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# Periodic Solutions Design for Feedback Bilinear Systems

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**Abstract:**In this paper, we propose a feedback control for a class of bilinear systems. Based on the Bellman-Gronwall inequality, the exponentially stable periodic solutions or limit cycles, appearing as an ellipsoid in the phase plane, are guaranteed. Moreover, the frequency of oscillation and convergence rate can be correctly estimated for such bilinear control systems. Finally, a numerical example is presented to exhibit the use of the main result.

**Keywords:** periodic solution; bilinear control systems; oscillator; exponentially stable; ellipsoid manifold

## 1 Introduction

Limit cycles are important physical phenomena, which often exists in nonlinear dynamical systems, such as Duffing equation, Van der Pol equation have been of great interest for many years. The studies of their properties such as magnitude, frequency and stability and the prediction of limit cycle are getting more important; see e.g., [1–8] and the reference therein.

We often apply the features of limit cycle to the design of electronic oscillator. There are some techniques to check the phenomenon of limit cycles, for example, harmonic balance techniques [9], Piecewise-linearized methods [10], Poincare-Bendixson theorem, and Lyapunov-like approach [11]. The harmonic balance method is not able to accurately predict the limit cycle. The Poincare-Bendixson theorem only gives a necessary condition for guaranteeing the existence of limit cycles [12]. Limit cycle may not exist in those systems satisfying the theorem.

In this paper, we propose a feedback control to a class of bilinear systems to generate the limit cycle with two different amplitudes of oscillation at same time, appeared as an ellipsoid in the phase plane. Applying the Bellman-Gronwall inequality, the exponentially stable limit cycle is guaranteed, and the frequency of oscillation and convergence rate can be correctly estimated for such bilinear control systems.

## 2 Problem Formulation and Main Results

In this paper, we consider the following bilinear control system [13]:

$$\dot{x}(t) = Fx(t) + u(t)Nx(t), \quad \forall t \geq t_0 \geq 0, \tag{1}$$

$$x(t_0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \tag{2}$$

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where

$$x(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \Re^2, \quad u(t)\in\Re, \quad F:=\begin{bmatrix} p & -\omega_0 \\ \omega_0 & p \end{bmatrix}, \quad N:=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

with $p > 0$ and $x\,(t_0) \neq 0$. Particularly, the feedback control law is selected as

$$u(t) = s\left[(lx_1(t))^2 + (mx_2(t))^2\right], with \quad s,l,m > 0 \tag{3}$$

Then, the closed-loop system is deduced as

$$\dot{x_1}(t) = -sx_1(t)\left[(lx_1(t))^2 + (mx_2(t))^2 - \frac{p}{s}\right] - \omega_0 x_2(t), \quad \forall\, t \geq t_0, \tag{4}$$

$$\dot{x_2}(t) = \omega_0 x_1(t) - sx_2(t)\left[(lx_1(t))^2 + (mx_2(t))^2 - \frac{p}{s}\right], \quad \forall\, t \geq t_0, \tag{5}$$

Apparently, $x\,(t) = 0$ is an equilibrium point of system Eq.(4) and Eq.(5) i.e., the solution of system Eq.(4) and Eq.(5) is given by $x\,(t) = 0$ if $x\,(t_0) = 0$. To avoid the static case of $x\,(t) = 0$, in the following, we only examine system Eq.(1), Eq.(2) and Eq.(3) under the case of $x\,(t_0) \neq 0$.

**Definition 1** *The closed and bounded manifold $v\,(x\,(t)) = 0$, in the $x_1 - x_2$ plane, is said to be an exponentially stable limit cycle if there exist two positive numbers $k$ and $\lambda$ such that the manifold of $v\,(x\,(t)) = 0$ along the trajectories of system Eq.(4) and Eq.(5) satisfies the following inequality*

$$|v(x(t))| \leq ke^{-\lambda(t-t_0)}, \quad \forall\, t \geq t_0, \tag{6}$$

*In this case, the positive number $\lambda$ is called the guaranteed convergence rate.*

Now, we demonstrate the main result for the existence of limit cycles of system Eq.(1), Eq.(2) and Eq.(3) as follows.

