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Rapid Convergence for Nonlinear Periodic Boundary-Value Problem *

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Abstract: In this paper, we investigate the convergence of a sequence of approximate solutions for a nonlinear mixed boundary value problem and prove that it is possible to construct two sequences of approximate solutions converging to the extremal solution with rate of order k.

Keywords: quasilinearization; rapid convergence; upper and lower solutions; periodic boundary value problem.

1 Introduction

The problem of existence of solutions of various differential equations has received a great deal of attention during the last few years. For example, Ni and Lin [1,2] studied Duffing and Riccati differential equations and obtained the existence of almost periodic solutions; Dong, Fan and Li [3,4] obtained the existence of mild solutions to nonlocal neutral functional differential, integrodifferential equations and nonlocal Cauchy problem. It is well known [5,6] that the method of quasilinearization has been employed to obtain the existence of approximate solutions to nonlinear differential problems. Recently in [7], this quasilinearization method has been extended widely to a variety of initial-value and boundary-value problems for different types of differential equations. In this paper, we study the following second order periodic boundary value problem (PBVP)

$$x'' = f(t, x', x),\ x(0) = a,\ x'(0) = x'(2\pi), \tag{1.1}$$

where $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $J = [0, 2\pi]$, and construct sequences of approximate solutions that converge rapidly to the extremal solutions of (1.1) by using the quasilinearization method with rate of convergence of order $k > 2$.

The result in this paper is inspired by Lakshmikantham and Vatsala [5], Lakshmikantham and Rama Mohana Rao [6], Cabada and Nieto [8]. In Ref. [6], Lakshmikantham and Rama Mohana Rao discussed a PBVP for nonlinear integro-differential equation. Here we establish an integro-differential inequality as a comparison principle, then show that the problem (1.1) has extremal solutions which can be approximated via monotone sequences with rate of convergence of order k.

2 Preliminary Results

In this section, we present some known results relative to the existence of extremal solutions for the problem (1.1) lying between a pair of lower and upper solutions.

We see that the problem (1.1) is equivalent to the following integro-differential equation

$$u' = f(t, u, Tu),\ u(0) = u(2\pi), \tag{2.1}$$

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where $Tu = a + \int_0^t u(s)\,\mathrm{d}s$. Then, we will consider the problem (2.1).

Assume that the following hypotheses hold: (H_0) $\alpha, \beta \in C^1(J, R)$, such that $\alpha(t) \le \beta(t)$,

$$\alpha' \le f(t, \alpha, T\alpha), \ \ \alpha(0) \le \alpha(2\pi),$$

and

$$\beta' \ge f(t, \beta, T\beta), \ \ \beta(0) \ge \beta(2\pi);$$

(H_1) $f(t, x, y)$ is nonincreasing in x, y and

$$f(t, x, Tx) - f(t, y, Ty) \ge -M(x - y) - NT(x - y),$$

whenever $x \ge y, M > 0$ and $N \ge 0$ are constants such that $2N\pi e^{2M\pi} < M$.

Lemma 2.1 (see [6]) *Let $m \in C^1(J, R)$ be such that*

$$m' \le -Mm - NTm, \ \ m(0) = m(2\pi),$$

where $M > 0, N \ge 0$. Then, $m(t) \le 0$ for $0 \le t \le 2\pi$ provided one of the conditions hold (a) $2N\pi(e^{2M\pi} - 1) \le M;$ *or* (b) $2\pi[M + (2\pi)N] \le 1$.

Theorem 2.1 (see [6]) *Assume that there exist $\alpha \le \beta$ lower and upper solutions respectively for the problem (2.1) and (H_0) and (H_1) hold. Then, there exist two monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ with $\alpha_0 = \alpha$ and $\beta_0 = \beta$ such that $\lim_{n\to\infty} \alpha_n(t) = \rho(t)$ and $\lim_{n\to\infty} \beta_n(t) = r(t)$ uniformly on J, and ρ and r are minimal and maximal solutions of PBVP (2.1) respectively, satisfying $\alpha(t) \le \rho(t) \le r(t) \le \beta(t)$ on J.*

3 Main Result

As we have seen in Section 2, the extremal solutions of problem (2.1) are approximated, via the monotone method, by two sequences starting at the lower solution and upper solution.

