ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.8(2009) No.1,pp.103-109

Weak Ergodic Theorem for General Semigroups in Banach Spaces

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(Received 6 April 2009, accepted 2 July 2009)

Abstract: This paper studies the problem of weak ergodic theorem for asymptotically nonexpansive type mappings of general semigroups in real reflexive Banach spaces with Frechet differential norm by a new method. Furthermore , we obtain the weak ergodic theorem for the almost orbits of asymptotically nonexpansive type semigroups without almost asymptotically isometric.

Keywords: semigroup; asymptotically nonexpensive type semigroup; invariant mean

1 Introduction

In 1975, Baillon[1] proved the first nonlinear ergodic theorem for nonexpensive mappings in the framework of a Hilbert space: let T be a nonexpensive mapping of C into itself and suppose that the set $F(T)$ of all fixed points of T is nonempty. Then the cesaro means

$$S_n(x) = \frac{1}{n}\sum_{i=0}^{n-1} T^i x$$

converges weakly as $n \to \infty$ to a fixed point of T for each $x \in C$. Similarly results were also obtained in uniformly convex Banach space with Frechet differential norm[8,9]. The analogous results were given for nonexpensive semigroups[2,3,4,6,7,10]. Li and Ma[5] proved the weak ergodic theorem for commutative asymptotically almost nonexpensive type semigroups in reflexive Banach space. In our paper, we study weak ergodic theorems of general asymptotically nonexpensive type semigroups in reflexive Banach space.

2 Preliminaries

Throughout this paper, we assume that X is a real reflexive Banach space and G is a general semigroup with unit element. Let C be a nonempty closed convex subset of X. Let X^* be dual space of X. Then the value of $x^* \in X^*$ at $x \in X$ will be denoted by $\langle x, x^* \rangle$, and the normalized duality mapping will be defined by

$$J(x) = \left\{ x^* \in X^*, \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\}$$

We say that X is (F) if the norm of X is Frechet differential, i.e. for each $x \in X \backslash \{0\}$, $\lim\limits_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists uniformly in $y \in \mathrm{B}_r$, where $\mathrm{B}_r = \{x \in X, \|x\| \le r\}$, $r > 0$. It is easy to see that X is (F) if and only if for any B_r and $x \in X$, $\lim\limits_{t\to 0} \frac{\|x+ty\|^2-\|x\|^2}{2t} = \langle y, J(x) \rangle$, uniformly in $y \in \mathrm{B}_r$.

Let $m(G)$ be the Banach space of all bounded real-valued functions on G with the supremum norm. Then for each $s \in G$ and $f \in m(G)$, we define $r_s f$ and $l_s f$ in $m(G)$ by $(r_s f)(t) = f(ts)$ and $(l_s f)(t) = f(st)$ for all $t \in G$ respectively.

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IJNS.2009.08.15/252

Let D be a subspace of $m(G)$ including constant functions and D^* be dual space of it. Then the value of $\mu \in D^*$ at $f \in D$ will be denoted by $\mu(f) = \int f(t)\, d\mu(t)$ or $\mu(t)\langle f(t)\rangle$. μ is called a mean on D if $\|\mu\| = \mu(1) = 1$. Further, a mean μ on D is left invariant if $\mu(l_s f) = \mu(f)$ for all $s \in G$ and $f \in D$. For $s \in G$, similarly, we can define right invariant mean. A mean μ is invariant if μ is right and left invariant. Since X is reflexive, there exists unique $u_\mu \in X$ such that for all $\mu \in D^*$ and $x^* \in X^*, \langle u_\mu, x^*\rangle = \int \langle u(t), x^*\rangle d\mu(t)$. We denote $\mu(t)\langle u(t)\rangle$ or $\int u(t)\, d\mu(t)$ by u_μ. Specially, if $u(t) = T(t)x$, we denote u_μ by $T_\mu x$.

