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B-splines Collocation Algorithms for Solving Numerically the MRLW Equation

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Abstract: Collocation methods using sextic B-splines have been developed for solving numerically the modified regularized long wave (MRLW) equation. A linear stability analysis shows that the scheme based on finite difference method is unconditionally stable. Use of Runge-Kutta method for solving the first order systems of ordinary differential equations at each time level results in much more accurate results. Three invariants of motion are evaluated to study the conservation properties of the algorithm. Interaction of two and three solitary waves have been investigated through computer simulations. The development of the Maxwellian initial condition into solitary waves is also studied. Figures and comparisons have been presented for clarity.

Keywords: solitary waves; MRLW equation; splines; collocation; Runge-kutta

1 Introduction

Solitary waves are wave packets or pulses which propagate in nonlinear dispersive media. Due to dynamical balance between the nonlinear and dispersive effects these waves retain a stable waveform. The regularized long wave equation (denoted by RLW) is given by

$$U_t + U_x + \varepsilon\, U\, U_x - \mu\, U_{xxt} = 0, \tag{1}$$

where ε and μ are positive constants and the lower scripts x and t denote space and time derivatives, respectively. It is the model equation governing the wave phenomena in which long wave incorporate the competing effects of nonlinearity and dispersion. This equation was originally introduced by Peregrine [1] to describe the development of an undular bore and describe the wave motion as an alternative to more usual Korteweg-de Vries equation which was introduced in 1895 as a model for the unidirectional propagation of the water waves of small amplitude and long wavelength [2]. The solutions of this equation are kinds of solitary waves named solitons whose shapes are not affected by collision. The RLW equation was solved numerically by various methods such as Galerkin method [3–5], least squares method [6], and collocation method with quadratic, cubic, quartic, and quintic B-splines [7, 8]. Indeed, the RLW equation is a special case of the generalized regularized long wave equation (denoted by GRLW) of the form

$$U_t + U_x + \varepsilon\, U^q\, U_x - \mu\, U_{xxt} = 0, \tag{2}$$

where q is a positive integer. The GRLW equation was studied by Zhang [9] via applying the finite difference method for a Cauchy problem and by Kaya [10] by using the Adomian decomposition method. Another special case of the GRLW equation is the modified regularized long wave equation (denoted by MRLW) with $q = 2$. Comparatively little work has been done for this case. Some of these can be found in Khalifa et al. [11] and Gardner et al. [12]. Additional techniques for similar equations are in [13–15]. In the

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present work we apply sextic B-spline collocation algorithm with Runge-Kutta method in addition to the finite difference method to find the solitary wave solutions of the MRLW equation. Runge-Kutta fourth order scheme will be used for solving the first order systems of ordinary differential equations resulting from applying the collocation method without discretizing the time dependent variables at each time level. This efficient method significantly increases the accuracy of results and greatly simplifies the computational work.

2 Governing equations and collocation solution

Consider the MRLW equation in the form

$$U_t + U_x + 6\,U^2\,U_x - \mu\,U_{xxt} = 0, \tag{3}$$

with the boundary conditions $U \to 0$ as $x \to \pm\infty$. The exact solution of this equation is given by [12]

$$U(x,\,t) = \sqrt{c}\;sech\left[p\,(x - x_0 - (1+c)\,t)\right], \tag{4}$$

where x_0 and c are arbitrary constants and $p = \sqrt{\frac{c}{\mu\,(1+c)}}$. Here the knots $a = x_0 < x_1 < \cdots < x_n = b$ are used over the solution domain $a \le x \le b$ and $h = x_{m+1} - x_m,\ m = 0, \cdots, N$ where N is the approximation order.

2.1 Sextic B-splines collocation method (Denoted by SBSC)

The sextic B-splines $Q_m(x),\ m = -3, \cdots, N+2$ are given by

$$Q_m(x) = \frac{1}{h^6}\begin{cases} (x-x_{m-3})^6, & [x_{m-3},\,x_{m-2}],\\ (x-x_{m-3})^6 - 7(x-x_{m-2})^6, & [x_{m-2},\,x_{m-1}],\\ (x-x_{m-3})^6 - 7(x-x_{m-2})^6 + 21(x-x_{m-1})^6, & [x_{m-1},\,x_m],\\ (x-x_{m-3})^6 - 7(x-x_{m-2})^6 + 21(x-x_{m-1})^6 & \\ -35(x-x_m)^6, & [x_m,\,x_{m+1}],\\ (x-x_{m+4})^6 - 7(x-x_{m+3})^6 + 21(x-x_{m+2})^6, & [x_{m+1},\,x_{m+2}],\\ (x-x_{m+4})^6 - 7(x-x_{m+3})^6, & [x_{m+2},\,x_{m+3}],\\ (x-x_{m+4})^6, & [x_{m+3},\,x_{m+4}],\\ 0, & otherwise. \end{cases} \tag{5}$$

For sextic B-splines near end boundaries, it is necessary to introduce 12-additional knots outside the solution domain to provide the support for the sextic B-spline functions, positioned at $x_{-6} < x_{-5} < x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_0$, and $x_N < x_{N+1} < x_{N+2} < x_{N+3} < x_{N+4} < x_{N+5} < x_{N+6}$. The set of sextic B-splines $\{Q_{-3},\,Q_{-2},\,\cdots,\,Q_{N+2}\}$ form a basis over the problem domain $[a,\,b]$ [13]. Thus $Q_m(x) = Q(x_m)$ is the B-spline of degree 6 defined upon knots $x_i,\ i = m-3, \cdots, m+4$. A global approximation solution $U^N(x,t)$ will be sought in the following form

