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New Peaked Solitary Wave Solutions and Solitons of a Generalized Dispersive Camassa-Holm Model

Lu Sun *
Nonlinear Scientific Research Center, Jiangsu University
Zhenjiang, Jiangsu, 212013, P.R. China
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Abstract: A new type of nonlinear generalized dispersive Camassa-Holm model is investigated in this paper. Through computations of integrations , we deduced that under different variables and the strength of nonlinearities, the model has different nonlimit zero point and different integral expressions of solitary wave solutions. A lot of peaked solitary wave solutions and their figures are given. We make four forms of ansatz solutions to derive some new solutions of the model, such as, pair compacton, multiple compacton, periodic compacton, blow-up solitary waves etc. Higher dimension generalized dispersive Camassa-Holm model is investigated too.

Keywords:Camassa-Holm model;peaked solitary wave; periodic soliton; multiple soliton

1 Introduction

Our model system is based entirely on the completely integrable Hamiltonian Camassa-Holm model which is arising in the context of small amplitude shallow water waves. The system is over a flat bottom for inviscid fluid moving under the influence of gravity. The investigation of this model is the same as the KdV model, which has been intensively studied for a century.

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}.$$

The fluid velocity in the x-direction or equivalently the height of the water's free surface is represented by the parameter u, and $4k = c_0$ is the critical shallow water wave speed which is proportional to the square root of the wave depth. In dispersive processes, the presence of even a weak singularity poses a formidable numerical challenge, which in recent literature devoted to the numerical aspects of the problem [1]. The conserved quantities and the initial value problem of the Camassa-Holm model are investigated in [2]. Symmetry properties are discussed in [3]. Integrable perturbation is investigated in [4]. The soliton solution of the Camassa-Holm model is investigated with variation method in [5]. Tian et al. [6] discussed the traveling wave solutions and double soliton solutions for the nonlinear models, and introduced the definitions of concave, convex peaked soliton and smooth soliton solution. Chen et al. [7] proposed the viscous Camassa-Holm equation as a closure approximate for the Reynolds- averaged equation of the incompressible Navier-Stokes fluid. This approximation is tested on turbulent channel flows with steady mean. Dullin et al. [8] studied a class of 1+1 quadratically nonlinear water wave equations that combine the linear dispersion of the KdV equation with the nonlinear/nonlocal dispersion of the C-H equation, yet still preserved integrability via the inverse scatting transform method. And [9] got the integrable equation derived by asymptotic expansion at one order higher approximation than KdV equation. The equation discussed in the work is in this class for unidirectional water waves with fluid velocity $u(x, y)$ as follows,

$$m_t + c_0 u_x + um_x + 2mu_x = -\gamma u_{xxx}, \tag{1.1}$$

*Corresponding author. *E-mail address*: flowersl@yahoo.cn

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where $m = u - \alpha^2 u_{xx}$ is a momentum variable, the constants α^2 and γ/c_0 are squares of length scales, and $c_0 = \sqrt{gh}$ is the linear wave speed for undisturbed water at rest at spatial infinity, where u and m are taken to vanish. Instead, taking $\gamma \longrightarrow 0$ in the Eq. (1.1) implies the CH equation

$$u_t + c_0 u_x - \alpha^2 u_{xxt} = 3uu_x = \alpha^2(2u_x u_{xx} + uu_{xxx}), \tag{1.2}$$

which for $c_0 = 0$ has peakon soliton solution $u(x,t) = ce^{-|x-ct|}$. When $\alpha^2 = 1$, the Eq. (1.2) becomes Eq. (1.1). Foias et al. [9] reviewed the properties of the nonlinear Navier-Stokes-Alpha $(NS - \alpha)$ model of incompressible fluid turbulence, or called the viscous Camassa-Holm equation in the literature. The $(NS - \alpha)$ model are derived by filtering the velocity of the fluid loop in Kelvin's circulation theorem for the Navier-Stokes equation. [9] also explained how the $(NS - \alpha)$ model is related to large eddy simulation (LES) turbulence modeling and to the stress tensor for second-grade fluids.

