepsilon-1)}$ is constant, $\varepsilon > \max\left\{\frac{1}{2\lambda_3}, \frac{1}{2}\right\}$ and λ_3 is a Poincare coefficient. We know that

$$\int_t^{t+r} C_{20}\|u_m\|_{H^1}\|Au_m\|_{L^2}\,ds \leq \frac{C_{20}}{2}, \int_t^{t+r}\left(\|u_m\|_{H^1}^2 + \|Au_m\|_{L^2}^2\right)ds \leq \frac{C_{20}r_2}{2\varepsilon} \tag{3.10}$$

Next, by uniform Growall inequality, it follows that

$$\|Au_m\|_{L^2}^2 + \|Au_m\|_{H^1}^2 \leq \left(\frac{r_4}{\varepsilon r}\right)\exp\left(\frac{C_{20}r_2}{2\varepsilon}\right) \triangleq r_5. \tag{3.11}$$

We integrate (3.8) in the interval$[t, t+r]$, and have

$$\left(\varepsilon - \frac{1}{2}\right)\int_t^{t+r}\left(\|Au_m\|_{H^1}^2 + \left\|A^2u_m\right\|_{L^2}^2\right)ds \leq \frac{C_{19}r_3r_4}{4\varepsilon} + \frac{C_{19}r_4}{2\varepsilon} + \frac{r_5}{2} \triangleq r_6, \tag{3.12}$$

where $\varepsilon > \max\left\{\frac{1}{2\lambda_3}, \frac{1}{2}\right\}$.

Taking the inner productof (3.1) to with A^3u_m in Ω, we can get

$$\|Au_m\|_{H^1}^2 + \left\|A^2u_m\right\|_{L^2}^2 \leq r_7 \tag{3.13}$$

from the method discussed above and uniform Gronwall lemma similarly.

Now, we have obtain that$\|u_m\|_{L^2}$,$\|u_m\|_{H^1}$,$\|Au_m\|_{L^2}$, $\|Au_m\|_{H^1}$and$\left\|A^2u_m\right\|_{L^2}$are bounded. Then$\|v_m\|_{L^2}$, $\|v_m\|_{H^1}$ and$\|Av_m\|_{L^2}$ are bounded. Hence we can conclude that $\|u_{m,t}\|_{L^2}$ and $\|v_{m,t}\|_{L^2}$ are bounded. By $Aubin's$ Compactness Theorem, we conclude that there is a subsequenceu_m', such that$u_m' \to u$, or equivalently $v_m' \to v$. Let us replace u_m' and v_m' by u_m andv_m. Now we prove u, v satisfy equation (2.1).

Let $\omega \in D(A)$. We know $\|\omega\|_{L^2}$ is bounded from the above discussion. From ordinary differential equations (3.1), we get

$$(v_m(t), w) + \varepsilon\int_0^t (Av_m(s), P_m w)ds + 2\int_0^t (B(v_m(s), u_m(s)), p_m w)\,ds + \int_0^t (B(u_m(s), v_m(s)), p_m w)\,ds$$

$$+\int_0^t (C(u_m), P_m\omega)ds + \int_0^t ((u_m^3)_x, P_m\omega)ds - \frac{3}{2}\int_0^t (u_m u_m, x, P_m\omega)ds = (v_m(0), w).$$

Now, it is clear that $\lim\limits_{m\to+\infty}|p_m\omega - \omega| = 0$, $\lim\limits_{m\to+\infty}|p_m A\omega - A\omega| = 0$, and so

$$(Av_m(t), \omega) \to (Av(t), \omega),$$

$\int_0^t (Av_m(s), P_m\omega)ds \to \int_0^T (Av(s), \omega)ds$,as$m \to +\infty$,

$$\left|\int_0^t (C(u_m(s)), P_m\omega)\,ds - \int_0^t (C(u(s)), \omega)\,ds\right|$$

$$\leq \left|\int_0^t \left(C\left(u_m\left(s\right)\right), P_m\omega-\omega\right)ds\right| + \left|\int_0^t \left(C\left(u_m\left(s\right)\right) - C\left(u\left(s\right)\right), \omega\right)ds\right|$$