$$U^N(x,\,t) = \sum_{i=-3}^{N+2} \delta_i(t)\,Q_i(x), \tag{6}$$

where $\delta_i(t)$ are the time dependent variables to be determined from the sextic B-spline collocation form of the MRLW equation together with the boundary conditions. The nodal values U and its 1^{st}, 2^{nd}, and 3^{rd} derivatives at the knots x_i are obtained from Eq.(5) and Eq.(6) as follows:

$$\begin{aligned} U_m &= U(x_m) = \delta_{m-3} + 57\delta_{m-2} + 302\delta_{m-1} + 302\delta_m + 57\delta_{m+1} + \delta_{m+2},\\ U'_m &= U'(x_m) = \tfrac{6}{h}(-\delta_{m-3} - 25\delta_{m-2} - 40\delta_{m-1} + 40\delta_m + 25\delta_{m+1} + \delta_{m+2}),\\ U''_m &= U''(x_m) = \tfrac{30}{h^2}(\delta_{m-3} + 9\delta_{m-2} - 10\delta_{m-1} - 10\delta_m + 9\delta_{m+1} + \delta_{m+2}),\\ U'''_m &= U'''(x_m) = \tfrac{120}{h^3}(-\delta_{m-3} - \delta_{m-2} + 8\delta_{m-1} - 8\delta_m + \delta_{m+1} + \delta_{m+2}). \end{aligned} \tag{7}$$

Collocation points are selected to coincide with the knots. Substituting nodal values U_m, U'_m and U''_m from Eq.(7) into Eq.(3) and using usual finite difference formula for the time derivatives of the element parameters $\dot{\delta}_m = \frac{\delta_m^{n+1} - \delta_m^n}{\Delta t}$ and Crank-Nicolson approach for the element parameters $\delta_m = \frac{\delta_m^{n+1} + \delta_m^n}{2}$ in the resulting system of equations leads to the following recurrence relation of element parameters between two time levels n and $n+1$

$$\begin{array}{l} \alpha_{1m}\,\delta_{m-3}^{n+1} + \alpha_{2m}\,\delta_{m-2}^{n+1} + \alpha_{3m}\,\delta_{m-1}^{n+1} + \alpha_{4m}\,\delta_{m}^{n+1} + \alpha_{5m}\,\delta_{m+1}^{n+1} + \alpha_{6m}\,\delta_{m+2}^{n+1} = \\ \alpha_{6m}\,\delta_{m-3}^{n} + \alpha_{5m}\,\delta_{m-2}^{n} + \alpha_{4m}\,\delta_{m-1}^{n} + \alpha_{3m}\,\delta_{m}^{n} + \alpha_{2m}\,\delta_{m+1}^{n} + \alpha_{1m}\,\delta_{m+2}^{n}, \end{array} \tag{8}$$

where

$$\begin{array}{ll} \alpha_{1m} = h^2 - 3dh\Delta t - 30\mu, & \alpha_{2m} = 57h^2 - 75dh\Delta t - 270\mu, \\ \alpha_{3m} = 302h^2 - 120dh\Delta t + 300\mu, & \alpha_{4m} = 302h^2 + 120dh\Delta t + 300\mu, \\ \alpha_{5m} = 57h^2 + 75dh\Delta t - 270\mu, & \alpha_{6m} = h^2 + 3dh\Delta t - 30\mu, \end{array} \tag{9}$$

and $d = 1 + 6\,Z_m,\ Z_m = (\delta_{m-3} + 57\delta_{m-2} + 302\delta_{m-1} + 302\delta_m + 57\delta_{m+1} + \delta_{m+2})^2$, is nonlinear term. A system of $N+1$ equations in $N+6$ unknown parameters $\delta_m,\ m = -3, \cdots, N+2$ has been obtained. The parameters $\delta_{-3}, \delta_{-2}, \delta_{-1}, \delta_{N+1}, \delta_{N+2}$ can be eliminated by using the boundary conditions $U(a,\,t) = U_x(a,\,t) = U_{xxx}(a,\,t) = 0$, and $U(b,\,t) = U_x(b,\,t) = 0$, so that we have to solve $(N+1) \times (N+1)$ six-banded matrix system by using a variant of Thomas algorithm. Before solving the recurrence system Eq.(8), the nonlinear term is linearized by replacing the element parameters δ_m by δ_m^n. This linearized system is recovered by applying the following corrector for two or three times at each time level

$$(\delta^\star)^{n+1} = \delta^n + \frac{1}{2}(\delta^{n+1} - \delta^n), \tag{10}$$

where $(\delta^\star)^{n+1}$ is a new approximation to δ^{n+1}. To start the recurrence relation system Eq.(8), initial parameters must be determined with the help of initial conditions and the five boundary conditions as follows

$$\begin{array}{l} U_x^N(a,\,0) = -\delta_{-3}^0 - 25\delta_{-2}^0 - 40\delta_{-1}^0 + 40\delta_0^0 + 25\delta_1^0 + \delta_2^0 = 0, \\ U_{xx}^N(a,\,0) = \delta_{-3}^0 + 9\delta_{-2}^0 - 10\delta_{-1}^0 - 10\delta_0^0 + 9\delta_1^0 + \delta_2^0 = 0, \\ U_{xxx}^N(a,\,0) = -\delta_{-3}^0 - \delta_{-2}^0 + 8\delta_{-1}^0 - 8\delta_0^0 + \delta_1^0 + \delta_2^0 = 0, \\ U^N(x_m,\,0) = \delta_{m-3}^0 + 57\delta_{m-2}^0 + 302\delta_{m-1}^0 + 302\delta_m^0 + 57\delta_{m+1}^0 + \delta_{m+2}^0 = \\ \qquad\qquad u(x_m, 0), \quad\quad m = 0,\,1,\,2, \cdots, N, \\ U_x^N(b,\,0) = \delta_{N+2}^0 + 25\delta_{N+1}^0 + 40\delta_N^0 - 40\delta_{N-1}^0 - 25\delta_{N-2}^0 - \delta_{N-3}^0 = 0, \\ U_{xx}^N(b,\,0) = \delta_{N+2}^0 + 9\delta_{N+1}^0 - 10\delta_N^0 - 10\delta_{N-1}^0 + 9\delta_{N-2}^0 + \delta_{N-3}^0 = 0. \end{array} \tag{11}$$