Consider the Banach space $C(J)$ with the usual supremum norm $\|.\|_\infty$. We recall the following definitions. For a given sequence $\{x_n\} \subset C(J)$, we say that $\{x_n\}$ converges to x with order of convergence k, if $\{x_n\}$ converges to x in $C(J)$ and there exist $m_0 \in \mathbb{N}$ and $\lambda > 0$ such that $\|x_{m+1} - x\|_\infty \le \|x_m - x\|_\infty^k$ for all $m \ge m_0$.

Theorem 3.1 *Suppose that there exist $\alpha \le \beta$ lower and upper solutions respectively for the problem (2.1) and for some $k \ge 1$, $\frac{\partial^i f}{\partial u^i}$, $\frac{\partial^i f}{\partial (Tu)^i}$ exist for every $i = 0, 1, \ldots, k$, and are continuous on $\Omega = \{(t, u,) : t \in J, \alpha(t) \le u \le \beta(t)\}$. Assume that*

$$\sum_{i-1}^{k} iM_i\|\beta - \alpha\|_\infty^{i-1} \le M, \ \ \sum_{i-1}^{k} iN_i\|T(\beta - \alpha)\|_\infty^{i-1} \le N, \tag{3.1}$$

where $M_i \ge 0$ and $N_i \ge 0$ are constants with

$$\frac{\partial^i f}{\partial u^i}(t, u, Tu) \ge -(i!)M_i, \ \ \frac{\partial^i f}{\partial (Tu)^i}(t, u, Tu) \ge -(i!)N_i \tag{3.2}$$

for all $i \in \{1, \ldots, k\}$ and $(t, u, Tu) \in \Omega$. Here, $M > 0$ and $N \ge 0$ are constants in Lemma 2.1. Moreover, assume that there exist $P_1 > 0$ and $Q_1 > 0$ such that

$$\frac{\partial f}{\partial u}(t, u, Tu) \le P_1, \ \ \frac{\partial f}{\partial Tu}(t, u, Tu) \le Q_1, \ \ (t, u) \in \Omega. \tag{3.3}$$

Then, there exists a monotone sequence $\{\gamma_m\}$ with $\gamma_0 = \alpha$, which converges uniformly to the minimal solution γ of (2.1) in $[\alpha, \beta]$, and the convergence is of order k.

Proof. First, we note that condition (3.2) implies that

$$\frac{\partial f}{\partial u}(t,u,Tu) \geq -M_1, \;\; \frac{\partial f}{\partial Tu}(t,u,Tu) \geq -N_1,$$

with $0 \leq M_1 \leq M,\; 0 \leq N_1 \leq N$. Thus, Theorem 2.1 implies that problem (2.1) has extremal solutions in the sector $[\alpha, \beta]$. Let γ be the minimal solution.

To construct the sequence $\{\gamma_m\}$, let $\alpha(t) \leq v \leq u \leq \beta(t), t \in J$. We note that

$$\begin{aligned} f(t,u,Tu) = \;& \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial u^i}(t,v,Tv)\frac{(u-v)^i}{i!} + \frac{\partial^k f}{\partial u^k}(t,\chi,T\chi)\frac{(u-v)^k}{k!} \\ & + \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial (Tu)^i}(t,v,Tv)\frac{T(u-v)^i}{i!} + \frac{\partial^k f}{\partial (Tu)^k}(t,\chi,T\chi)\frac{T(u-v)^k}{k!}, \end{aligned} \tag{3.4}$$

where $\chi(t) \in [v,u]$. We define the following auxiliary function

$$\begin{aligned} g(t,u,Tu;v,Tv) = \;& \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial u^i}(t,v,Tv)\frac{(u-v)^i}{i!} - M_k(u-v)^k \\ & + \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial (Tu)^i}(t,v,Tv)\frac{T(u-v)^i}{i!} - N_k T(u-v)^k. \end{aligned} \tag{3.5}$$