$\leq \int_0^t \left\|u_m\right\|_{H^1} \left\|P_m\omega-\omega\right\|_{L^2} ds + \int_0^t \left\|u_m-u\right\|_{H^1} \left\|\omega\right\|_{L^2} ds \to 0$,as $m \to +\infty$,

$$\left|\int_0^t \left(B\left(v_m\left(s\right), u_m\left(s\right)\right), P_m\omega\right)ds - \int_0^t \left(B\left(v\left(s\right), u\left(s\right)\right), \omega\right)ds\right| \leq I_m^{(1)} + I_m^{(2)} + I_m^{(3)},$$

where

$$I_m^{(1)} = \left|\int_0^t \left(B\left(v_m\left(s\right), u_m\left(s\right)\right), P_m\omega-\omega\right)ds\right| \leq \int_0^t \left|B\left(v_m\left(s\right), u_m\left(s\right)\right)\right| \left|P_m\omega-\omega\right| ds \to 0,$$

$$I_m^{(2)} = \left|\int_0^t \left(B\left(v\left(s\right), u_m\left(s\right)-u\left(s\right)\right), \omega\right)ds\right| \leq \int_0^t \left\|v\left(s\right)\right\|_{L^2} \left\|u_m\left(s\right)-u\left(s\right)\right\|_{H^1} \left\|\omega\right\|_{L^2} ds \to 0,$$

$$I_m^{(3)} = \left|\int_0^t \left(B\left(v_m\left(s\right)-v\left(s\right), u_m\left(s\right)\right), \omega\right)ds\right| \leq \int_0^t \left\|v_m\left(s\right)-v\left(s\right)\right\|_{L^2} \left\|u_m\left(s\right)\right\|_{H^1} \left\|\omega\right\|_{L^2} ds \to 0$$

as $m \to +\infty$.

According to above discussion and Lebesgue Control Convergence Theorem, we have

$$\lim_{m\to\infty} \int_0^t \left(B\left(v_m\left(s\right), u_m\left(s\right)\right), P_m\omega\right)ds = \int_0^t \left(B\left(v\left(s\right), u\left(s\right)\right), \omega\right)ds.$$

Similarly, $\left|\int_0^t \left(B\left(u_m\left(s\right), v_m\left(s\right)\right), P_m\omega\right)ds - \int_0^t \left(B\left(u\left(s\right), v\left(s\right)\right), \omega\right)ds\right| \leq I_m^{(4)} + I_m^{(5)} + I_m^{(6)}$,

where $I_m^{(4)} \to 0$, $I_m^{(5)} \to 0$,$I_m^{(6)} \to 0$ as $n \to \infty$.

According to above discussion and Lebesgue Control Convergence Theorem, we have

$$\lim_{m\to\infty} \int_0^t \left(B\left(u_m\left(s\right), v_m\left(s\right)\right), P_m\omega\right)ds = \int_0^t \left(B\left(u\left(s\right), v\left(s\right)\right), \omega\right)ds.$$

We can also get

$$\left|\int_0^t ((u_m^3)_x, P_m\omega)ds - \int_0^t ((u^3)_x, \omega)ds\right| \leq \left|\int_0^t ((u_m^3)_x, P_m\omega-\omega)ds\right| + \left|\int_0^t ((u_m^3)_x - (u^3)_x, \omega)ds\right|$$