Eliminating $\delta_{-3}^0, \delta_{-2}^0, \delta_{-1}^0, \delta_{N+1}^0$, and δ_{N+2}^0 from system Eq.(8) we get the $(N+1) \times (N+1)$ six-banded matrix system which can be solved for $\delta_0^0, \delta_1^0, \delta_2^0, \cdots, \ \delta_N^0$ by using a variant of Thomas algorithm and then $\delta_{-3}^0, \delta_{-2}^0, \delta_{-1}^0, \delta_{N+1}^0$, and δ_{N+2}^0 can be computed from the first three equations and the last two equations of the system Eq.(8).

2.2 Stability analysis of the SBSC method

To investigate the stability of the difference scheme Eq.(8) we apply the Von Neumann stability analysis. Assuming U in the nonlinear term $U^2\,U_x$ of the MRLW equation as locally constant ζ then the corresponding values of Z_m are also constant and equal to ζ. Substituting the Fourier mode

$$\delta_m^n = \zeta^n\,e^{im\phi}, \quad \phi = kh, \quad \text{and} \quad i = \sqrt{-1} \tag{12}$$

where k is the mode number and h the element size into the linearized form of the difference Eq.(8) we obtain

$$\zeta^{n+1} = g\,\zeta^n, \tag{13}$$

where the growth factor is determined by

$$g = \frac{A - iB}{A' - iB'}, \tag{14}$$

with

$$
\begin{aligned}
A &= 2h^2\left[\cos^2(\tfrac{3\phi}{2}) + 58\cos^2(\phi) + 359\cos^2(\tfrac{\phi}{2}) - 58\right] - 60\mu\left[\cos^2(\tfrac{3\phi}{2}) + 10\cos^2(\phi)\right.\\
&\quad\left. - \cos^2(\tfrac{\phi}{2}) - 10\right] + 6dh\Delta t\left[\cos^2(\tfrac{3\phi}{2}) + 24\cos^2(\phi) + 15\cos^2(\phi) - 40\right],\\
B &= h^2\left[\sin(3\phi) + 56\sin(2\phi) + 245\sin(\phi)\right] - 30\mu\left[\sin(3\phi) + 8\sin(2\phi) - 19\sin(\phi)\right]\\
&\quad +3dh\Delta t\left[\sin(3\phi) + 26\sin(2\phi) + 65\sin(\phi)\right],\\
A' &= 2h^2\left[\cos^2(\tfrac{3\phi}{2}) + 58\cos^2(\phi) + 359\cos^2(\tfrac{\phi}{2}) - 58\right] - 60\mu\left[\cos^2(\tfrac{3\phi}{2}) + 10\cos^2(\phi)\right.\\
&\quad\left. - \cos^2(\tfrac{\phi}{2}) - 10\right] - 6dh\Delta t\left[\cos^2(\tfrac{3\phi}{2}) + 24\cos^2(\phi) + 15\cos^2(\tfrac{\phi}{2}) - 40\right],\\
B' &= h^2\left[\sin(3\phi) + 56\sin(2\phi) + 245\sin(\phi)\right] - 30\mu\left[\sin(3\phi) + 8\sin(2\phi) - 19\sin(\phi)\right]\\
&\quad -3dh\Delta t\left[\sin(3\phi) + 26\sin(2\phi) + 65\sin(\phi)\right].
\end{aligned}
\tag{15}
$$

Since $|g| \leq 1$, then the difference scheme Eq.(8) is unconditionally stable.

2.3 Sextic B-splines Runge-Kutta method (denoted by SBSRK)

Substituting the approximation solution $U^N(x,\, t)$ from Eq.(6) into the MRLW Eq.(3) we obtain a sextic diagonal matrix system of the following form

$$
A\,\dot{\delta}(t) = H\,\delta(t), \tag{16}
$$

where $\dot{\delta}(t) = [\dot{\delta}_0(t)\ \dot{\delta}_1(t)\ \cdots\ \dot{\delta}_N(t)]^T$. First, we solve this system for $\dot{\delta}_i(t),\ \ i = 0,\, 1,\, \cdots,\, N$ by using a variant of Thomas algorithm only once at each time level $t > 0$ then we get a first order system of ordinary differential equations which can be solved for $\delta_i(t),\ i = 0,\, 1,\, \cdots,\, N$ by using the famous fourth order Runge-Kutta method and consequently the solution $U^N(x,\, t)$ is completely known.

3 Numerical applications

The MRLW Eq.(3) has the following three conservation laws [12]

$$
\begin{aligned}
I_1 &= \textstyle\int_a^b U dx \simeq h \sum\limits_{j=1}^{N} U_j,\\
I_2 &= \textstyle\int_a^b (U^2 + \mu(U_x)^2) dx \simeq h \sum\limits_{j=1}^{N} (U_j^2 + \mu(U_x)_j^2),\\
I_3 &= \textstyle\int_a^b (U^4 - \mu(U_x)^2) dx \simeq h \sum\limits_{j=1}^{N} (U_j^4 - \mu(U_x)_j^2).
\end{aligned}
\tag{17}
$$

These integrals are evaluated to measure the conservation properties of the collocation schemes. The analytical values of these invariants can be found to be [12]

$$
I_1 = \tfrac{\pi\sqrt{c}}{p}, \qquad I_2 = \tfrac{2c}{p} + \tfrac{2\mu p c}{3}, \qquad I_3 = \tfrac{4c^2}{3p} - \tfrac{2\mu p c}{3}. \tag{18}
$$

The L_2 and L_∞-error norms are used to measure the accuracy of the present schemes and to compare our results with both exact values, Eq.(4), as well as other results in literature whenever available.