Throughout this paper, we assume that $m(G)$ has invariant mean. Let $S = \{T(t); t \in G\}$ be a semigroup of nonlinear mappings on C, i.e. $T(ts)x = T(t)T(s)x$ for all $t, s \in G$ and $x \in C$. Then S is said to be

(1)nonexpansive if $\|T(t)x - T(t)y\| \le \|x - y\|$ for $x, y \in C$ and $t \in G$;
(2) asymptotically nonexpensive if there exists a function $K : G \to [0, \infty)$ with $\inf\limits_{s\in G} \sup\limits_{t\in G} K(ts) \le 1$ such that $\|T(t)x - T(t)y\| \le K(t)\|x - y\|$ for all $x, y \in C$ and $t \in G$;
(3) asymptotically nonexpensive type if there exists a function $\alpha(\cdot) : G \to [0, +\infty)$ with $\inf\limits_{s\in G} \sup\limits_{t\in G} \alpha(ts) = 0$ such that $\|T(t)x - T(s)y\| \le \|x - y\| + \alpha(t)$ for all $x, y \in C$ and $t \in G$.

A function $u(\cdot) : G \to C$ is said to be an almost orbit of S, if $\inf\limits_{s\in G} \sup\limits_{t\in G}[\sup\limits_{h\in G} \|u(hts) - T(h)u(ts)\|] = 0$. We denote by $\varphi(t) = \sup\limits_{h\in G} \|u(ht) - T(h)u(t)\|$, then $\inf\limits_{s\in G} \sup\limits_{t\in G} \varphi(ts) = 0$.

Let Γ be the set of strictly increasing continuous convex functions $f(0) = 0$ with $f : [0, +\infty) \to [0, +\infty)$. S is said to be of type (Γ) if there exists $\gamma \in \Gamma$ and a function $\varphi_1(\cdot) : G \to [0, +\infty)$ with $\inf\limits_{s\in G} \sup\limits_{t\in G} \varphi_1(ts) = 0$ such that

$$\gamma\left(\left\|\sum_{i=1}^n \lambda_i T(t)x_i - T(t)\sum_{i=1}^n \lambda_i x_i\right\|\right) \le \max_{1\le i,j\le n}\{\|x_i - x_j\| - \|T(t)x_i - T(t)x_j\|\} + \varphi_1(t)$$

for all $t \in G$, $x_i \in C\,(1 \le i \le n)$ and $\lambda_i \ge 0$, $\sum\limits_{i=1}^n \lambda_i = 1$.

Remark 1 *By Bruck[2,Theorem2.1], we know if X is a uniformly convex Banach space , then Lipschitzian semigroup is type* (Γ).

We denote by $\beta(t) = \max\{\alpha(t), \varphi_1(t)\}$, then $\beta(t) \ge 0$ and $\inf\limits_{s\in G} \sup\limits_{t\in G} \beta(ts) = 0$, by $F(S)$ the set of common fixed points of S, by $AO(S)$ the set of all almost-orbits of S, let $d = 2\sup\{\|x\|; x \in C\}$, and $LAO(S) = \{T(h)u(\cdot); h \in G, u(\cdot) \in AO(S)\}$.

3 Lemmas and Main Results

Lemma 2 *Let X is (F), then $\bigcap\limits_{s\in G} \overline{co}\{u(ts); t \in G\} \cap F(S)$ consists of at most one point. Specially, for any $x \in C$, then $\bigcap\limits_{s\in G} \overline{co}\{T(ts)x; t \in G\} \cap F(S)$ consists of at most one point.*

Proof. We firstly prove that for all $\lambda \in (0, 1), f, g \in F(S)$

$$\inf_{s\in G} \sup_{t\in G} \|\lambda u(ts) + (1 - \lambda)f - g\| \le \sup_{s\in G} \inf_{t\in G} \|\lambda u(ts) + (1 - \lambda)f - g\|$$