$\leq \int_0^t \left\|u_m^3\right\|_{H^1} \left\|P_m\omega-\omega\right\|_{L^2} ds + \int_0^t \left\|u_m^3-u^3\right\|_{H^1} \left\|\omega\right\|_{L^2} ds$
$\leq \int_0^t \left\|u_m^3\right\|_{H^1} \left\|P_m\omega-\omega\right\|_{L^2} ds + \int_0^t \left\|(u_m-u)(u_m^2+u_mu+u^2)\right\|_{H^1} \left\|\omega\right\|_{L^2} ds \to 0$, as $m \to +\infty$,

$$\left|\int_0^t ((\frac{3}{2}u_m^2)_x, P_m\omega)ds - \int_0^t ((\frac{3}{2}u^2)_x, \omega)ds\right|$$

$$\leq \left|\int_0^t ((\frac{3}{2}u_m^2)_x, P_m\omega-\omega)ds\right| + \left|\int_0^t ((\frac{3}{2}u_m^2)_x - (\frac{3}{2}u^2)_x, \omega)ds\right|$$

$\leq \int_0^t \left\|\frac{3}{2}u_m^2\right\|_{H^1} \left\|P_m\omega-\omega\right\|_{L^2} ds + \int_0^t \left\|\frac{3}{2}(u_m-u)(u_m+u)\right\|_{H^1} \left\|\omega\right\|_{L^2} ds \to 0$, as $m \to +\infty$.

Finally, for all $\omega \in D\left(A\right)$, we can conclude

$$\int_0^t \left(v_t\left(t\right), \omega\right)ds + \varepsilon \int_0^t \left(Av\left(s\right), \omega\right)ds + 2\int_0^t \left(B\left(v\left(s\right), u\left(s\right)\right), \omega\right)ds + \int_0^t \left(B\left(u\left(s\right), v\left(s\right)\right), \omega\right)ds$$

$$+ \int_0^t ((u^3)_x, \omega)ds - \int_0^t ((\frac{3}{2}u^2)_x, \omega)ds + \int_0^t \left(C\left(u\left(s\right)\right), \omega\right)ds = (v(0), \omega). \tag{3.14}$$

Above all, we can conclude that the existence of global solution to Eq.(2.1).

Uniqueness of the solution is an immediate consequence.(see [13])

Second, we will prove the global exponential stability of the solution.

We take the inner product of (2.1) with u in $(0,1)$ to obtain,

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2+\|u\|_{H^1}^2\right)+\varepsilon\left(\|u\|_{H^1}^2+\|Au\|_{L^2}^2\right)=0 \tag{3.15}$$

By Poincare inequality, we have $\|u\|_{H^1}^2\geq\lambda_1\|u\|_{L^2}^2$, $\|Au\|_{L^2}^2\geq\lambda_1\|u\|_{H^1}^2$,

$$\frac{d}{dt}\left(\|u\|_{L^2}^2+\|u\|_{H^1}^2\right)+2\varepsilon\lambda_1\left(\|u\|_{L^2}^2+\|u\|_{H^1}^2\right)\leq 0,$$

$$\frac{d}{dt}\left(\|u\|_{L^2}^2+\|u\|_{H^1}^2\right)\leq-2\varepsilon\lambda_1\left(\|u\|_{L^2}^2+\|u\|_{H^1}^2\right).$$

Thus we get (2.2)

$\|u\|_{L^2}^2+\|u\|_{H^1}^2\leq(\|u(0)\|_{L^2}^2+\|u(0)\|_{H^1}^2)\exp\left(-2\varepsilon\lambda_1\right)t$, where λ_1 is a Poincare coefficient.

Now, take the inner product of (2.1) with Au in $(0,1)$ to obtain,

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{H^1}^2+\|Au\|_{L^2}^2\right)+\varepsilon\left(\|Au\|_{L^2}^2+\|Au\|_{H^1}^2\right)+\left(B\left(u,v\right),Au\right)+2\left(B\left(v,u\right),Au\right)$$

$$+\left(C\left(u\right),Au\right)+((u^3)_x,Au)-((\frac{3u^2}{2})_x,Au)=0.$$

By computing, we have

$$\left(C\left(u\right),Au\right)=0,$$

$$\left|((u^3)_x,Au)\right|=\left|3\int_0^1u^2u_xu_{xx}dx\right|\leq 3\|u\|_{L^2}^2\|u\|_{H^1}\|Au\|_{L^2}\leq a_1\left(\|u\|_{H^1}^2+\|Au\|_{L^2}^2\right),$$