3.1 Single solitary wave

Two sets of parameters have been considered for computational work. The first set is chosen to be $c = 1$, $h = 0.2$, $\Delta t = 0.025$, $\mu = 1$, and $x_0 = 40$ with range $[0, 100]$ to coincide with the cubic scheme of Khalifa et al. [11] and the quintic scheme of Gardner et al. [12]. Thus the solitary wave has amplitude 1 and the simulations are done up to $t = 10$. Values of the three invariants as well as L_2 and L_∞-error norms due to SBSC method have been computed and reported in Table 1. Analytical values of the invariants are $I_1 = 4.442883$, $I_2 = 3.299832$, and $I_3 = 1.414214$. The changes of the invariants $I_1 \times 10^6$, $I_2 \times 10^6$, and $I_3 \times 10^6$ from their initial values are less than 0.213668, 0.313724, and 0.208092, respectively. Error deviations are changed in the ranges of $-0.0014054149 \leq error \leq 0.0013203318$. The second set will be chosen to be $c = 0.3$, $h = 0.1$, $\Delta t = 0.01$, $\mu = 1$, and $x_0 = 40$ with range $[0,\, 100]$, then the amplitude is 0.54772. The simulations are done up to $t = 20$. Table 2 represent values of the three invariants and error norms due to SBSC method in this case. Analytical values of the invariants are $I_1 = 3.581967$, $I_2 = 1.345077$, and $I_3 = 0.153723$. The changes of the invariants $I_1 \times 10^7$, $I_2 \times 10^7$, and $I_3 \times 10^7$ from their initial values are less than 2.2827, 0.1×10^{-4}, and 5.4×10^{-4}, respectively, i.e. approach zero throughout, indicating the efficiency of our scheme. The error deviations are changed in the ranges of $-1.878 \times 10^{-5} \leq error \leq 1.6447 \times 10^{-5}$ in this case.

Table 1: Invariants and error norms for single solitary waves of MRLW equation by using SBSC method with $c = 1$, $h = 0.2$, $\Delta t = 0.025$, $\mu = 1$, $x_0 = 40$, and $[a,\, b] = [0,\, 100]$

Time	I_1	I_2	I_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
0	4.442883	3.299832	1.414214	0.0	0.0
1	4.442883	3.299832	1.414214	0.31438	0.20533
2	4.442883	3.299832	1.414214	0.61002	0.36665
3	4.442883	3.299832	1.414214	0.88536	0.50539
4	4.442883	3.299832	1.414214	1.14843	0.63731
5	4.442883	3.299832	1.414214	1.40495	0.76675
6	4.442883	3.299832	1.414214	1.65797	0.89511
7	4.442883	3.299832	1.414214	1.90913	1.02297
8	4.442883	3.299832	1.414214	2.15926	1.15057
9	4.442883	3.299832	1.414214	2.40886	1.27803
10	4.442883	3.299832	1.414214	2.65819	1.40541

Table 2: Invariants and error norms for single solitary waves of MRLW equation by using SBSC method with $c = 0.3$, $h = 0.1$, $\Delta t = 0.01$, $\mu = 1$, $x_0 = 40$, and $[a, b] = [0, 100]$

Time	I_1	I_2	I_3	$L_2 \times 10^4$	$L_\infty \times 10^4$
0	3.581967	1.345077	0.153723	0.0	0.0
2	3.581967	1.345077	0.153723	0.051966	0.030378
4	3.581967	1.345077	0.153723	0.100545	0.055213
6	3.581967	1.345077	0.153723	0.145293	0.075259
8	3.581967	1.345077	0.153723	0.187240	0.093098
10	3.581967	1.345077	0.153723	0.227315	0.109782
12	3.581967	1.345077	0.153723	0.266132	0.125858
14	3.581967	1.345077	0.153723	0.304091	0.141582
16	3.581967	1.345077	0.153723	0.341453	0.157094
18	3.581967	1.345077	0.153723	0.378394	0.172471
20	3.581967	1.345077	0.153723	0.415036	0.187761

Use of fourth order Runge-Kutta method, as a variant of the collocation method without discretizing the time dependent variables, to solve the resulting first order system of ordinary differential equations only once at each time level $t > 0$ for the parameters $\dot{\delta}_m^{n+1}$ in terms of δ_m^n, $m = 0,\, 1,\, \cdots,\, N$ not only reduces the computational efforts but also increases the accuracy of results up to a great extent. Invariants and error norms due to SBSRK method corresponding to the 1^{st} and the 2^{nd} chosen sets of parameters are computed and reported in Tables 3 and 4, respectively. The error deviations corresponding to the first set of parameters are changed in the ranges of $-0.1505 \times 10^{-6} \leq error \leq 0.2676 \times 10^{-6}$ and those corresponding to the 2^{nd} set are changed in the ranges of $-0.253 \times 10^{-8} \leq error \leq 0.8835 \times 10^{-7}$. Changes of the invariants $I_1 \times 10^7$, $I_2 \times 10^7$, and $I_3 \times 10^7$ from their initial values are less than 2.68×10^{-5}, 1.0199237, and 1.0199283, respectively, for the first set of parameters and less than 1.8557718, 4.72×10^{-5}, and 1.43×10^{-5}, respectively, for the second set of parameters. It is clear that the error deviations and the changes of invariants approach zero throughout indicating the efficiency of the present scheme. The motion of a single solitary wave using SBSRK scheme corresponding to the 2^{nd} set of parameters has been computed at different time levels and plotted in Fig. 1. Figures corresponding to the 1^{st} set of parameters are almost the same.