Since $\gamma^{-1}(0) = 0$ and γ^{-1} is continuous, then for any$\varepsilon > 0$, there exists $\delta \in \left(0, \frac{\varepsilon}{3}\right)$ such that

$$\gamma^{-1}(5\delta) < \frac{\varepsilon}{3} \tag{1}$$

For such an δ, by definitions of φ and β, there exists $s_0, t_0 \in G$ such that, for all $t \in G$,

$$\varphi(ts_0) < \delta \quad \beta(tt_0) < \delta \tag{2}$$

By Eq.(2), for all $a \in G$, we have

$$\begin{aligned}
&\inf_{s\in G}\sup_{t\in G}\left\|u\left(tss_0\right)-f\right\| \leq \sup_{t\in G}\left\|u\left(tt_0as_0\right)-f\right\| \\
&\leq \sup_{t\in G}\left\|u\left(tt_0as_0\right)-T\left(tt_0\right)u\left(as_0\right)\right\| + \sup_{t\in G}\left\|T\left(tt_0\right)u\left(as_0\right)-f\right\| \\
&\leq \varphi\left(as_0\right)+\sup_{t\in G}\beta\left(tt_0\right)+\left\|u\left(as_0\right)-f\right\| \\
&< \delta+\delta+\left\|u\left(as_0\right)-f\right\| \\
&= 2\delta+\left\|u\left(as_0\right)-f\right\|
\end{aligned}$$

Since $a \in G$ is arbitrary, it is then easily seen that

$$\inf_{s\in G}\sup_{t\in G}\left\|u\left(tss_0\right)-f\right\| \leq 2\delta+\inf_{a\in G}\left\|u\left(as_0\right)-f\right\|$$

Then there is $s_1 \in G$ such that, for all $t \in G$,

$$\left\|u\left(ts_1s_0\right)-f\right\| < 3\delta+\inf_{a\in G}\left\|u\left(as_0\right)-f\right\| \tag{3}$$

By Eq.(1,2,3), for all $a \in G$, we have

$$\begin{aligned}
&\inf_{s\in G}\sup_{t\in G}\left\|\lambda u\left(ts\right)+\left(1-\lambda\right)f-g\right\| \\
&\leq \sup_{t\in G}\left\|\lambda u\left(tt_0as_1s_0\right)+\left(1-\lambda\right)f-g\right\| \\
&\leq \lambda\varphi\left(as_1s_0\right)+\sup_{t\in G}\left\|\lambda T\left(tt_0\right)u\left(as_1s_0\right)+\left(1-\lambda\right)f-T\left(tt_0\right)\left(\lambda u\left(as_1s_0\right)+\left(1-\lambda\right)\right)f\right\| \\
&+\sup_{t\in G}\left\|T\left(tt_0\right)\left(\lambda u\left(as_1s_0\right)+\left(1-\lambda\right)f\right)-g\right\| \\
&\leq \delta+\sup_{t\in G}\gamma^{-1}\left(\left\|u\left(as_1s_0\right)-f\right\|-\left\|T\left(tt_0\right)u\left(as_1s_0\right)-f\right\|+\beta\left(tt_0\right)\right) \\
&+\left\|\lambda u\left(as_1s_0\right)+\left(1-\lambda\right)f-g\right\|+\sup_{t\in G}\beta\left(tt_0\right) \\
&\leq 2\delta+\sup_{t\in G}\gamma^{-1}\left(\left\|u\left(as_1s_0\right)-f\right\|-\left\|u\left(tt_0as_1s_0\right)-f\right\|+\varphi\left(as_1s_0\right)+\beta\left(tt_0\right)\right) \\
&+\left\|\lambda u\left(as_1s_0\right)+\left(1-\lambda\right)f-g\right\| \\
&\leq 2\delta+\gamma^{-1}\left(3\delta+\inf_{a\in G}\left\|u\left(as_0\right)-f\right\|-\inf_{t\in G}\left\|u\left(ts_0\right)-f\right\|+2\delta\right)+\left\|\lambda u\left(as_1s_0\right)+\left(1-\lambda\right)f-g\right\| \\
&\leq 2\delta+\gamma^{-1}\left(5\delta\right)+\left\|\lambda u\left(as_1s_0\right)+\left(1-\lambda\right)f-g\right\| \\
&< \varepsilon+\left\|\lambda u\left(as_1s_0\right)+\left(1-\lambda\right)f-g\right\|
\end{aligned}$$