**Theorem 2** *For the feedback bilinear system Eq.(1), Eq.(2) and Eq.(3), all of phase trajectories tend to the exponentially stable limit cycle $v\,(x\,(t)) = \left[(lx_1(t))^2 + (mx_2(t))^2 - \frac{p}{s}\right] = 0$ in the $x_1 - x_2$ plane, with the guaranteed convergence rate*

$$\lambda := \begin{cases} \infty, & if\, (lx_{10})^2 + (mx_{20})^2 = \frac{p}{s}, \\ 2p, & if\, (lx_{10})^2 + (mx_{20})^2 > \frac{p}{s}, \\ 2s[(lx_{10})^2 + (mx_{20})^2], & if\, (lx_{10})^2 + (mx_{20})^2 < \frac{p}{s}. \end{cases}$$

*Furthermore, the states $x_1(t)$ and $x_2(t)$ exponentially follow, respectively, the trajectories $\frac{1}{l}\sqrt{\frac{p}{s}}cos[\omega_0(t - t_0) + tan^{-1}(\frac{x_{20}}{x_{10}})]$ and $\frac{1}{m}\sqrt{\frac{p}{s}}sin[\omega_0(t - t_0) + tan^{-1}(\frac{x_{20}}{x_{10}})]$, in the time domain, with the guaranteed convergence rate $\frac{\lambda}{2}$.*

**Proof.** Define a smooth ellipsoid manifold $v\,(x\,(t)) = 0$ and a continuous function $\theta(x(t)) := tan^{-1}(\frac{x_2(t)}{x_1(t)})$ with $v\,(x\,(t)) = (lx_1(t))^2 + (mx_2(t))^2 - \frac{p}{s}$. Then the time derivatives of $v^2\,(x\,(t))$ and $\theta(x(t))$ along the trajectories of system Eq. (4) and Eq. (5) are given by

$$\frac{dv^2(x(t))}{dt} = 2v(x(t))[2l^2x_1(t)\dot{x}_1(t) + 2m^2x_2(t)\dot{x}_2(t)] = -4s[(lx_1(t))^2 + (mx_2(t))^2]v^2(x(t)). \tag{7}$$

$$\frac{d\theta(x(t))}{dt} = \frac{\dot{x}_2(t)x_1(t) - \dot{x}_1(t)x_2(t)}{x_1^2(t) + x_2^2(t)} = \omega_0,$$

which imply

$$\theta(x(t)) = \omega_0(t - t_0) + tan^{-1}(\frac{x_{20}}{x_{10}}) \tag{8}$$

In the following, three cases are to be discussed about the trajectories of the feedback control system Eq. (4) and Eq. (5).

**Case 1:** $(lx_1(t_0))^2 + (mx_2(t_0))^2 = p/s$ (or equivalently; $v(x(t_0)) = 0$)

In this case, from Eq. (7), it can be obtained that $\frac{dv^2(x(t))}{dt} = 0$ which implies

$$(lx_1(t))^2 + (mx_2(t))^2 = p/s, \quad \forall\, t \geq t_0. \tag{9}$$

Hence, we deduce that

$$x_1(t) = \frac{1}{l}\sqrt{\frac{p}{s}}cos[\omega_0(t - t_0) + tan^{-1}(\frac{x_{20}}{x_{10}})], \quad \forall\, t \geq t_0,$$

$$x_2(t) = \frac{1}{m}\sqrt{\frac{p}{s}}sin[\omega_0(t - t_0) + tan^{-1}(\frac{x_{20}}{x_{10}})], \quad \forall\, t \geq t_0,$$

$$v(x(t)) = 0, \quad \forall\, t \geq t_0,$$

in view of Eq. (8) and Eq. (9).