Table 3: Invariants and error norms for single solitary waves of MRLW equation by using SBSRK method with $c = 1$, $h = 0.2$, $\Delta t = 0.025$, $\mu = 1$, $x_0 = 40$, and $[a,\, b] = [0,\, 100]$

Time	I_1	I_2	I_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
0	4.442883	3.299832	1.414214	0.0	0.0
1	4.442883	3.299832	1.414214	0.000178	0.000137
2	4.442883	3.299832	1.414214	0.000176	0.000151
3	4.442883	3.299832	1.414214	0.000186	0.000145
4	4.442883	3.299832	1.414214	0.000202	0.000131
5	4.442883	3.299832	1.414214	0.000225	0.000148
6	4.442883	3.299832	1.414213	0.000254	0.000170
7	4.442883	3.299832	1.414213	0.000287	0.000192
8	4.442883	3.299832	1.414213	0.000325	0.000216
9	4.442883	3.299832	1.414213	0.000367	0.000241
10	4.442883	3.299832	1.414213	0.000413	0.000268

Table 4: Invariants and error norms for single solitary waves of MRLW equation by using SBSRK method with $c = 0.3$, $h = 0.1$, $\Delta t = 0.01$, $\mu = 1$, $x_0 = 40$, and $[a, b] = [0, 100]$

Time	I_1	I_2	I_3	$L_2 \times 10^4$	$L_\infty \times 10^4$
0	3.581967	1.345077	0.153723	0.0	0.0
2	3.581967	1.345077	0.153723	0.000021	0.000021
4	3.581967	1.345077	0.153723	0.000036	0.000027
6	3.581967	1.345077	0.153723	0.000047	0.000027
8	3.581967	1.345077	0.153723	0.000057	0.000027
10	3.581967	1.345077	0.153723	0.000063	0.000026
12	3.581967	1.345077	0.153723	0.000067	0.000026
14	3.581967	1.345077	0.153723	0.000071	0.000026
16	3.581967	1.345077	0.153723	0.000083	0.000073
18	3.581967	1.345077	0.153723	0.000163	0.000253
20	3.581967	1.345077	0.153723	0.000511	0.000884

Comparisons of our results due to different methods with exact solution as well as the recorded values in [11, 12] have been made and tabulated in Table 5 at $t = 10$. The superiority of the present schemes over the previous ones is clear. Observations of L_2 and L_∞-error norms indicate that SBSRK method is much more satisfactorily conservative than SBSC method and both are better than the collocation methods with cubic and quintic B-splines used in [11, 12] at all time levels.

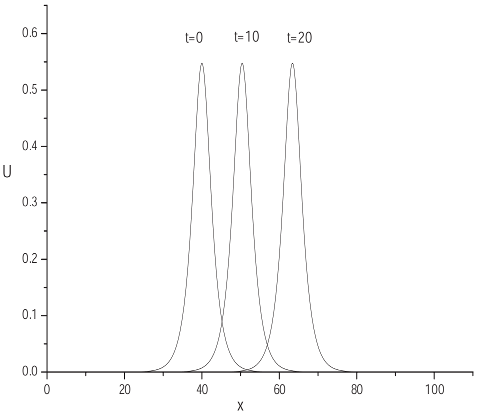

Figure 1: Motion of a single solitary wave of MRLW equation with $c = 0.3$, $h = 0.1$, $\Delta t = 0.01$, $\mu = 1$, $x_0 = 40$, and $0 \leq x \leq 100$ at $t = 0$, $t = 10$, and $t = 20$.

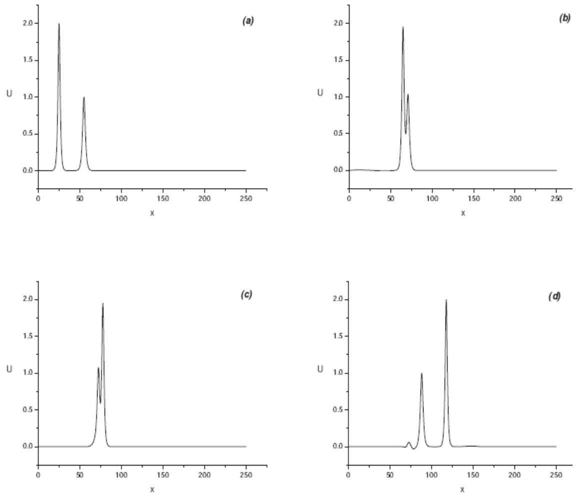

Figure 2: Interaction of two solitary waves of MRLW equation with $h = 0.1$, $\Delta t = 0.01$ and $0 \leq x \leq 250$ at: (a) $t = 0$, (b) $t = 10$, (c) $t = 12$, and (d) $t = 20$.

Table 5: Comparisons of results for a single solitary wave of MRLW equation with $c = 1$, $h = 0.2$, $\Delta t = 0.025$, and $0 \leq x \leq 100$ at $t = 10$

Ref.	Method	$L_2 \times 10^3$	$L_\infty \times 10^3$	I_1	I_2	I_3
Exact	Eq.(6)	0.000000	0.000000	4.442883	3.299832	1.414214
Ours	SBSRK	0.000413	0.000268	4.442883	3.299832	1.414214
Ours	SBSC	2.65819	1.40541	4.442883	3.299832	1.414214
[11]	Cubic-BS	9.30196	5.43718	4.4288	3.29983	1.41421
[11]	Cubic B-spline coll-CN	16.39	9.24	4.442	3.299	1.413
[11]	Cubic B-splines coll+PA-CN	20.3	11.2	4.440	3.296	1.411
[12]	Quadratic B-spline Galerkin-CN	3.80	1.98	4.443	3.301	1.411
[12]	Quintic B-spline coll+PA-CN	3.94	2.03	4.44290	3.29978	1.41417