Since $a \in G$ is arbitrary, we have

$$\begin{aligned}
&\inf_{s\in G}\sup_{t\in G}\left\|\lambda u\left(ts\right)+\left(1-\lambda\right)f-g\right\| \leq \varepsilon+\inf_{a\in G}\left\|\lambda u\left(as_1s_0\right)+\left(1-\lambda\right)f-g\right\| \\
&\leq \varepsilon+\sup_{s\in G}\inf_{t\in G}\left\|\lambda u\left(ts\right)+\left(1-\lambda\right)f-g\right\|
\end{aligned}$$

Since ε is arbitrary, we have

$$\inf_{s\in G}\sup_{t\in G}\left\|\lambda u\left(ts\right)+\left(1-\lambda\right)f-g\right\| \leq \sup_{s\in G}\inf_{t\in G}\left\|\lambda u\left(ts\right)+\left(1-\lambda\right)f-g\right\| \tag{4}$$

Let

$$h\left(\lambda\right)=\inf_{s\in G}\sup_{t\in G}\left\|\lambda u\left(ts\right)+\left(1-\lambda\right)f-g\right\| \quad \forall f,g\in \bigcap_{s\in G}\overline{co}\left\{u\left(ts\right);t\in G\right\}\cap F\left(S\right)$$

For any $\varepsilon > 0$, then there is $s_2 \in G$ such that

$$\left\|\lambda u\left(ts_2\right)+\left(1-\lambda\right)f-g\right\| < h\left(\lambda\right)+\varepsilon \quad \forall t\in G$$

and hence

$$\left\langle \lambda u\left(ts_2\right)+\left(1-\lambda\right)f-g, J\left(f-g\right)\right\rangle < \left(h\left(\lambda\right)+\varepsilon\right)\left\|f-g\right\| \quad \forall t\in G$$

From $f \in \overline{co}\left\{u\left(ts_2\right); t \in G\right\}$, we have

$$\left\langle \lambda f + (1-\lambda) f - g, J(f-g)\right\rangle \le (h(\lambda) + \varepsilon)\|f-g\|$$

i.e.

$$\|f-g\| \le h(\lambda) + \varepsilon$$

Since ε is arbitrary , we have

$$\|f-g\| \le h(\lambda) \tag{5}$$

Since X is (F), then

$$\lim_{\lambda \to 0} (2\lambda)^{-1}\left(\|\lambda u(ts) + (1-\lambda) f - g\|^2 - \|f-g\|^2\right) = \left\langle u(ts) - f, J(f-g)\right\rangle$$

uniformly for $t, s \in G$.

By Eq.(4,5),

$$\sup_{s \in G} \inf_{t \in G} \left\langle u(ts) - f, J(f-g)\right\rangle \ge 0 \tag{6}$$

For any $\varepsilon > 0$, then there is $s_3 \in G$ such that $\left\langle u(ts_3) - f, J(f-g)\right\rangle > -\varepsilon$ for all $t \in G$.

From $g \in \overline{co}\left\{u\left(ts_3\right); t \in G\right\}$, then

$$\left\langle g - f, J(f-g)\right\rangle \ge -\varepsilon$$

i.e.

$$\|f-g\|^2 \le \varepsilon$$

Since ε is arbitrary, we have $f = g$.

From X is (F), then

$$\bigcap_{s \in G} \overline{co}\left\{u(ts); t \in G\right\} \cap F(S)$$

consists of at most one point. ■

In order to convenience, foe any $\varepsilon \in (0,1]$ and $h \in G$, let $F_\varepsilon(T(h)) = \{x \in C; \|T(h)x - x\| \le \varepsilon\}$, let $a(\varepsilon) = \frac{\varepsilon^2}{4R^2}\min\left\{\frac{\varepsilon}{4}, \varepsilon\gamma\left(\frac{\varepsilon}{4}\right)\right\}, G_\varepsilon = \{h \in G; \beta(h) \le \varepsilon\}, R = 2d+1.$

Lemma 3 $\left(F_{\varepsilon/4}(T(h)) + B\left(O, \frac{\varepsilon}{4}\right)\right) \cap C \subset F_\varepsilon(T(h))$ *for all* $\varepsilon > 0$ *and* $h \in G_{\varepsilon/4}$.