$$((\frac{3u^2}{2})_x,Au)=\left|3\int_0^1uu_xu_{xx}dx\right|\leq 3\|u\|_{L^2}\|u\|_{H^1}\|Au\|_{L^2}\leq a_2\left(\|u\|_{H^1}^2+\|Au\|_{L^2}^2\right),$$

$$\left|\left(B\left(u,v\right),Au\right)+2\left(B\left(v,u\right),Au\right)\right|$$
$$=\left|3\left(B\left(u,u\right),Au\right)+\left(B\left(Au,u\right),Au\right)+\left(B\left(u,Au\right),Au\right)\right|,$$

$$\left|\left(B\left(u,u\right),Au\right)\right|\leq\|u\|_{L^2}\|u\|_{H^1}^2\|Au\|_{L^2}\leq a_3\left(\|u\|_{H^1}^2+\|Au\|_{L^2}^2\right),$$

$$\left|\left(B\left(Au,u\right),Au\right)\right|\leq\|u\|_{H^1}\|Au\|_{L^2}^2\leq a_4\left(\|u\|_{H^1}^2+\|Au\|_{L^2}^2\right),$$

$$\left|\left(B\left(u,Au\right),Au\right)\right|\leq a_5\|Au\|_{L^2}^2\leq a_5\left(\|u\|_{H^1}^2+\|Au\|_{L^2}^2\right).$$

So we have

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{H^1}^2+\|Au\|_{L^2}^2\right)\leq\left(a_1+a_2+a_3+a_4+a_5-\varepsilon\lambda_2\right)\left(\|u\|_{H^1}^2+\|Au\|_{L^2}^2\right),$$

$$\begin{aligned}&\|u\|_{H^1}^2+\|Au\|_{L^2}^2\leq\left(\|u(0)\|_{H^1}^2+\|Au(0)\|_{L^2}^2\right)\exp 2\left(a_1+a_2+a_3+a_4+a_5-\varepsilon\lambda_2\right)t\\&\triangleq\left(\|u(0)\|_{H^1}^2+\|Au(0)\|_{L^2}^2\right)\exp(b_1-2\varepsilon\lambda_2)t,\end{aligned} \tag{3.16}$$

where $\varepsilon\in\left(\frac{b_1}{2\lambda_2},+\infty\right)$ and b_1 is a nonnegative constant, λ_2 is a Poincare coefficient.

Then we get (2.3).

Take the inner product of (2.1) with A^2u in $(0,1)$ to obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|Au\|_{L^2}^2+\|Au\|_{H^1}^2\right)+\varepsilon\left(\|Au\|_{H^1}^2+\left\|A^2u\right\|_{L^2}^2\right)+\left(B\left(u,v\right),A^2u\right)+2\left(B\left(v,u\right),A^2u\right)$$

$$+\left(C\left(u\right),A^2u\right)+((u^3)_x,A^2u)-((\frac{3u^2}{2})_x,A^2u)=0,$$

$$\left(C\left(u\right),A^2u\right)=0,$$

$$\left|((u^3)_x, A^2u)\right| = \left|15\int_0^1 uu_xu_{xx}^2dx\right| \leq 15\|u\|_{L^\infty(\Omega)}\|u_x\|_{L^\infty(\Omega)}\|Au\|_{L^2}^2$$

$$\leq 15(a_6\|u\|_{L^2}^{\frac{1}{2}}\|u\|_{H^1}^{\frac{1}{2}})(a_7\|u\|_{H^1}^{\frac{1}{2}}\|Au\|_{L^2}^{\frac{1}{2}})(\|Au\|_{L^2}^2+\|Au\|_{H^1}^2) \leq a_8\|u\|_{H^1}^{\frac{1}{2}}\|Au\|_{L^2}^{\frac{1}{2}}(\|Au\|_{L^2}^2+\|Au\|_{H^1}^2)$$