3.2 Interaction of two MRLW solitary waves

Here we study the interaction of two well separated MRLW solitary waves having different amplitudes and traveling in the same direction. The initial condition is given by

$$U(x,\, 0) = \sum_{i=1}^{2} \sqrt{c_i}\ sech\left(p_i\left(x - x_i\right)\right), \tag{19}$$

where $p_i = \sqrt{\frac{c_i}{\mu\,(1+c_i)}}$, $i = 1,\ 2,\ c_i$ and x_i are arbitrary constants. The analytical values of the conservation laws in this case are as follows [11]

$$I_1 = \sum_{i=1}^{2} \frac{\pi\sqrt{c_i}}{p_i}, \quad I_2 = \sum_{i=1}^{2} \left(\frac{2\,c_i}{p_i} + \frac{2\,\mu\,p_i\,c_i}{3}\right), \quad I_3 = \sum_{i=1}^{2} \left(\frac{4\,c_i^2}{3\,p_i} - \frac{2\,\mu\,p_i\,c_i}{3}\right). \tag{20}$$

For computational work we choose $c_1 = 4$, $c_2 = 1$, $x_1 = 25$, $x_2 = 55$, and $\mu = 1$ with interval $[0, 250]$. The amplitudes are in the ratio of $2 : 1$, i.e. $\sqrt{c_1} = 2\sqrt{c_2}$. The analytical values of the invariants are $I_1 = 11.467698$, $I_2 = 14.629243$, and $I_3 = 22.880466$. Numerical values of the three invariants have been computed using SBSC and SBSRK methods and reported in Table 6 for $h = 0.2$ and $\Delta t = 0.025$ and in Table 7 for $h = 0.1$ and $\Delta t = 0.01$. The simulation is done up to $t = 20$. Changes of the invariants I_1, I_2, and I_3 from their initial values in Table 6 are less than 0.1704×10^{-2}, 0.4591×10^{-2}, and 1.8286×10^{-2}, respectively, for SBSC but less than 0.5×10^{-8}, 0.807267×10^{-3}, and 3.231×10^{-3}, respectively, for SBSRK scheme. Figs. $2(a)$ up to $2(d)$ show the computer plot of the interaction of two solitary waves at different time levels. These has been done by using SBSRK method for the second set of parameters. Figures corresponding to the first set of parameters are almost the same. Changes of the invariants in Table 7 from their initial values are less than 0.1786×10^{-4}, 0.4814×10^{-4}, and 1.8189×10^{-4} for SBSC method but less than 0.71×10^{-8}, 0.8327×10^{-5}, and 0.3332×10^{-4} for SBSRK method. Decreasing the values of h and Δt to 0.1 and 0.01, respectively, results in more accurate results as it is clear from the Tables.

Table 6: Invariants for interaction of two solitary waves of MRLW equation with $c_1 = 4$, $c_2 = 1$, $x_1 = 25$, $x_2 = 55$, $h = 0.2$, $\Delta t = 0.025$, $\mu = 1$, and $[a, b] = [0, 250]$

	SBSC method			SBSRK method		
Time	I_1	I_2	I_3	I_1	I_2	I_3
0	11.4677	14.6292	22.8805	11.4677	14.6292	22.8805
2	11.4675	14.6287	22.8784	11.4677	14.6292	22.8801
4	11.4673	14.6282	22.8764	11.4677	14.6291	22.8798
6	11.4671	14.6277	22.8743	11.4677	14.6290	22.8794
8	11.4670	14.6273	22.8682	11.4677	14.6289	22.8791
10	11.4669	14.6271	22.8660	11.4677	14.6289	22.8789
12	11.4667	14.6267	22.8703	11.4677	14.6288	22.8786
14	11.4666	14.6262	22.8683	11.4677	14.6287	22.8783
16	11.4664	14.6257	22.8662	11.4677	14.6286	22.8779
18	11.4662	14.6252	22.8642	11.4677	14.6285	22.8776
20	11.4660	14.6247	22.8622	11.4677	14.6284	22.8772

Table 7: Invariants for interaction of two solitary waves of MRLW equation with $c_1 = 4$, $c_2 = 1$, $x_1 = 25$, $x_2 = 55$, $h = 0.1$, $\Delta t = 0.01$, $\mu = 1$, and $[a, b] = [0, 250]$

	SBSC method			SBSRK method		
Time	I_1	I_2	I_3	I_1	I_2	I_3
0	11.46770	14.62924	22.88047	11.46770	14.62924	22.88047
2	11.46770	14.62924	22.88044	11.46770	14.62924	22.88046
4	11.46769	14.62923	22.88042	11.46770	14.62924	22.88046
6	11.46769	14.62923	22.88039	11.46770	14.62924	22.88046
8	11.46769	14.62922	22.87965	11.46770	14.62924	22.88045
10	11.46769	14.62922	22.87947	11.46770	14.62924	22.88045
12	11.46769	14.62922	22.88037	11.46770	14.62924	22.88045
14	11.46769	14.62921	22.88035	11.46770	14.62924	22.88044
16	11.46768	14.62921	22.88033	11.46770	14.62924	22.88044
18	11.46768	14.62920	22.88031	11.46770	14.62924	22.88044
20	11.46768	14.62919	22.88028	11.46770	14.62923	22.88043

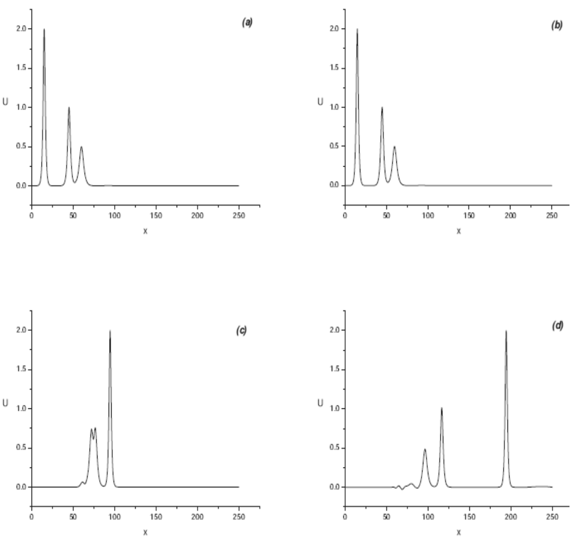

Figure 3: Interaction of three solitary waves of MRLW equation with $h = 0.1$, $\Delta t = 0.01$ and $0 \leq x \leq 250$ at: (a) $t = 0$, (b) $t = 5$, (c) $t = 20$, and (d) $t = 40$.