Proof. For any $h \in G_{\varepsilon/4}$ and $x \in \left(F_{\varepsilon/4}(T(h)) + B\left(O, \frac{\varepsilon}{4}\right)\right) \cap C$, we assume that $x = y + z$, where $y \in F_{\varepsilon/4}(T(h)), z \in B\left(0, \varepsilon/4\right)$,
and hence

$$\begin{aligned} \|T(h)x - x\| &\le \|T(h)x - T(h)y\| + \|T(h)y - y\| + \|y - x\| \\ &\le 2\|x-y\| + \beta(h) + \|T(h)y - y\| \le \varepsilon \end{aligned}$$

i.e. $x \in F_\varepsilon(T(h)) \quad \forall h \in G_{\varepsilon/4}$, we have

$$\left(F_{\varepsilon/4}(T(h)) + B\left(O, \frac{\varepsilon}{4}\right)\right) \cap C \subset F_\varepsilon(T(h)) \quad \forall h \in G_{\varepsilon/4}.$$

■

The following lemma is very important in our proof of main result:

Lemma 4 *For any* $\varepsilon \in (0,1]$ *and* $h \in G_{a(a(\varepsilon))}$, *there exists large enough* $n_0 \in \mathbb{Z}^+$ *such that*

$$\frac{1}{n}\sum_{i=1}^{n} T\left(h^i\right)x \in F_\varepsilon(T(h)) \quad \forall n > n_0, x \in C.$$

Proof. For any $\varepsilon \in (0,1]$, let $m = \left[\frac{2d}{a(\varepsilon)}\right] + 1$, then there exists large enough $n_0 \in \mathrm{Z}^+$ such that $n_0 \geq \max\left\{\frac{8md}{\varepsilon}, \left(\gamma\left(\frac{a(\varepsilon)}{2}\right)\varepsilon\right)^{-1} 16m^2d\right\}$ for all $n \geq n_0$ and $h \in G_{a(a(\varepsilon))}$, and let

$$N = \left[\left(\gamma\left(\frac{a(\varepsilon)}{2}\right)\right)^{-1}\left(m^2d + m^2n\beta(h)\right)\right]\left(< \frac{n}{2}\right),$$

$$\begin{aligned}
&\gamma\left(\left\|\tfrac{1}{m}\sum_{j=1}^{m} T(h)^{i+j+1}x - T(h)\tfrac{1}{m}\sum_{j=1}^{m} T(h)^{i+j}x\right\|\right) \\
&\leq \max_{1\leq j,k\leq m}\left(\left\|T(h)^{i+j}x - T(h)^{i+k}x\right\| - \left\|T(h)^{i+j+1}x - T(h)^{i+k+1}x\right\| + \beta(h)\right) \\
&\leq \sum_{1\leq j<k\leq m}\left(\left\|T(h)^{i+j}x - T(h)^{i+k}x\right\| - \left\|T(h)^{i+j+1}x - T(h)^{i+k+1}x\right\| + \beta(h)\right)
\end{aligned}$$

Let $a_i(x) = \gamma\left(\left\|\frac{1}{m}\sum_{j=1}^{m} T(h)^{i+j+1}x - T(h)\frac{1}{m}\sum_{j=1}^{m} T(h)^{i+j}x\right\|\right), (i = 1, 2, \cdots, n)$.

