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Existence of Periodic Solutions for n-dimessions Systems of Duffing Type

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Abstract: In this paper, we use the Minimax Theorem to investigate periodic solutions of n-dimessions Systems of Duffing'Type , and give out some sufficient conditions of existence for periodic solutions, the results generalizes and improves some known results.

Key words: subconvex potential; periodic solution; subquadratic; critical points; the Saddle Point Theorem

1 Introduction

Solution theory is one of the most important aspect in nonlinearity, which is widely applied in many natural sciences such as chemistry, biology, mathematics, communication, and particularly in almost all branches of physics like fluid dynamics, plasma physics, field theory, optics,and condensed matter physics.In order to find some new exact solutions of nonlinear equations, a wealth of effective methods has been set up [14-17].

In this paper,we consider the following second-order systems:

$$\begin{cases} \ddot{u}(t)+A\dot{u}(t)+\nabla F(t,u(t))=0 \quad a.e. t\in[0,T], \\ u(0)-u(T)=\dot{u}(0)-\dot{u}(T)=0, \end{cases} \tag{1}$$

where A is antisymmetry matric with $\| A \| < \frac{2\pi}{T}$;$T>0$ and $F:[0,T]\times R^N \to R$ satisfies the following assumption:

(H) $F(t,x)$ is measurable in t for each $x\in R^N$ and continuously differentiable in x for a.e. $t\in[0,T]$, and there exists $a\in C(R^+,R^+), b\in L^1(0,T;R^+)$ such that

$$|F(t,x)|\le a(|x|)b(t), |\nabla F(t,x)|\le a(|x|)b(t),$$

for all $x\in R^N$ and a.e. $t\in[0,T]$.

In the case $A=0$, there are many solvability conditions for problem (1), [2]consider problem (1)with coercivity condition; [3]consider problem (1) with periodicity condition; [4]consider problem (1)with the convexity condition; [5]consider (1) with $\gamma-$subadditive potential; [6]consider problem (1) with sublinear nonlinearity:that is,there exist $f,g\in L^1(0,T;R^+)$ and $\alpha\in[0,1)$ such that $|\nabla F(t,x)|\le f(t)|x|^\alpha+g(t)$ for all $x\in R^N$ and a.e. $t\in[0,T]$. [10]consider problem (1)with a potential which is the sum of a subconvex function and a subquadratic function.

Recently, Tang [12]consider problem(1), where $A=0$, when $F(t,x)$ is subquadratic in Rabinowitz's sense, that is , there exists $0<\gamma<2$, $M>0$ such that

$$(\nabla F(t,x),x)\le\gamma F(t,x), \tag{2}$$

for all $x\in R^N, |x|>M$ and a.e. $t\in[0,T]$. In this paper, under the same condition in[12], we consider problem (1) where A is antisymmetry matric with $\| A \| < \frac{2\pi}{T}$.

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2 Main results

we first recall a definition due to Wu-Tang [10]:A function $G : R^N \to R$ is called to be (λ, μ)-subconvex,if

$$G(\lambda(x + y)) \le \mu(G(x) + G(y)),$$

for some$\lambda, \mu > 0$ and all $x, y \in R^N$.

Theorem 1 *Suppose that F satisfies assumption (H),(2),and the following conditons:*
(i)$F(t, .)$ is β-subadditive with $\beta > 0$ for $a.e.t \in [0, T]$,that is

$$F(t, x + y) \le \beta F(t, x) + F(t, y)),$$

for all $x, y \in R^N$;
(ii)$\int_0^T F(t, x)dt \to +\infty$ as $|x| \to \infty$,
then problem (1) has at least one solutions in H_T^1.

We shall prove more general results than Theorem 1.

Theorem 2 *Suppose that F satisfies assumption (H),(2), and (ii). Assume that $F(t, .)$ is $(\lambda, \mu)-$subconvex for a.e. $t \in [0, T]$, then problem (1) has at least one solution in H_T^1.*

3 Proofs of Theorems

$$H_T^1 = \left\{u : [0, T] \to R^N | u \text{ is absolutely continous }, \ u(0) = u(T) \text{ and } \dot{u} \in L^2(0, T; R^N)\right\}$$

is a Hilbert space with the norm defined by

$$\|u\| = (\int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt)^{\frac{1}{2}},$$

for $u \in H_T^1$.

For $u \in H_T^1$, Let $\bar{u} = (\frac{1}{T}) \int_0^T u(t)dt$,and $\tilde{u} = u(t) - \bar{u}$.

Then one has Sobolev inequality

$$||\tilde{u}||_\infty^2 \le \frac{T}{12} ||\dot{u}||_{L^2}^2,$$

and Wirtinger inequality

$$\int_0^T | \tilde{u}(t) |^2 dt \le \frac{T^2}{4\pi^2} \int_0^T | \dot{u} |^2 dt.$$

Then we have

$$|| \dot{u}(t) ||_{L^2}^2 \le || \tilde{u}(t) ||_{L^2}^2 \le (1 + \frac{T^2}{4\pi^2}) || \dot{u}(t) ||_{L^2}^2,$$

for all $u \in H_T^1$ (see proposition 1.3 in [1]).