As a consequence, using (3.2), we obtain that

$$g(t,u,Tu;v,Tv) \leq f(t,u,Tu), \;\; t \in J, \;\; v,u \in \Omega. \tag{3.6}$$

Now, let $\gamma_0 = \alpha$, and consider the following boundary value problem

$$\begin{aligned} & u'(t) = g(t,u(t),Tu(t);\gamma_0(t),T\gamma_0(t)), \quad t \in J, \\ & u(0) = u(2\pi). \end{aligned} \tag{3.7}$$

We see that α and γ are lower and upper solutions for (3.7), respectively. For $t \in J$ and $\alpha(t) \leq x \leq y \leq \gamma(t)$, we have that

$$\begin{aligned} & g(t,x,Tx;\alpha,T\alpha) - g(t,y,Ty;\alpha,T\alpha) \\ & = \sum_{i=1}^{k-1} \frac{\partial^i f}{\partial u^i}(t,\alpha,T\alpha)\Big(\frac{(x-\alpha)^i - (y-\alpha)^i}{i!}\Big) - M_k\Big((x-\alpha)^k - (y-\alpha)^k\Big) \\ & \quad + \sum_{i=1}^{k-1} \frac{\partial^i f}{\partial (Tu)^i}(t,\alpha,T\alpha)\left(\frac{T(x-\alpha)^i - T(y-\alpha)^i}{i!}\right) - N_k\left(T(x-\alpha)^k - T(y-\alpha)^k\right) \\ & = (x-y)\Big\{\sum_{i=1}^{k-1} \frac{\partial^i f}{\partial u^i}(t,\alpha,T\alpha)\Big(\frac{1}{i!}\sum_{j=0}^{i-1}(x-\alpha)^{i-1-j}(y-\alpha)^j\Big) - M_k(x-\alpha)^{i-1-j}(y-\alpha)^j\Big\} \\ & \quad + T(x-y)\Big\{\sum_{i=1}^{k-1} \frac{\partial^i f}{\partial (Tu)^i}(t,\alpha,T\alpha)\Big(\frac{1}{i!}\sum_{j=0}^{i-1}T(x-\alpha)^{i-1-j}T(y-\alpha)^j\Big) \\ & \quad - N_k T(x-\alpha)^{i-1-j}T(y-\alpha)^j\Big\} \\ & \geq -(x-y)\sum_{i=1}^{k} M_i\Big(\sum_{j=0}^{i-1}(\beta-\alpha)^{i-j}\Big) - T(x-y)\sum_{i=1}^{k} N_i\Big(\sum_{j=0}^{i-1}T(\beta-\alpha)^{i-j}\Big) \\ & \geq -M(x-y) - NT(x-y). \end{aligned}$$

Thus, Theorem 2.1 shows that problem (3.7) has extremal solutions in $[\alpha, \gamma]$. Let $\gamma_1 \in [\alpha, \gamma]$ be the maximal solution of the mentioned problem.

Now, suppose we have constructed $\gamma_0 = \alpha \leq \gamma_1 \leq ... \leq \gamma_m \leq \gamma$, with γ_m the maximal solution in $[\gamma_{m-1}, \gamma]$ of the problem

$$\begin{aligned} & u'(t) = g(t,u(t),Tu(t);\gamma_{m-1}(t),T\gamma_{m-1}(t)), \;\; t \in J, \\ & u(0) = u(2\pi). \end{aligned} \tag{3.8}$$

In this case, by (3.6) we also have that γ and γ_m are upper and lower solution respectively for the following problem

$$\begin{array}{l} u'(t) = g(t, u(t), Tu(t); \gamma_m(t), T\gamma_m(t)), \ \ t \in J, \\ u(0) = u(2\pi). \end{array} \tag{3.9}$$

As in problem (3.7), we have that $g(t, x, Tx; \gamma_m, T\gamma_m)$ satisfies condition (H_1) for some $M > 0$, $N \geq 0$ such that Lemma 2.1 is verified. Thus, using again Theorem 2.1, we conclude that problem (3.9) has extremal solutions in $[\gamma_m, \gamma]$. Let γ_{m+1} be the maximal one. Hence, the constructed sequence $\{\gamma_m\}$ is nondecreasing and bounded in $C^1(J)$. In consequence, it converges uniformly in $C(J)$ to some continuous function $\psi(t) \in [\alpha, \gamma]$.