**Case 2:** $(lx_1(t_0))^2 + (mx_2(t_0))^2 > p/s$ (or equivalently; $v(x(t_0)) > 0$)

In this case, from Eq. (7), it can be obtained that $v^2\,(x\,(t))$ is a strictly decreasing function of $t$ with $v(x(t)) \geq 0, \forall\, t \geq t_0$, and

$$\frac{dv^2(x(t))}{dt} = -4s[(lx_1(t))^2 + (mx_2(t))^2]v^2(x(t)) = -4s\frac{p}{s}v^2(x(t)) = -4pv^2(x(t)), \qquad \forall\, t \geq t_0.$$

Applying the Bellman-Gronwall inequality to the above differential inequality, one has

$$v^2(x(t)) \leq v^2(x(t_0))exp[-4p(t - t_0)], \forall\, t \geq t_0,$$

this implies

$$|v(x(t))| \leq |v(x(t_0))|exp[-2p(t - t_0)], \forall\, t \geq t_0,$$

$$|\sqrt{(lx_1(t))^2 + (mx_2(t))^2} - \sqrt{\frac{p}{s}}|^2\sqrt{(lx_1(t))^2 + (mx_2(t))^2} - \sqrt{\frac{p}{s}}|\;|\sqrt{(lx_1(t))^2 + (mx_2(t))^2} + \sqrt{\frac{p}{s}}|$$

$$= |(lx_1(t))^2 + (mx_2(t))^2 - \frac{p}{s}| = |v(x(t))| \leq |v(x(t_0))|exp[-2p(t - t_0)], \forall\, t \geq t_0.$$

It yields

$$|\sqrt{(lx_1(t))^2 + (mx_2(t))^2} - \sqrt{\frac{p}{s}}| \leq \sqrt{|v(x(t_0)}|exp[-p(t - t_0)], \forall\, t \geq t_0. \tag{10}$$

Consequently, by Eq. (8) and Eq. (10), we deduce that

$$|lx_1(t) - \sqrt{\frac{p}{s}}cos[\omega_0(t - t_0) + tan^{-1}(\frac{x_{20}}{x_{10}})]$$

$$= |\sqrt{(lx_1(t))^2 + (mx_2(t))^2}cos[\omega_0(t - t_0) + tan^{-1}(\frac{x_{20}}{x_{10}})] - \sqrt{\frac{p}{s}}cos[\omega_0(t - t_0) + tan^{-1}(\frac{x_{20}}{x_{10}})]|$$

$$= |\sqrt{(lx_1(t))^2 + (mx_2(t))^2} - \sqrt{\frac{p}{s}}||cos[\omega_0(t - t_0) + tan^{-1}(\frac{x_{20}}{x_{10}})]|$$

$$\leq |\sqrt{(lx_1(t))^2 + (mx_2(t))^2} - \sqrt{\frac{p}{s}}|$$

$$\leq \sqrt{|v(x(t_0)}|exp[-p(t - t_0)], \forall\, t \geq t_0.$$

$$|x_1(t) - \frac{1}{l}\sqrt{\frac{p}{s}}cos[\omega_0(t - t_0) + tan^{-1}(\frac{x_{20}}{x_{10}})]| \leq \frac{1}{l}\sqrt{|v(x(t_0)}|exp[-p(t - t_0)], \forall\, t \geq t_0.$$

$$|mx_2(t)-\sqrt{\frac{p}{s}}sin[\omega_0(t-t_0)+tan^{-1}(\frac{x_{20}}{x_{10}})]|$$

$$=|\sqrt{(lx_1(t))^2+(mx_2(t))^2}sin[\omega_0(t-t_0)+tan^{-1}(\frac{x_{20}}{x_{10}})]-\sqrt{\frac{p}{s}}sin[\omega_0(t-t_0)+tan^{-1}(\frac{x_{20}}{x_{10}})]|$$

$$=|\sqrt{(lx_1(t))^2+(mx_2(t))^2}-\sqrt{\frac{p}{s}}||sin[\omega_0(t-t_0)+tan^{-1}(\frac{x_{20}}{x_{10}})]$$

$$\leq|\sqrt{(lx_1(t))^2+(mx_2(t))^2}-\sqrt{\frac{p}{s}}|$$

$$\leq\sqrt{|v(x(t_0)}|exp[-p(t-t_0)],\forall\, t\geq t_0.$$

$$|x_2(t)-\frac{1}{m}\sqrt{\frac{p}{s}}sin[\omega_0(t-t_0)+tan^{-1}(\frac{x_{20}}{x_{10}})]|\leq\frac{1}{m}\sqrt{|v(x(t_0)}|exp[-p(t-t_0)],\forall\, t\geq t_0.$$