Based on the above research works, we considered the strength of nonlinearity and introduce the dispersive term of Eq. (1.1), we call it generalized dispersive Camassa-Holm equation $C(m, n, p, l)$,

$$u_t + ku_x + \beta_1 u_{xxt} + \beta_2 (u^m)_x + \beta_3 u_x (u^n)_{xx} + \beta_4 u(u^p)_{xxx} + (u^l)_{xxx} = 0. \tag{1.3}$$

The term $(u^l)_{xxx}$ is the nonlinear dispersive term. In order to get the integral expressions of the model, we discussed the nonlinearity variables l, m, n, p when $n = 1, p = 1, l = 2$ and $\beta_1 = -1, \beta_2 = a, \beta_3 = 2, \beta_4 = -\frac{1}{2}, \beta_5 = -\frac{1}{4}$,we get the following equation,

$$u_t + ku_x - u_{xxt} + a(u^m)_x + 2u_x u_{xx} - \frac{1}{2}uu_{xxx} - \frac{1}{4}(u^2)_{xxx} = 0. \tag{1.4}$$

From researches of the above equation, peaked solitary wave solutions and lots of different type solitary waves are obtained in this paper. This paper is consisted of four parts. The first section is introduction. In the second section we discuss the traveling wave solutions and peakons of GCH equation. In the third section, we find the examples of peakon solutions. In the fourth section, we derived the exact solutions of the generalized dispersive Camassa-Holm equation.

2 The integral expressions of the generalized dispersive Camassa-Holm model

2.1 The traveling wave solution of the generalized dispersive Camassa-Holm model

To seek the traveling wave solutions of the generalized dispersive model, we first change the partial differential equation into the ordinary differential equation. We let $u(x,t) = \nu(\xi), \xi = x - ct$, then the equation changed into the following form,

$$(k-c)\nu_\xi + cv_{\xi\xi\xi} + a(\nu^m)_\xi - \frac{1}{2}(\nu\nu_{\xi\xi})_\xi - \frac{1}{4}(\nu^2)_{\xi\xi\xi} = 0.$$

Through the integration of ξ , and we take the integral constant to be zero, we get,

$$(k-c)\nu + cv_{\xi\xi} + a\nu^m - \frac{1}{2}\nu\nu_{\xi\xi} - \frac{1}{4}(\nu^2)_{\xi\xi} = 0.$$

And then we take the integration of ν , By taking the integral constant to be zero, we get,

$$(\frac{c}{2} - \frac{1}{2}\nu)(\nu_\xi)^2 = \frac{c-k}{2}\nu^2 - \frac{a}{m+1}\nu^{m+1}.$$

Then we can deduce,

$$\nu_\xi = \pm\sqrt{\frac{(c-k)(m+1)\nu^2 - 2a\nu^{m+1}}{(c-\nu)(m+1)}}.$$

We get the integral expression of the generalized dispersive Camassa-Holm model,
(1) when m is even, at this moment $k \neq c$,

$$|\xi| = \int_\nu^{(m-1)\sqrt{\frac{(c-k)(m+1)}{2a}}} \sqrt{\frac{(c-\varphi)(m+1)}{(c-k)(m+1)\varphi^2 - 2a\varphi^{(m+1)}}} d\varphi,$$

for $\nu \in (0, (m-1)\sqrt{\frac{(c-k)(m+1)}{2a}})$

$$|\xi| = \int_{(m-1)\sqrt{\frac{(c-k)(m+1)}{2a}}}^{\nu} \sqrt{\frac{(c-\varphi)(m+1)}{(c-k)(m+1)\varphi^2 - 2a\varphi^{(m+1)}}} d\varphi,$$

for $\nu \in ((m-1)\sqrt{\frac{(c-k)(m+1)}{2a}}, 0)$

(2) when m is odd, at this moment $k < c$,

$$|\xi| = \int_{\nu}^{-(m-1)\sqrt{\frac{(c-k)(m+1)}{2a}}} \sqrt{\frac{(c-\varphi)(m+1)}{(c-k)(m+1)\varphi^2 - 2a\varphi^{(m+1)}}} d\varphi,$$

for $\nu \in (0, (m-1)\sqrt{\frac{(c-k)(m+1)}{2a}})$

$$|\xi| = \int_{-(m-1)\sqrt{\frac{(c-k)(m+1)}{2a}}}^{\nu} \sqrt{\frac{(c-\varphi)(m+1)}{(c-k)(m+1)\varphi^2 - 2a\varphi^{(m+1)}}} d\varphi,$$

for $\nu \in ((m-1)\sqrt{\frac{(c-k)(m+1)}{2a}}, 0)$.