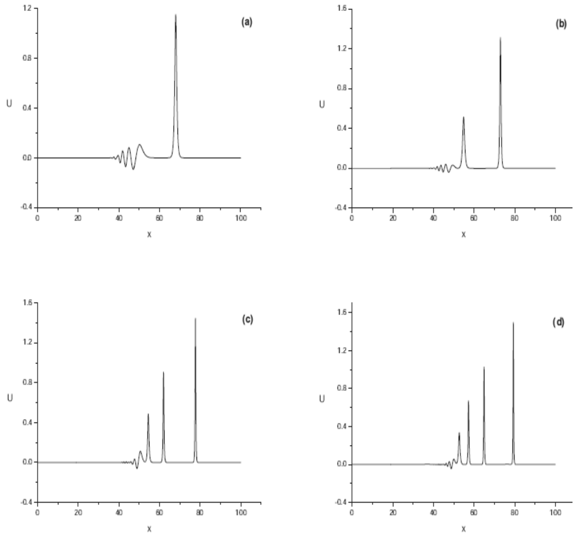

Figure 4: Maxwellian initial condition with (a) $\mu = 0.1$, (b) $\mu = 0.04$, (c) $\mu = 0.015$, and (d) $\mu = 0.01$ at $t = 14.5$.

3.3 Interaction of three MRLW solitary waves

For interaction of three waves of different amplitudes and traveling in the same direction we take

$$U(x,\, 0) = \sum_{i=1}^{3} \sqrt{c_i}\, sech\left(p_i\left(x - x_i\right)\right), \tag{21}$$

with c_i, p_i and x_i, $i = 1,\, 2,\, 3$ are as before. The analytical values of invariants in this case are [11]:

$$I_1 = \sum_{i=1}^{3} \frac{\pi \sqrt{c_i}}{p_i}, \quad I_2 = \sum_{i=1}^{3} (\frac{2\,c_i}{p_i} + \frac{2\,\mu\,p_i\,c_i}{3}), \quad I_3 = \sum_{i=1}^{3} (\frac{4\,c_i^2}{3\,p_i} - \frac{2\,\mu\,p_i\,c_i}{3}). \tag{22}$$

For numerical computations we choose $c_1 = 4$, $c_2 = 1$, $c_3 = 0.25$, $x_1 = 15$, $x_2 = 45$, $x_3 = 60$, and $\mu = 1$ with interval $[0, 250]$. Thus the amplitudes are in the ratio of $4 : 2 : 1$, i.e. $\sqrt{c_1} = 2\sqrt{c_2} = 4\sqrt{c_3}$. The analytical values of the invariants are $I_1 = 14.9801$, $I_2 = 15.8218$, and $I_3 = 22.9923$. Numerical values of the invariants are calculated using SBSC and SBSRK methods and reported in Table 8 for $h = 0.2$ and $\Delta t = 0.025$ and in Table 9 for $h = 0.1$ and $\Delta t = 0.01$. The simulation is done up to $t = 40$. Changes of I_1, I_2, and I_3 from their initial values in Table 8 are less than 0.677×10^{-2}, 1.8871×10^{-2}, and 7.5175×10^{-2}, respectively, for SBSC method but less than 0.7935×10^{-4}, 1.6091×10^{-3}, and 6.4608×10^{-3}, respectively, for SBSRK method. In Table 9 these changes are less than 3.243×10^{-5}, 0.958×10^{-4}, and 4.011×10^{-4}, respectively, for SBSC but less than 0.828×10^{-4}, 1.6613×10^{-5}, and 6.6648×10^{-5}, respectively, for SBSRK method. Figs. $3(a)$ up to $3(d)$ show the details of interaction of three solitary waves corresponding to the second set of parameters and SBSRK method at different time levels.

Table 8: Invariants for interaction of three solitary waves of MRLW equation with $c_1 = 4$, $c_2 = 1$, $c_3 = 0.25$, $x_1 = 15$, $x_2 = 45$, $x_3 = 60$, $h = 0.2$, $\Delta t = 0.025$, $\mu = 1$, and $[a, b] = [0, 250]$

Time	SBSC method			SBSRK method		
	I_1	I_2	I_3	I_1	I_2	I_3
0	14.9801	15.8375	23.0082	14.9801	15.8375	23.0082
5	14.9810	15.8401	23.0185	14.9801	15.8373	23.0073
10	14.9817	15.8420	23.0123	14.9801	15.8371	23.0066
15	14.9822	15.8434	23.0312	14.9801	15.8370	23.0060
20	14.9831	15.8460	23.0417	14.9801	15.8367	23.0051
25	14.9841	15.8486	23.0522	14.9801	15.8365	23.0043
30	14.9850	15.8512	23.0625	14.9802	15.8363	23.0034
35	14.9859	15.8538	23.0729	14.9802	15.8361	23.0026
40	14.9869	15.8564	23.0833	14.9802	15.8359	23.0017