**Case 3:** $(lx_1(t_0))^2+(mx_2(t_0))^2<p/s$ (or equivalently; $v(x(t_0))<0$)

In this case, from Eq. (7), it can be obtained that $v^2(x(t))$ is a strictly decreasing function of $t$ with $v^2(x(t))\geq 0,\forall\, t\geq t_0$, and

$$\frac{dv^2(x(t))}{dt}=-4s[(lx_1(t))^2+(mx_2(t))^2]v^2(x(t))$$

$$\leq-4s[(lx_{10})^2+(mx_{20})^2]v^2(x(t)),\forall\, t\geq t_0.$$

Applying the Bellman-Gronwall inequality to the above differential inequality, one has

$$v^2(x(t))\leq v^2(x(t_0))exp[-4s((lx_{10})^2+(mx_{20})^2)(t-t_0)],\forall\, t\geq t_0,$$

which implies

$$|v(x(t))|\leq|v(x(t_0))|exp[-2s((lx_{10})^2+(mx_{20})^2)(t-t_0)],\forall\, t\geq t_0,$$

$$\sqrt{(lx_1(t))^2+(mx_2(t))^2}-\sqrt{\frac{p}{s}}|^2\leq\sqrt{(lx_1(t))^2+(mx_2(t))^2}-\sqrt{\frac{p}{s}}||\sqrt{(lx_1(t))^2+(mx_2(t))^2}+\sqrt{\frac{p}{s}}|$$

$$=|(lx_1(t))^2+(mx_2(t))^2-\frac{p}{s}|=|v(x(t))|\leq|v(x(t_0)|exp[-2s((lx_{10})^2+(mx_{20})^2)(t-t_0)],\forall\, t\geq t_0.$$

It yields

$$|\sqrt{(lx_1(t))^2+(mx_2(t))^2}-\sqrt{\frac{p}{s}}|\leq\sqrt{|v(x(t_0)}|exp[-s((lx_{10})^2+(mx_{20})^2)(t-t_0)],\forall\, t\geq t_0. \quad (11)$$

Consequently, by Eq. (8) and Eq. (11), we deduce that

$$|lx_1(t)-\sqrt{\frac{p}{s}}cos[\omega_0(t-t_0)+tan^{-1}(\frac{x_{20}}{x_{10}})]|$$

$$=|\sqrt{(lx_1(t))^2+(mx_2(t))^2}cos[\omega_0(t-t_0)+tan^{-1}(\frac{x_{20}}{x_{10}})]-\sqrt{\frac{p}{s}}cos[\omega_0(t-t_0)+tan^{-1}(\frac{x_{20}}{x_{10}})]|$$

$$=|\sqrt{(lx_1(t))^2+(mx_2(t))^2}-\sqrt{\frac{p}{s}}||cos[\omega_0(t-t_0)+tan^{-1}(\frac{x_{20}}{x_{10}})]|$$

$$\leq|\sqrt{(lx_1(t))^2+(mx_2(t))^2}-\sqrt{\frac{p}{s}}|$$

$$\leq\sqrt{|v(x(t_0)}|exp[-s((lx_{10})^2+(mx_{20})^2)(t-t_0)],\forall\, t\geq t_0,$$

$$|x_1(t)-\frac{1}{l}\sqrt{\frac{p}{s}}cos[\omega_0(t-t_0)+tan^{-1}(\frac{x_{20}}{x_{10}})]|$$
$$\leq\frac{1}{l}\sqrt{|v(x(t_0)|}exp[-s((lx_{10})^2+(mx_{20})^2)(t-t_0)],\forall t\geq t_0,$$