We have

$$\begin{aligned}
&\sum_{i=1}^{n} a_i(x) \leq \sum_{i=1}^{n}\sum_{1\leq j<k\leq m}\left(\left\|T(h)^{j+i}x - T(h)^{k+i}x\right\| - \left\|T(h)^{j+i+1}x - T(h)^{k+i+1}x\right\| + \beta(h)\right) \\
&= \sum_{1\leq j<k\leq m}\sum_{i=1}^{n}\left(\left\|T(h)^{j+i}x - T(h)^{k+i}x\right\| - \left\|T(h)^{j+i+1}x - T(h)^{k+i+1}x\right\| + \beta(h)\right) \\
&\leq \sum_{1\leq j<k\leq m}(d + n\beta(h)) \leq m^2d + m^2n\beta(h)
\end{aligned}$$

We assume that the number of terms is K such that $a_i(x) \geq \gamma\left(2^{-1}a(\varepsilon)\right)$ in $\{a_i\}_{i=1}^{n}$, then $K\gamma\left(2^{-1}a(\varepsilon)\right) \leq m^2d + m^2n\beta(h)$. i.e. $K \leq \left(\gamma\left(2^{-1}a(\varepsilon)\right)\right)^{-1}\left(m^2d + m^2n\beta(h)\right)$ i.e. the number of terms is no more than N such that $a_i(x) \geq \gamma\left(2^{-1}a(\varepsilon)\right)$. For any i, there is at most one term such that , for $j \in [0, N]$,$a_{i+j}(x) < \gamma\left(2^{-1}a(\varepsilon)\right)$ in $\{a_{i+j}(x)\, 0 \leq j \leq N\}$. Let $l_i = \min\left\{j; a_{i+j}(x) < \gamma\left(2^{-1}a(\varepsilon)\right)\right\} 0 \leq j \leq N$, then easily seen that the number of terms no more than N such that $l_i \neq 0$, hence

$$\begin{aligned}
&\left\|\tfrac{1}{m}\sum_{j=1}^{m} T(h)^{i+l_i+j}x - T(h)\tfrac{1}{m}\sum_{j=1}^{m} T(h)^{i+l_i+j}x\right\| \\
&\leq \left\|\tfrac{1}{m}\sum_{j=1}^{m} T(h)^{i+l_i+j}x - \tfrac{1}{m}\sum_{j=1}^{m} T(h)^{i+l_i+j+1}x\right\| + \left\|\tfrac{1}{m}\sum_{j=1}^{m} T(h)^{i+l_i+j+1}x - T(h)\tfrac{1}{m}\sum_{j=1}^{m} T(h)^{i+l_i+j}x\right\| \\
&\leq \tfrac{d}{m} + \tfrac{a(\varepsilon)}{2} < a(\varepsilon)
\end{aligned}$$

Then $\frac{1}{m}\sum_{j=1}^{m} T(h)^{i+l_i+j}x \in F_{a(\varepsilon)}(T(h)), \forall h \in G_{a(a(\varepsilon))}$.

By lemma 3, we have

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{m}\sum_{j=1}^{m} T(h)^{i+l_i+j}x \in coF_{a(\varepsilon)}(T(h)) \subset F_{\frac{\varepsilon}{4}}(T(h)), \forall h \in G_{a(a(\varepsilon))} \subset G_{a(\varepsilon)}$$

$$\begin{aligned}
&\left\|\tfrac{1}{n}\sum_{i=1}^{n} T(h)^{i}x - \tfrac{1}{n}\sum_{i=1}^{n}\tfrac{1}{m}\sum_{j=1}^{m} T(h)^{i+l_i+j}x\right\| \leq \tfrac{1}{mn}\sum_{j=1}^{m}\left\|\sum_{i=1}^{n} T(h)^{i}x - \sum_{i=1}^{n} T(h)^{i+l_i+j}x\right\| \\
&\leq \tfrac{1}{mn}\sum_{j=1}^{m}\left\|\sum_{i=1}^{n} T(h)^{i}x - \sum_{i=1}^{n} T(h)^{i+j}x\right\| + \tfrac{1}{mn}\sum_{j=1}^{m}\left\|\sum_{i=1}^{n} T(h)^{i+j}x - \sum_{i=1}^{n} T(h)^{i+l_i+j}x\right\| \leq \tfrac{m}{n}d + \tfrac{N}{n}d \\
&\leq \tfrac{\varepsilon}{8} + \tfrac{m^2\beta(h)d}{\gamma\left(\frac{1}{2}a(\varepsilon)\right)} + \tfrac{m^2d^2}{n\gamma\left(\frac{1}{2}a(\varepsilon)\right)} \leq \tfrac{\varepsilon}{8} + \tfrac{\varepsilon}{16} + \tfrac{\varepsilon}{16} = \tfrac{\varepsilon}{4}
\end{aligned}$$