Table 9: Invariants for interaction of three solitary waves of MRLW equation with $c_1 = 4$, $c_2 = 1$, $c_3 = 0.25$, $x_1 = 15$, $x_2 = 45$, $x_3 = 60$, $h = 0.1$, $\Delta t = 0.01$, $\mu = 1$, and $[a, b] = [0, 250]$

Time	SBSC method			SBSRK method		
	I_1	I_2	I_3	I_1	I_2	I_3
0	14.98010	15.83749	23.00817	14.98010	15.83749	23.00817
5	14.98010	15.83748	23.00811	14.98012	15.83749	23.00816
10	14.98009	15.83747	23.00591	14.98014	15.83749	23.00815
15	14.98009	15.83746	23.00798	14.98015	15.83749	23.00815
20	14.98009	15.83745	23.00797	14.98015	15.83749	23.00814
25	14.98008	15.83744	23.00793	14.98016	15.83748	23.00813
30	14.98008	15.83742	23.00787	14.98017	15.83748	23.00812
35	14.98007	15.83741	23.00782	14.98017	15.83748	23.00811
40	14.98007	15.83740	23.00777	14.98018	15.83748	23.00810

3.4 Maxwellian initial condition

Evolution of a train of solitary waves of the MRLW equation has been studied using the Maxwellian initial condition

$$U(x,\, 0) = \exp\left(-(x - 40)^2\right), \tag{23}$$

for various values of μ. The computations are carried out and values of the invariants are reported in Table 10 for the cases $\mu = 0.1$, 0.04, 0.015, and 0.01 at fixed space-time steps of $h = 0.025$ and $\Delta t = 0.002$, respectively. It is to be noted here that when μ is reduced more and more solitary waves are formed. A single solitary wave is formed when $\mu = 0.1$ as shown in Fig. $4(a)$. A well developed osculating tail is observed just before the solitary wave. For $\mu = 0.04$, the Maxwellian pulse breaks up into a train of at least two solitary waves as shown in Fig. $4(b)$. For $\mu = 0.015$ three stable solitons are generated as shown in Fig. $4(c)$. Finally, when $\mu = 0.01$ the Maxwellian initial condition has split up into four stable solitary waves as shown in Fig. $4(d)$. All these have been computed at $t = 14.5$. Changes of the invariants I_1, I_2, and I_3 from their initial values are less than 0.95×10^{-9}, 0.193×10^{-8}, and 0.7119×10^{-5} for $\mu = 0.1$ but less than 0.2033×10^{-4}, 0.4327×10^{-4}, and 0.3823×10^{-3} for $\mu = 0.01$ both when using SBSC method. Also the changes of these invariants from their initial values are less than 0.47×10^{-11}, 0.7244×10^{-9}, and 0.9281×10^{-9} for $\mu = 0.1$ but less than 0.61×10^{-8}, 0.5486×10^{-5}, and 1.2397×10^{-5} for $\mu = 0.01$ both when using SBSRK method. Invariants remain almost constant for larger values of μ and occur a little more when we use smaller values of μ. Simulations are done up to $t = 15$.

4 Conclusion

The main purpose of this work has been to illustrate the significant improvements in calculations when sextic B-spline collocation algorithm is applied using the fourth order Runge-Kutta method. This powerful

Table 10: Invariants for Maxwellian initial condition of MRLW equation with $h = 0.025$ and $\Delta t = 0.002$

Time	μ	SBSC method			SBSRK method			μ	SBSC method			SBSRK method		
		I_1	I_2	I_3	I_1	I_2	I_3		I_1	I_2	I_3	I_1	I_2	I_3
0	0.1	1.77245	1.37865	0.76090	1.77245	1.37865	0.76090	0.015	1.77245	1.27211	0.86743	1.77245	1.27211	0.86743
3		1.77245	1.37865	0.76090	1.77245	1.37865	0.76090		1.77245	1.27212	0.86761	1.77245	1.27211	0.86743
6		1.77245	1.37865	0.76090	1.77245	1.37865	0.76090		1.77246	1.27212	0.86761	1.77245	1.27211	0.86743
9		1.77245	1.37865	0.76090	1.77245	1.37865	0.76090		1.77246	1.27212	0.86761	1.77245	1.27211	0.86743
12		1.77245	1.37865	0.76090	1.77245	1.37865	0.76090		1.77246	1.27212	0.86762	1.77245	1.27211	0.86742
15		1.77245	1.37865	0.76090	1.77245	1.37865	0.76090		1.77246	1.27212	0.86762	1.77245	1.27211	0.86742
0	0.04	1.77245	1.30345	0.83609	1.77245	1.30345	0.83609	0.01	1.77245	1.26585	0.87369	1.77245	1.26585	0.87369
3		1.77245	1.30345	0.83614	1.77245	1.30345	0.83609		1.77246	1.26585	0.87399	1.77245	1.26585	0.87369
6		1.77245	1.30345	0.83614	1.77245	1.30345	0.83609		1.77246	1.26586	0.87401	1.77245	1.26585	0.87369
9		1.77245	1.30345	0.83614	1.77245	1.30345	0.83609		1.77247	1.26587	0.87404	1.77245	1.26584	0.87369
12		1.77245	1.30345	0.83614	1.77245	1.30345	0.83609		1.77247	1.26588	0.87406	1.77245	1.26484	0.87368
15		1.77245	1.30345	0.83614	1.77245	1.30345	0.83609		1.77247	1.26589	0.87408	1.77245	1.26484	0.87368

technique proves to be very suitable for solving such type of problems. The present numerical findings shows and proves the superiority of these schemes over earlier quadratic and cubic ones. Use of Runge-Kutta method makes the calculations not only very accurate but also straight, forward, and efficiently applicable for computing machines. Moreover, sextic B-spline functions are handy in writing the approximate solutions in the numerical methods if the partial differential equation involve higher order derivatives.

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