2.2 The integral expression of peakon solution of the generalized dispersive Camassa-Holm model

Let $c^{(m-1)} = \frac{(c-k)(m+1)}{2a}$, we obtain the following expression,

$$|\xi| = \int_{\nu}^{c} \frac{1}{\varphi} \sqrt{\frac{(m+1)(c+4\varphi)}{2a(c^{(m-1)-\varphi^{(m-1)}})}} d\varphi.$$

Thus the bifurcation equation of peakon is $a = \frac{(c-k)(m+1)}{2c^{(m-1)}}$, a is the bifurcation variable of peakons.

3 Examples of peakons

(1)When $m = 2, k \neq c$, we have $|\xi| = \sqrt{\frac{(m+1)}{2a}} \int_{\nu}^{c} \frac{1}{\varphi} d\varphi$ and $\nu = ce^{-\sqrt{\frac{2a}{3}}|\xi|}$. In this situation peakon solution of the generalized dispersive Camassa-Holm model changed into the form $u(x,t) = ce^{-\sqrt{\frac{2a}{3}}|x-ct|}$, so we can obviously reduce the bifurcation equation into $(3-2a)c = 3k$. In this case the model also should satisfy $a \neq \frac{3}{2}, c = \frac{3k}{3-2a}$.

The peakon solution of the constant coefficient dispersive Camassa-Holm model is,

$$u(x,t) = \frac{3k}{3-2a} exp(-\sqrt{\frac{2a}{3}}|x - \frac{3k}{3-2a}t|),$$

when $a = \frac{3}{2}$ and $k = 0$, c is arbitrary constant. At this time the model has peakons in the following forms $u(x,t) = ce^{-|x-ct|}$.

(2)When $m = 3, k < c$, we have $|\xi| = \sqrt{\frac{2}{a}} \int_{\nu}^{c} \frac{1}{\varphi\sqrt{\varphi+c}} d\varphi$. At this moment we obtain the peakon solution $u(x,t) = c\tanh(\sqrt{2} + \sqrt{\frac{ac}{8}}|x-ct|) - c$. The bifurcation equation can obviously changed into the form $c^2a - 2c + 2k = 0$.

When $k = 0$ or $a = \frac{1}{2k}$, the bifurcation equation has a constant solution $c = \frac{1}{a}$, so we get $\nu = \frac{1}{a}\tanh(\sqrt{2} + \frac{1}{2\sqrt{2}}|\xi|)^2 - \frac{1}{a}$. Then the peakons of the constant coefficient dispersive Camassa-Holm model is in the following form,

$$u(x,t) = \frac{1}{a}\tanh(\sqrt{2} + \frac{1}{2\sqrt{2}}|x - \frac{1}{a}t|)^2 - \frac{1}{a}.$$

Thus we can deduce that the constant coefficient dispersive Camassa-Holm model has two peakons,

$$u(x,t) = \frac{1 \pm \sqrt{(1-2ak)}}{a} \tanh(\sqrt{2} + \sqrt{\frac{1 \pm \sqrt{(1-2ak)}}{8}} |x - \frac{1 \pm \sqrt{(1-2ak)}}{a} t|)^2 - c,$$

when $k < 0, a > 0$, the model do not have solutions.