By standard arguments, it is easy to see that ψ is a solution of (2.1). Since γ is the minimal solution of (2.1) in $[\alpha, \beta]$ and $\gamma_m \leq \gamma$, it is clear that $\psi = \gamma$.

Now, we prove that the convergence is indeed of order k. First, note that, for all $m \geq 1, t \in J$.

$$\begin{aligned} \gamma' &= f(t, \gamma, T\gamma) \\ &= \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial u^i}(t, \gamma_m, T\gamma_m) \frac{(\gamma - \gamma_m)^i}{i!} + \frac{\partial^k f}{\partial u^k}(t, \chi_m, T\chi_m) \frac{(\gamma - \gamma_m)^k}{k!} \\ &\quad + \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial (Tu)^i}(t, \gamma_m, T\gamma_m) \frac{T(\gamma - \gamma_m)^i}{i!} + \frac{\partial^k f}{\partial (Tu)^k}(t, \chi_m, T\chi_m) \frac{T(\gamma - \gamma_m)^k}{k!}, \end{aligned}$$

where $\chi_m \in [\gamma_m, \gamma]$. Let $p_{m+1}(t) = \gamma(t) - \gamma_{m+1}(t)$. In consequence,

$$\begin{aligned} p'_{m+1}(t) &= \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial u^i}(t, \gamma_m, T\gamma_m) \left(\frac{p_m^i - (\gamma_{m+1} - \gamma_m)^i}{i!} \right) \\ &\quad + \frac{\partial^k f}{\partial u^k}(t, \chi_m, T\chi_m) \left(\frac{p^k}{k!} \right) + M_k(\gamma_{m+1} - \gamma_m)^k \\ &\quad + \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial (Tu)^i}(t, \gamma_m, T\gamma_) \left(\frac{Tp_m^i - T(\gamma_{m+1} - \gamma_m)^i}{i!} \right) \\ &\quad + \frac{\partial^k f}{\partial (Tu)^k}(t, \chi_m, T\chi_m) \left(\frac{Tp^k}{k!} \right) + N_k T(\gamma_{m+1} - \gamma_m)^k \\ &= p_{m+1} \sum_{i=1}^{k-1} \frac{\partial^i f}{\partial u^i}(t, \gamma_m, T\gamma_m) \left(\frac{1}{i!} \sum_{j=0}^{i-1} p_m^{i-1-j} (\gamma_{m+1} - \gamma_m)^j \right) \\ &\quad + \frac{\partial^k f}{\partial u^k}(t, \chi_m, T\chi_m) \left(\frac{p_m^k}{k!} \right) + M_k(\gamma_{m+1} - \gamma_m)^k \\ &\quad + Tp_{m+1} \sum_{i=1}^{k-1} \frac{\partial^i f}{\partial u^i}(t, \gamma_m, T\gamma_m) \left(\frac{1}{i!} \sum_{j=0}^{i-1} Tp_m^{i-1-j} T(\gamma_{m+1} - \gamma_m)^j \right) \\ &\quad + \frac{\partial^k f}{\partial u^k}(t, \chi_m, T\chi_m) \left(\frac{Tp_m^k}{k!} \right) + N_k T(\gamma_{m+1} - \gamma_m)^k \end{aligned}$$