Lemma 3 *([13]) Define the corresponding functional φ on H_T^1 by:*

$$\varphi(u) = \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \frac{1}{2} \int_0^T (Au(t), \dot{u}(t))dt - \int_0^T F(t, u(t))dt.$$

It follows from (H) that φ is continuously differentiable. Moreover the solutions of problem (1) corresponding to the critical points of φ .

Lemma 4 *([12]) Suppose $F(t,x)$ satisfies assumption(H) (2), then there exist*

$$a_0 = max_{|x|\leq M} a(\mid x \mid),$$

such that

$$F(t,x) \leq a_0 b(t)((\frac{\mid x \mid}{M})^\gamma + 1),$$

for all $x \in R^N$ and a.e.$t \in [0,T]$,where $a(x), b(t)$is the same as in the assumption (H).

Proof. For every $\mid x \mid\geq M$ and a.e.$t \in [0,T]$, let

$$y(s) = F(t,sx), Q(s) = y'(s) - \frac{\gamma}{s} y(s). \tag{3}$$

Then by (2), we have

$$Q(s) = \frac{1}{s}[(\nabla F(t,sx), sx) - \gamma F(t,sx)] \leq 0. \tag{4}$$

for all $s \geq \frac{M}{|x|}$.

It follows from (3) that $y(s) = F(t,sx)$ is a solution of first order linear ordinary differential equation

$$y'(s) = \frac{\gamma}{s} y(s) + Q(s),$$

which implies that

$$F(t,sx) = s^\gamma (\int_1^s r^{-\gamma} Q(r) dr + F(t,x)).$$

for $s \geq \frac{M}{|x|}$.

Moreover, by assumption (H) and (4), we have

$$a_0 b(t) \geq F(t, \frac{Mx}{\mid x \mid}) \geq (\frac{M}{\mid x \mid})^\gamma F(t,x),$$

for all $\mid x \mid\geq M$ and a.e. $t \in [0,T]$,which implies that

$$F(t,x) \leq a_0 b(t)((\frac{\mid x \mid}{M})^\gamma + 1),$$

for all $\mid x \mid\geq M$ and a.e. $t \in [0,T]$.

Lemma 5 *Under conditions (H),(2),(ii) and $F(t,.)$ is $(\lambda,\mu)-$subconvex for a.e.$t \in [0,T]$, the functional φ satisfies condition (C), that is $\{u_n\}$ has a convergent subsequence in H^1_T where $\{\varphi(u_n)\}$ is bounded and $\| \varphi'(u_n) \| (1+ \| u_n \|) \to 0$ as $n \to \infty$.*

Proof. Let $\{u_n\}$ be a sequence in H^1_T such that $\{\varphi(u_n)\}$ is bounded and $\| \varphi'(u_n) \| (1+ \| u_n \|) \to 0$ as $n \to \infty$. Then there exists a constant C_1 such that

$$\mid \varphi(u_n) \mid\leq C_1, \| \varphi'(u_n) \| (1+ \| u_n \|) \leq C_1. \tag{5}$$

for all $n \in N$. In a way similar to the proof of Proposition 4.1 in [1],we only need to prove that $\{u_n\}$ is bounded. By assumption (H) and (2), we have

$$-h(t) + (\nabla F(t,x), x) \leq \gamma F(t,x), \tag{6}$$

for all $x \in R^N$ and a.e. $t \in [0,T]$, where $h(t) = (2+M)b(t)a_0 \geq 0$. a_0 is the same as in Lemma 4.

It follows from(5)(6), that

$$\begin{aligned}
3C_1 &\geq \| \varphi'(u_n) \| (1+ \| u_n \|) - 2\varphi(u_n) \\
&\geq (\varphi'(u_n))(u_n) - 2\varphi(u_n) \\
&= \int_0^T | \dot{u}_n(t) |^2 dt - \int_0^T (A\dot{u}_n(t), u_n(t))dt - \int_0^T (\nabla F(t, u_n(t)), u_n(t))dt \\
&\quad - \int_0^T | \dot{u}_n(t) |^2 dt - \int_0^T (Au_n(t), \dot{u}_n(t))dt + 2\int_0^T F(t, u_n(t))dt \\
&= 2\int_0^T F(t, u_n(t))dt - \int_0^T (\nabla F(t, u_n(t)), u_n(t))dt \\
&\geq (2-\gamma)\int_0^T F(t, u_n(t))dt - \int_0^T h(t)dt,
\end{aligned}$$