$$\frac{1}{n}\sum_{i=1}^{n} T(h)^{i}x \in \left(F_{\varepsilon/4}(T(h)) + B\left(O, \frac{\varepsilon}{4}\right)\right) \cap C \subset F_{\varepsilon}(T(h)) \quad \forall h \in G_{a(a(\varepsilon))}, x \in C$$

■

Lemma 5 *Let μ be a invariant mean, then $T_\mu x \in \underset{s\in G}{\cap} \overline{co}\{T(ts)x; t\in G\} \cap F(S)$.*

Furthermore , if μ and ν are invariant means and X is (F), then $T_\mu x = T_\nu x$.

Proof. Since μ is a invariant mean, then $T_\mu x = \mu(t)\langle T(ts)x\rangle \in \overline{co}\{T(ts)x; t\in G\}$ for all $s\in G$, hence $T_\mu x \in \underset{s\in G}{\cap} \overline{co}\{T(ts)x; t\in G\}$. We prove that $T_\mu x \in F(S)$.

By lemma 4, for any $\varepsilon > 0$ and $h \in G_{a(a(a(\varepsilon)))}$ there is large enough $n \in Z^+$ such that

$$\frac{1}{n}\sum_{i=1}^{n} T(h)^i T(t)x \in F_{a(\varepsilon)}(T(h)) \quad \forall t\in G, x\in C. \qquad i.e. \quad \frac{1}{n}\sum_{i=1}^{n} T\left(h^i t\right)x \in F_{a(\varepsilon)}(T(h)) \quad \forall t\in G.$$

We have , for all $h \in G_{a(a(a(\varepsilon)))}\left(\subset G_{a(\varepsilon)}\right)$

$$T_\mu x = \mu(t)\langle T(t)x\rangle = \mu(t)\left\langle \frac{1}{n}\sum_{i=1}^{n} T\left(h^i t\right)x\right\rangle \in \overline{co}F_{a(\varepsilon)}(T(h)) \subset F_\varepsilon(T(h)).$$

For all $s\in G$, since $T(s)$ is continuous ,then , for any $\varepsilon > 0$, there exists $\varepsilon_0 \in \left(0, \frac{\varepsilon}{2}\right)$ such that

$$\|T(s)y - T(s)T_\mu x\| < \frac{\varepsilon}{2} \quad \forall y \in B(T_\mu x, \varepsilon_0).$$

By definition of $\beta(\cdot)$, there is $s_0 \in G$ such that $hs_0 \in G_{a(a(a(\varepsilon_0)))}$ for all $h\in G$.
We have $T_\mu x \in F_{\varepsilon_0}(T(hs_0))$ for all $h\in G$. i.e.

$$\|T(hs_0)T_\mu x - T_\mu x\| \le \varepsilon_0$$

For all $s\in G$,we have $\|T(shs_0)T_\mu x - T(s)T_\mu x\| < \frac{\varepsilon}{2}$, then

$$\|T(s)T_\mu x - T_\mu x\| \le \|T(s)T_\mu x - T(shs_0)T_\mu x\| + \|T(shs_0)T_\mu x - T_\mu x\| < \frac{\varepsilon}{2} + \varepsilon_0 < \varepsilon$$

Since ε is arbitrary, we have $T_\mu x = T(s)T_\mu x$.
Since s is arbitrary , we have $T_\mu x \in F(S)$.
Hence

$$T_\mu x \in \underset{s\in G}{\cap} \overline{co}\{T(ts)x; t\in G\} \cap F(S).$$