$$\left|((\frac{3u^2}{2})_x, A^2u)\right| = \left|\frac{15}{2}\int_0^1 u_xu_{xx}^2dx\right| \leq \frac{15}{2}\|u\|_{L^\infty(\Omega)}\|Au\|_{L^2}^2 \leq a_9\|u\|_{H^1}^{\frac{1}{2}}\|Au\|_{L^2}^{\frac{1}{2}}(\|Au\|_{L^2}^2+\|Au\|_{H^1}^2),$$

$$\left|\left(B\left(u,v\right),A^2u\right)+2\left(B\left(v,u\right),A^2u\right)\right| = \left|3(B(u,u),A^2u)+(B(u,Au),A^2u)+(B(Au,u),A^2u)\right|,$$

$$\left|\left(B\left(u,u\right),A^2u\right)\right| \leq \|u_x\|_{L^\infty}\|Au\|_{L^2}^2 \leq a_{10}\|u\|_{H^1}^{\frac{1}{2}}\|Au\|_{L^2}^{\frac{1}{2}}\left(\|Au\|_{L^2}^2+\|Au\|_{H^1}^2\right),$$

$$\left|\left(B\left(Au,u\right),A^2u\right)\right| \leq \|u_x\|_{L^\infty}\|Au\|_{H^1}^2 \leq a_{11}\|u\|_{H^1}^{\frac{1}{2}}\|Au\|_{L^2}^{\frac{1}{2}}\left(\|Au\|_{L^2}^2+\|Au\|_{H^1}^2\right),$$

$$\left|\left(B\left(u,Au\right),A^2u\right)\right| \leq \|u_x\|_{L^\infty}\|Au\|_{H^1}^2 \leq a_{12}\|u\|_{H^1}^{\frac{1}{2}}\|Au\|_{L^2}^{\frac{1}{2}}\left(\|Au\|_{L^2}^2+\|Au\|_{H^1}^2\right)$$

By (3.16), we get $\|u\|_{H^1} \leq a_{13}, \|Au\|_{H^1} \leq a_{14}$.

From above inequalities, we have

$$\begin{array}{l}\frac{1}{2}\frac{d}{dt}\left(\|Au\|_{L^2}^2+\|Au\|_{H^1}^2\right)+\varepsilon\left(\|Au\|_{H^1}^2+\left\|A^2u\right\|_{L^2}^2\right)\\ \leq (a_8+a_9+3a_{10}+a_{11}+a_{12})\|u\|_{H^1}^{\frac{1}{2}}\|Au\|_{L^2}^{\frac{1}{2}}\left(\|Au\|_{L^2}^2+\|Au\|_{H^1}^2\right)\\ \leq (a_8+a_9+3a_{10}+a_{11}+a_{12})a_{13}^{\frac{1}{2}}a_{14}^{\frac{1}{2}}\left(\|Au\|_{L^2}^2+\|Au\|_{H^1}^2\right) \triangleq b_2\left(\|Au\|_{L^2}^2+\|Au\|_{H^1}^2\right)\end{array},$$

where b_2 is a nonnegative constant.

By Poincare inequality, we have

$$\frac{d}{dt}\left(\|Au\|_{L^2}^2+\|Au\|_{H^1}^2\right) \leq (2b_2-2\varepsilon\lambda_3)\left(\|Au\|_{L^2}^2+\|Au\|_{H^1}^2\right),$$

$$\|Au\|_{L^2}^2+\|Au\|_{H^1}^2 \leq \left(\|Au(0)\|_{L^2}^2+\|Au(0)\|_{H^1}^2\right)\exp\left(2b_2-2\varepsilon\lambda_3\right)t,$$

where $\varepsilon \in (\frac{b_2}{\lambda_3},+\infty)$ and b_2 is a nonnegative constant, λ_3 is a Poincare coefficient.

We get (2.4) of the theorem. The proof of Theorem is completed.

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