$$|mx_2(t)-\sqrt{\frac{p}{s}}sin[\omega_0(t-t_0)+tan^{-1}(\frac{x_{20}}{x_{10}})]|$$
$$=|\sqrt{(lx_1(t))^2+(mx_2(t))^2}sin[\omega_0(t-t_0)+tan^{-1}(\frac{x_{20}}{x_{10}})]-\sqrt{\frac{p}{s}}sin[\omega_0(t-t_0)+tan^{-1}(\frac{x_{20}}{x_{10}})]|$$
$$=|\sqrt{(lx_1(t))^2+(mx_2(t))^2}-\sqrt{\frac{p}{s}}||sin[\omega_0(t-t_0)+tan^{-1}(\frac{x_{20}}{x_{10}})]|$$
$$\leq|\sqrt{(lx_1(t))^2+(mx_2(t))^2}-\sqrt{\frac{p}{s}}|$$
$$\leq\sqrt{|v(x(t_0)|}exp[-s((lx_{10})^2+(mx_{20})^2)(t-t_0)],\forall t\geq t_0,$$
$$|x_2(t)-\frac{1}{m}\sqrt{\frac{p}{s}}sin[\omega_0(t-t_0)+tan^{-1}(\frac{x_{20}}{x_{10}})]|$$
$$\leq\frac{1}{m}\sqrt{|v(x(t_0)|}exp[-s((lx_{10})^2+(mx_{20})^2)(t-t_0)],\forall t\geq t_0.$$

■

**Remark 3** *Apparently, by Theorem 2, the bilinear feedback systems Eqs. (1), (2) and (3) can be described as two nonlinear oscillators with the frequency $\omega_0$ and different amplitudes $\frac{1}{l}\sqrt{\frac{p}{s}}$ and $\frac{1}{m}\sqrt{\frac{p}{s}}$ , respectively. These types of oscillations are generally independent of the initial condition, and are not influenced by parameter variation.*

## 3 An Illustrative example

Consider the feedback bilinear system

$$\dot{x}(t)=\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}3 & -2\\ 2 & 3\end{bmatrix}x(t)+u(t)\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}x(t),\forall t\geq 0, \tag{12}$$

with the control law $u(t)=0.3[(3x_1(t))^2+(x_2(t))^2]$ and $x(0)=[1 \quad -1]^T$.

By Theorem 2, we conclude that the phase trajectories of system Eq. (12) tend to the exponentially stable limit cycle $v(x(t))=(3x_1(t))^2+(x_2(t))^2-10=0$ in the $x_1-x_2$ plane, with the guaranteed convergence rate $\lambda=6$. As well, the states $x_1(t)$ and $x_2(t)$ exponentially track, respectively, the trajectories $\frac{\sqrt{10}}{3}cos[2t-\frac{\pi}{4}]$ and $\sqrt{10}sin[2t-\frac{\pi}{4}]$ in the time domain, with the guaranteed convergence rate $\frac{\lambda}{2}=3$. The state trajectories of the feedback-controlled system are depicted in Fig.1. and Fig.2. Meantime, by Theorem 2 with the control law $u(t)=5[(3x_1(t))^2+(x_2(t))^2]$ and $x(0)=[-2,2]^T$, we conclude that the phase trajectories of system Eq. (12) tend to the exponentially stable limit cycle $v(x(t))=(3x_1(t))^2+(x_2(t))^2-\frac{3}{5}=0$ in the $x_1-x_2$ plane, with the guaranteed convergence rate $\lambda=6$. As well, the states $x_1(t)$ and $x_2(t)$ exponentially track, respectively, the trajectories $\frac{1}{3}\sqrt{\frac{3}{5}}cos[2t+\frac{3\pi}{4}]$ and $\sqrt{\frac{3}{5}}sin[2t+\frac{3\pi}{4}]$ in the time domain, with the guaranteed convergence rate $\frac{\lambda}{2}=3$. The state trajectories of the feedback-controlled system are depicted in Fig.3. and Fig.4. Meanwhile, by Theorem 2 with the control law $u(t)=0.1[(3x_1(t))^2+(x_2(t))^2]$ and $x(0)=[1,-1]^T$, we conclude that the phase trajectories of system Eq. (12) tend to the exponentially stable limit cycle $v(x(t))=(3x_1(t))^2+(x_2(t))^2-30=0$ in the $x_1-x_2$ plane, with the guaranteed convergence rate $\lambda=6$. As well, the states $x_1(t)$ and $x_2(t)$ exponentially track, respectively, the trajectories $\frac{\sqrt{30}}{3}cos[2t-\frac{\pi}{4}]$ and $\sqrt{30}sin[2t-\frac{\pi}{4}]$, in the time domain, with the guaranteed convergence rate $\frac{\lambda}{2}=3$. The state trajectories of the feedback-controlled system are depicted in Fig.5. and Fig.6.