The continuity of $\frac{\partial^k f}{\partial u^k}$ and $\frac{\partial^k f}{\partial (Tu)^k}$ in Ω implies that there exist $A_k \geq 0, B_k \geq 0$ such that

$$\frac{\partial^k f}{\partial u^k}(t, u, Tu) \leq k! A_k, \quad \frac{\partial^k f}{\partial (Tu)^k}(t, u, Tu) \leq k! B_k, \ \ (t, u) \in \Omega.$$

Therefore, we have that

$$p'_{m+1}(t) - P_m(t) p_{m+1}(t) - Q_m(t) T p_{m+1}(t) \leq C_k p_m^k(t) + D_k T p_m^k(t), \ \ t \in J,$$

where $C_k = A_k + M_k > 0, D_k = B_k + N_k > 0$, and

$$\begin{aligned} P_m(t) &= \sum_{i=1}^{k-1} \frac{\partial^i f}{\partial u^i}(t, \gamma_m(t), T\gamma_m(t)) \left(\frac{1}{i!} \sum_{j=0}^{i-1} p_m^{i-1-j}(t) (\gamma_{m+1} - \gamma_m)^j(t) \right), \\ Q_m(t) &= \sum_{i=1}^{k-1} \frac{\partial^i f}{\partial u^i}(t, \gamma_m(t), T\gamma_m(t)) \left(\frac{1}{i!} \sum_{j=0}^{i-1} Tp_m^{i-1-j}(t) T(\gamma_{m+1} - \gamma_m)^j(t) \right). \end{aligned}$$

Since γ_m converges uniformly to γ in J, condition (3.3) implies that there exist $m_0 \in \mathbb{N}$ and $P > 0$, $Q > 0$, such that $P_m(t) \leq -P < 0, \quad Q_m(t) \leq -Q < 0,$ for all $m > m_0$, and $t \in J$. Then, there exists a continuous function $\sigma_m \leq 0$ on J such that

$$p'_{m+1}(t) + Pp_{m+1}(t) + QTp_{m+1}(t) = C_k p_m^k(t) + \sigma_m(t) + D_k T p_m^k(t) + T\sigma_m(t), \;\; t \in J,$$
$$p_{m+1}(0) = p_{m+1}(2\pi);$$

or equivalently

$$p_{m+1}(t) = \int_0^{2\pi} G(t,s,P)\left[C_k p_m^k(t) + \sigma_m(t)\right] \mathrm{d}s + \int_0^{2\pi} G(t,s,Q)\left[D_k T p_m^k(t) + T\sigma_m(t)\right] \mathrm{d}s.$$

Here, G is the Green function associated to the linear boundary-value problem

$$u' + Pu + QTu = \sigma(t) + T\sigma(t), \;\; u(0) = u(2\pi).$$

Using Lemma 2.1, we have that G on $J \times J$ is positive, since the solution of problem (2.1) is given by

$$u(t) = \int_0^{2\pi} G(t,s,P)\sigma(s)\,\mathrm{d}s + \int_0^{2\pi} G(t,s,Q)T\sigma(s)\,\mathrm{d}s.$$

Using

$$\int_0^{2\pi} G(t,s,P)\,\mathrm{d}s = \frac{1}{P}, \quad \int_0^{2\pi} G(t,s,Q)\,\mathrm{d}s = \frac{1}{Q}.$$

we conclude that, for every $t \in J$ and $m \geq m_0$,

$$\begin{aligned} 0 \leq \gamma(t) - \gamma_{m+1}(t) &\leq \tfrac{C_k}{P} \max_{t\in J} p_m^k(t) + \tfrac{D_k}{Q} T p_m^k(t) \\ &\leq \tfrac{C_k}{P} \max_{t\in J} p_m^k(t) + \tfrac{D_k}{Q} 2\pi \max_{t\in J} p_m^k(t). \end{aligned}$$

Hence,

$$\|\gamma - \gamma_{m+1}\|_\infty \leq \lambda \|\gamma - \gamma_m\|_\infty^k, \;\; \text{for all} \;\; m \geq m_0 \;\; \text{and} \;\; \lambda = \frac{C_k}{P} + \frac{D_k}{Q} > 0.$$

The proof is complete. ■

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