for all $n \in N$, which implies that

$$\int_0^T F(t, u_n)dt \leq C_2. \tag{7}$$

for all $n \in N$, and some constant C_2. By (5) and (7) we have

$$\begin{aligned}
C_1 &\geq \varphi(u_n) \\
&= \frac{1}{2}\int_0^T | \dot{u}_n(t) |^2 dt + \frac{1}{2}\int_0^T (Au_n(t), \dot{u}_n(t))dt - \int_0^T F(t, u_n(t))dt \\
&= \frac{1}{2}\int_0^T | \dot{u}_n(t) |^2 dt - \frac{1}{2}\int_0^T (A\dot{u}_n(t), u_n(t))dt - C_2 \\
&\geq \frac{1}{2}(1 - \frac{T}{2\pi} \| A \|)\int_0^T | \dot{u}_n(t) |^2 dt - C_2.
\end{aligned}$$

for all $n \in N$. Hence we have

$$\int_0^T | \dot{u}_n(t) |^2 dt \leq C_3.$$

for all $n \in N$ and some constant C_3.

It follows from Sobolev'inequality that

$$\| \tilde{u}_n(t) \|_\infty \leq C_4. \tag{8}$$

for all $n \in N$ and some constant C_4. From (λ, μ)-subconvex of F(t,.), we obtain

$$F(t, \lambda(u(t) - \tilde{u}(t))) \leq \mu(F(t, u(t)) + F(t, -\tilde{u}(t))),$$

that is

$$F(t, u(t)) \geq \frac{1}{\mu}F(t, \lambda\bar{u}(t)) - F(t, -\tilde{u}(t)). \tag{9}$$

By (7)(8)(9), we have

$$\begin{aligned}
C_2 &\geq \int_0^T F(t, u_n)dt \geq \frac{1}{\mu}\int_0^T F(t, \lambda\bar{u}(t))dt - \int_0^T F(t, -\tilde{u}(t))dt \\
&\geq \frac{1}{\mu}\int_0^T F(t, \lambda\bar{u}(t))dt - max_{|x| \leq C_4} a(| x |)\int_0^T b(t)dt,
\end{aligned}$$

for all $n \in N$, which implies that $\{\bar{u}_n\}$ is bounded. Thus $\{u_n\}$is bounded. Hence φ satisfies condition (C).
■

Proof of Theorem 2:

Proof. Let $\tilde{H}_T^1 = \{u \in H_T^1 \mid \bar{u} = 0\}$. Then $H_T^1 = \tilde{H}_T^1 \bigoplus \bar{H}_T^1$ and $dim\bar{H}_T^1 = N < +\infty$. As shown in[9], a deformation lemma can be proved with the weaker condition (C) replacing the usual (PS) condition, and it turns out that the Saddle Point Theorem holds true under condition (C). By the Saddle Point Theorem, we only need to prove

(1) $\varphi(u) \to +\infty$ as $\| u \| \to \infty$ in $\tilde{H}_T^1$, which implies that $inf_{u\in\tilde{H}_T^1}\varphi(u) > -\infty$ and

(2) $\varphi(u) \to -\infty$ as $\| u \| \to \infty$ in $\bar{H}_T^1$.

For all $u \in \tilde{H}_T^1$ and $\| u \| \to \infty$, by Lemma 4, We have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T \mid \dot{u}(t) \mid^2 dt + \frac{1}{2}\int_0^T (Au(t), \dot{u}(t))dt - \int_0^T F(t, u(t))dt \\
&\geq \frac{1}{2}(1 - \frac{T}{2\pi} \| A \|)\int_0^T \mid \dot{u}(t) \mid^2 dt - \int_0^T a_0 b(t)(\frac{\mid u \mid}{M})^\gamma dt - a_0\int_0^T b(t)dt \\
&= \frac{1}{2}(1 - \frac{T}{2\pi} \| A \|)\int_0^T \mid \dot{u}(t) \mid^2 dt - \frac{1}{M^\gamma}\int_0^T a_0 b(t)\mid u \mid^\gamma dt - a_0\int_0^T b(t)dt \\
&\geq \frac{1}{2}(1 - \frac{T}{2\pi} \| A \|)\frac{4\pi^2}{T^2 + 4\pi^2}\| u \|_{L^2}^2 - C_5\| u \|_{L^2}^\gamma - C_6,
\end{aligned}$$

for some constant C_5 and C_6, which implies that

$$\varphi(u) \to +\infty.$$

Because $\gamma < 2$ and $\| A \| < \frac{2\pi}{T}$, for all $u \in \bar{H}_T^1$ and $\| u \| \to \infty$, by (ii) we have

$$\varphi(u) = -\int_0^T F(t, u)dt \to -\infty.$$

Then there exists $R > 0$, such that

$$\sup_{u\in\bar{S}_R}\varphi(u) < \inf_{u\in\tilde{H}_T^1}\varphi(u),$$

where $\bar{S}_R = \{u \mid u \in \tilde{H}_T^1, \mid u \mid = R\}$. It follows from the Saddle Point Theorem that the conclusion holds. ■

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