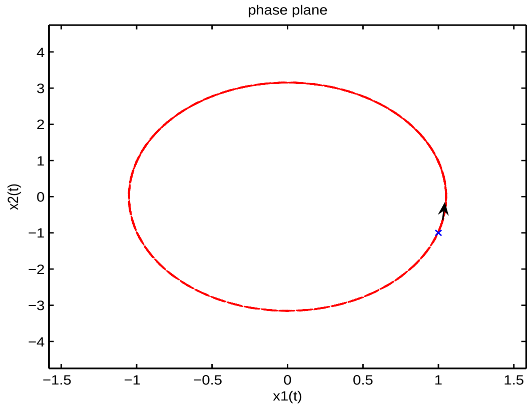

Fig1:Typical phase trajectory of the feedback-controlled

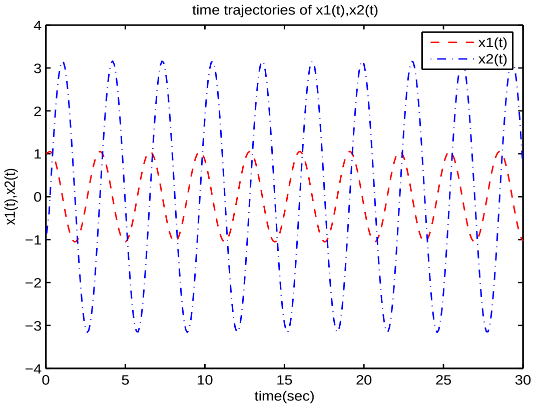

Fig2:Time trajectories of $x_1(t)$ and $x_2(t)$(case 1)

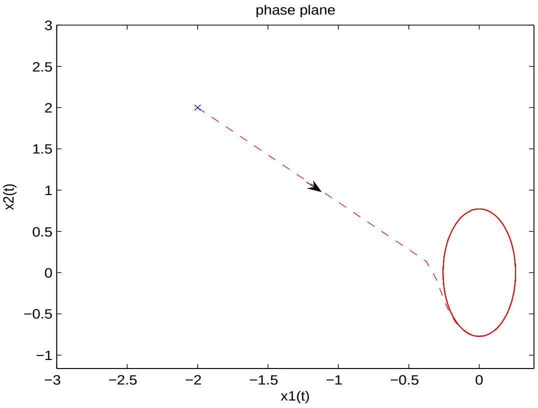

Fig. 3 Typical phase trajectory of the feedback-controlled

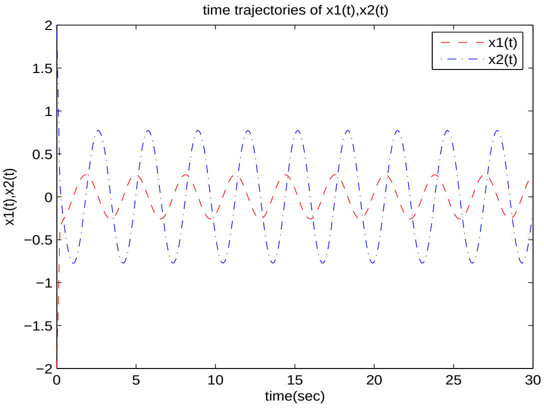

Fig. 4 Time trajectories of $x_1(t)$ and $x_2(t)$(case 2)