If X is (F), μ and ν are invariant means, by lemma 2, we have $T_\mu x = T_\nu x$. ■
Before we give our main results, we firstly give the following definition:

Definition 1 *Let $\{\mu_\alpha; \alpha\in \mathrm{A}\}$ be a family means where A is a directed set on D . We call $\{\mu_\alpha; \alpha\in \mathrm{A}\}$ be asymptotically invariant ,if $\mu_\alpha(f) - \mu_\alpha(l_s f) \to 0$, $\mu_\alpha(f) - \mu_\alpha(r_s f) \to 0$ for all $f\in D$ and $s\in G$.*

Theorem 6 *Let X is a real reflexive Banach space and X is (F), and let G be a general semigroup, and let C be a nonempty bounded and closed convex subset of X, and let $S = \{T(t); t\in G\}$ be asymptotically nonexpansive type semigroup and be (Γ) on C, and let D be an invariant subspace of $m(G)$ including constant functions, for all asymptotically invariant mean $\{\mu_\alpha; \alpha\in \mathrm{A}\}$ of D, then*

$$\int T(t)x d\mu_\alpha(t) \xrightarrow{w} p \in F(S).$$

Proof. Let W be a set of all weak limit points of net $\left\{\int T(t)x d\mu_\alpha(t) : \alpha\in \mathrm{A}\right\}$. Since C is bounded, W is nonempty.

Suppose $z\in W$, then there is a subset of $\left\{\int T(t)x d\mu_{\alpha_\beta}(t); \beta\in \mathrm{B}\right\}$ where B is a directed set such that $\int T(t)x d\mu_{\alpha_\beta}(t) \xrightarrow{w} z$. And from $\left\{\mu_{\alpha_\beta}; \beta\in \mathrm{B}\right\}$ be bounded, then it has w^* convergent subset.Without loss of generality, we assume that the w^* convergent subset is $\left\{\mu_{\alpha_\beta}; \beta\in \mathrm{B}\right\}$, we have $\mu_{\alpha_\beta} \overset{w^*}{\to} \mu \in D^*$, then μ is a mean.

For all $f \in D$, we have

$$|\mu(f) - \mu(l_s f)| \leq \left|\mu(f) - \mu_{\alpha_\beta}(f)\right| + \left|\mu_{\alpha_\beta}(f) - \mu_{\alpha_\beta}(l_s f)\right| + \left|\mu_{\alpha_\beta}(l_s f) - \mu(l_s f)\right| \to 0$$

hence $\mu(f) = \mu(l_s f)$.

Similarly, $\mu(f) = \mu(r_s f), \forall s \in G$, then μ is invariant.

From

$$\int T(t)\, x d\mu_{\alpha_\beta}(t) \xrightarrow{w} T_\mu x,$$

then $z = T_\mu x \in F(S)$. By lemma 5, W consists of one point. We have $\int T(t)\, x d\mu_\alpha(t) \xrightarrow{w} p \in F(S)$. ■

By Zhang [11, Theorem 2.1], we gain weak ergodic theorem of almost orbits.

Theorem 7 *Let X is a real reflexive Banach space and X is (F), and let G be a general semigroup, and let C be a nonempty bounded and closed convex subset of X, and let $S = \{T(t); t \in G\}$ be asymptotically nonexpansive type semigroup and be (Γ) on C, and let D be an invariant subspace of $m(G)$ including constant functions,f or any $x \in C$, and let $f(t) = \|u(t) - x\|^2$, $t \in G$, $f(\cdot) \in D$, for all asymptotically invariant mean $\{\mu_\alpha; \alpha \in \mathrm{A}\}$ of D, then*

$$\int u(t)\, d\mu_\alpha(t) \xrightarrow{w} p \in F(S).$$

Acknowledgements

Research was supported by foundation for the high-level talented person special subsidizes of Jiangsu University(No:05JDG047).

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