4 The exact solutions of the generalized dispersive Camassa-holm equation

We try to seek the solitary wave solutions of Eq.(1.3). Let Eq.(1.3) has the following solitary wave solution, $u(x,t) = u(\xi) = u(x - Dt)$ with D as wave speed, then Eq. (1.3) can be change into the following form,

$$(k - D)u_\xi + Du_{3\xi} + a(u^m)_\xi + 2u_\xi u_{2\xi} - \frac{1}{2}uu_{3\xi} - \frac{1}{4}(u^2)_{3\xi} = 0. \tag{4.1}$$

Ansatz 1: We get the compacton solution of the generalized dispersive Camassa-Holm model, when $|\sqrt{\frac{a(\beta+2)}{8\beta^3+4\beta}}\xi| \le \frac{\pi}{2}$, then

$$u = \frac{(D-k)[8\gamma^3 + 8\beta - 4\beta] + aD(\beta+2)(3\beta-2)}{a(\beta+2)^2(\beta+1)} \cos^4(\sqrt{\frac{a(\beta+2)}{8\beta^3+4\beta}}\xi).$$

Otherwise, $u = 0$. To prove that this solution is existed, we need to make the second derivation of u when $|\sqrt{\frac{a(\beta+2)}{8\beta^3+4\beta}}\xi| = \frac{\pi}{2}$,

$$\frac{\partial^2}{\partial x^2}u = \frac{(D-k)[8\beta^3 + 8\beta - 4\beta] + aD(\beta+2)(3\beta-2)}{a(\beta+2)^2(\beta+1)} \times$$

$$\{\frac{3a(\beta+2)}{2\beta^3+\beta} \sin^2[\sqrt{\frac{a(\beta+2)}{8\beta^3+4\beta}}(x-Dt)] \times \cos^2[\sqrt{\frac{a(\beta+2)}{8\beta^3+4\beta}}(x-Dt)] - \frac{a(\beta+2)}{2\beta^3+\beta} \sin^4[\frac{4D}{\beta-1}](x-Dt)\}.$$

Then we can obtain that the second derivative at $(-\infty, \infty)$ is existed. Thus the compacton is the solutions of (4.1) at the same time it is the solution of generalized dispersive Camassa-Holm equation.

Ansatz 2: We get that when $|\sqrt{\frac{a(\beta+2)}{8\beta^3+4\beta}}\xi| \le \pi$,

$$u = \frac{(D-k)[8\beta^3 + 8\beta - 4\beta] + aD(\beta+2)(3\beta-2)}{a(\beta+2)^2(\beta+1)} \sin^4(\sqrt{\frac{a(\beta+2)}{8\beta^3+4\beta}}\xi).$$

Otherwise $u = 0$, we call them pair compacton. In the same way, we need to make the second derivation of u when$|\sqrt{\frac{a(\beta+2)}{8\beta^3+4\beta}}\xi| = \pi$,

$$\frac{\partial^2}{\partial x^2}u = \frac{(D-k)[8\beta^3 + 8\beta - 4\beta] + aD(\beta+2)(3\beta-2)}{a(\beta+2)^2(\beta+1)} \times$$

$$\{\frac{3a(\beta+2)}{2\beta^3+\beta} \sin^2[\sqrt{\frac{a(\beta+2)}{8\beta^3+4\beta}}\xi] \times \cos^2[\sqrt{\frac{a(\beta+2)}{8\beta^3+4\beta}}\xi] - \frac{a(\beta+2)}{8\beta^3+\beta} \sin^4[\sqrt{\frac{a(\beta+2)}{8\beta^3+4\beta}}\xi\}.$$

Then we can obtain that the second derivative at $(-\infty, \infty)$ is existed. Thus the compacton is the solutions of Eq.(4.1) at the same time it is the solution of generalized Camassa-Holm equation.

Ansatz 3:We get the solitary pattern solution of Eq.(4.1),

$$u = \frac{(k-D)(\beta^3 - 8\beta - 16) - aD(\beta^3 - 4\beta^2 + 8 - 8\beta)}{a(\beta+2)^2(\beta+1)} \cosh^4(\sqrt{\frac{a(\beta+2)}{\beta^3 - 8\beta - 16}}\xi).$$

Ansatz 4:We derived the solitary pattern solution of Ansatz4,

$$u = \frac{(k-D)(\beta^3 - 8\beta - 16) - aD(\beta^3 - 4\beta^2 + 8 - 8\beta)}{a(\beta+2)^2(\beta+1)} \sinh^4(\sqrt{\frac{a(\beta+2)}{\beta^3 - 8\beta - 16}}\xi).$$

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