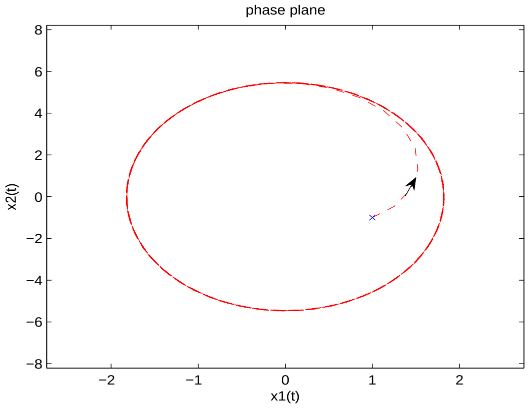

Fig. 5 Typical phase trajectory of the feedback-controlled

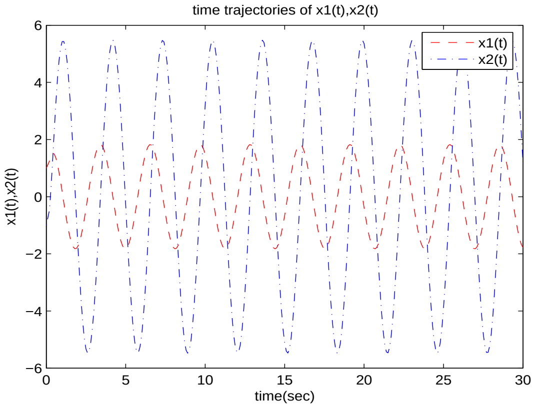

Fig. 6 Time trajectories of $x_1(t)$ and $x_2(t)$(case 3)

## 4 Conclusions

In this paper, we propose a feedback control for a class of bilinear systems to generate the limit cycle, appeared as an ellipsoid in the phase plane. Applying the Bellman-Gronwall inequality, it's shown that the exponentially stable limit cycle is guaranteed, and the frequency of oscillation and convergence rate can be correctly estimated for such bilinear control systems. A numerical example is offered to exhibit the use of the main result.

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## References

[1] Lixin Tian, Gaogao Dong: Predictive Control of Sudden Occurrence of Chaos. *International Journal of Nonlinear Science*. 5(4):99-105(2008)

[2] Hua Ni, Faxing Lin: The Existence of Almost Periodic Solutions of Several Classes of Second Order Differential Equations. *International Journal of Nonlinear Science*. 2(4):83-91(2006)

[3] Engelberg, S.: Limitations of the describing function for limit cycle prediction. *IEEE Trans. Automat. Control*. 47:814-819(2002)

[4] Genesio, R. and Nistri, P.: On the limit cycles in feedback bilinear systems. *The 27th Proceedings of the IEEE Conference on Decision and control*, Austin, Texas USA.:946-947(1988)

[5] Ying Li, Jianbao Zhang, Zengrong Liu: Circadian Oscillators and Phase Synchronization under a Light-Dark Cycle. *International Journal of Nonlinear Science*. 1(4):131-138(2006)

[6] Guoliang Cai, Weihuai Zhou, Zhenmei Tan: Stabilization of High Periodic Orbits of Discrete-time Chaotic Systems. *International Journal of Nonlinear Science*. 4(4):118-126(2007)

[7] Sh. Salem, K. R. Raslan: Oscillation of Some Second Order Damped Difference Equations. *International Journal of Nonlinear Science*. 5(5):246-254(2008)

[8] Palmor, Z.J., Halevi, Y. and Efrati, T.: A general and exact method for determining limit cycles in decentralized relay systems. *Automatica*. 31(:)1333-1339(1995).

[9] Genesio, R. and Tesi, A.: A Harmonic Balance Approach for Chaos Prediction: Chua's Circuit. *International Journal of Bifurcation and Chaos*. 2(:)61-79(1992)

[10] Ramos, J.I.: Piecewise-linearized methods for oscillators with limit cycles. *Chaos, Solitons & Fractals*. 27:1229-1238(2006)

[11] Sun, Y.J.: Prediction of limit cycles in feedback bilinear systemsGLyapunov-like approach. *The Proceeding of 2002 National Symposium on Automatic Control*.801-805(2002)

[12] Slotine, Jean-Jacques E. and Li, Weiping: Applied Nonlinear Control. *Prentice-Hall, Englewood Cliffs, NJ*(1991)

[13] Mohler, Ronald R.: Bilinear Control Process. *Academic Press, New York*(1973)

[14] Ji-Huan He: Variational approach for nonlinear oscillators. *Chaos, Solitons & Fractals*. 34:1430-1439(2007)