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The Solutions of the Generalized b-family Equation

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Abstract: This paper introduces a family of evolutionary 1+1 PDEs that describe the balance between convection and stretching in the dynamics of 1D nonlinear waves in fluids. It includes the Camassa-Holm equation and the Degasperis-Procesi equation as special cases, and it is reversible in time and parity invariant . In the paper, when $m = u - \alpha^2 u_{xx}$, special solutions including peakons, ramps and cliffs are discussed for $b = 0$ and for $b \neq 0$, the general solutions are given.
Keywords: b-family equation, traveling wave solution

1 Introduction

An one-dimensional version of active fluid transport is described by the following family of 1+1 evolutionary equations,

$$m_t + um_x + bu_x m + au_x - a\alpha^2 u_{xxx} = 0, u = g * m, m = u - \alpha^2 u_{xx} \tag{1}$$

with independent variable time t and one spatial coordinate x. It is the generalized $b-$family equation.

Holm and Staley [1] introduced the b-family PDEs that described the balance between convection and stretching for small viscosity in the dynamics of 1D nonlinear wave in fluids:

$$m_t + \underbrace{um_x}_{convection} + \underbrace{bu_x m}_{streching} = \underbrace{\varepsilon m_{xx}}_{viscosity}, u = g * m. \tag{2}$$

Here $u = g * m$ denotes $u(x) = \int_{-\infty}^{\infty} g(x-y)m(y)dy$. The convolution relates velocity u to momentum density m by integration against the kernel $g(x)$. They studied the effects of the balance parameter b and kernel $g(x)$ of the solitary wave structures and investigated their interactions analytically for $\varepsilon = 0$ and numerically for small viscosity $\varepsilon \neq 0$.

Camassa and Holm [2] derived the following equation for unidirectional motion of shallow water waves in a particular Galilean frame,

$$m_t + um_x + 2u_x m = -c_0 u_x - \gamma u_{xxx}, m = u - \alpha^2 u_{xx}. \tag{3}$$

Here $m = u - \alpha^2 u_{xx}$ is a momentum variable, partial derivatives are denoted by subscripts, the constants α^2 and γ/c_0 are squares of length scales, and $c_0 = \sqrt{g'h}$ is the linear wave speed for undisturbed water of depth h at rest under gravity g' at spatial infinity, where u and m are taken to vanish. Any constant value $u = u_0$ is also a solution of Eq.(3). Eq.(3) was derived using Hamiltonian methods in [2] and was also shown in [1] to appear as a water wave equation at quadratic order in the standard asymptotic expansion for shallow water waves in terms of their two small parameters (aspect ratio and wave height). The famous Korteweg-de Vries (KdV) equation appears at linear order in this asymptotic expansion and is recovered from Eq.(3) when $\alpha^2 \to 0$. Both KdV at linear order and its nonlocal, nonlinear generalization in Eq.(3) at quadratic order

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in this expansion have the remarkable property of being completely integrable by the isospectral transform (IST) method. The IST properties of KdV solitons are well known and these properties for Eq.(3) were studied in, e.g., [2] and [1]. When linear dispersion is absorbed by a Galilean transformation and a velocity shift, Eq.(3) reduces to an active transport equation that contains competing quadratically nonlinear terms representing convection and stretching,

$$m_t + um_x + 2u_x m = 0, \quad \text{and} \quad m = u - \alpha^2 u_{xx}. \tag{4}$$

A variant of Eq.(4) with coefficient $b = 3$

$$m_t + um_x + 3u_x m = 0, \quad \text{and} \quad m = u - \alpha^2 u_{xx} \tag{5}$$

was first singled-out for further analysis by Degasperis and Procesi [4]. Degasperis, et al [3] discovered that this 2 to 3 variant of Eq.(4) possessed superposed peakon solutions and also was completely integrable by the isospectral transform method. Fringer and Holm [5]extended the zero-dispersion Eq.(4) for the peakons to the "pulson" equation,

$$m_t + \underbrace{um_x}_{convection} + \underbrace{2u_x m}_{streching} = 0, u = g * m. \tag{6}$$

Fringer and Holm [5] chose $g(x)$ to be an even function, so that u and m have the same parity.

When setting small viscosity $\varepsilon = 0$ and adding two nonlinearities au_x, $a\alpha^2 u_{xxx}$, Eq.(2) turns into Eq.(1).

Zhou and Tian [6] employ the bifurcation method to dynamical systems to investigate the exact travelling wave solutions for the Fornberg-Whitham equation. The explicit expressions for peakons and periodic cusp wave solutions are also obtained. By using bifurcation method, Zhou and Tian [7] successfully find the Fornberg-Whitham equation has a type of traveling wave solutions called kink-like wave solutions and antikink-like wave solutions. Tian, et al [8] converge to the solution to the corresponding BBM equation as the parameter converges to zero. Zhou and Tian [9] obtain a conserbation law which enable us to present a blow-up result by using multiplier technique. A remarkable feature of the physical model is that it has peakon solution which has peak form[10]. Tian and Yin [11] introduce the concept of nonlinear intensity, study a fully nonlinear sine-Gordon equation SG(m, n, p) and obtain a new type of peakon solutions and kink solutions by a direct method. Tian and Yin [12] introduce the fully nonlinear generalized Camassa-Holm equation C(m,n,p) and by using four direct ansatzs, we obtain abundant solutions. Tian and Song[13] consider generalized Camassa-Holm equations and the generalized weakly dissipative Camassa-Holm equations and derive some new exact peaked solitary wave solutions.

The paper is organized as follows. In Section 2, we obtain the conservation law and some basic notation. In Section 3, peakons, ramps and cliffs are discussed for $b = 0$ and for $b \neq 0$, general solution is given.

2 Basic notations and conservation law

We seek solutions for the fluid velocity u that are defined either on the real line and vanishing at spatial infinity, or on a periodic one dimensional domain. Here $u = g * m$ denotes the convolution (or filtering),

$$u(x) = \int_{-\infty}^{\infty} g(x-y)m(y)dy, \tag{7}$$

which relates velocity u to momentum density m by integration against kernel $g(x)$ over the real line. We shall choose $g(x)$ to be an even function, so that u and m have the same parity.The family of Eq.(1) is characterized by the kernel g and the real dimensionless constant b, which is the ratio of stretching to convective transport. As we see, b is also the number of covariant dimensions associated with the momentum density m. The function $g(x)$ will determine the traveling wave shape and length scale for Eq.(1), while the constant b will provide a balance or bifurcation parameter for the nonlinear solution behavior. Special values of b will include the first few positive and negative integers.

Its invariance under space and time translations ensures that Eq.(1) admits traveling wave solutions for any b. Let us write the traveling wave solutions as $u = u(z)$ and $m = m(z)$, where $z = x - ct$, and let prime $'$ denote d/dz.

Eq.(1) implies a similar reversible, parity invariant equation for the absolute value $|m|$,

$$(1-a/c)|m|_t + u|m|_x + bu_x|m| = 0 \quad \text{and} \quad u = g * m. \tag{8}$$

If $m^{1/b}$ is well-defined, Eq.(1) may be written as the conservation law

$$\partial_t T m^{1/b} + \partial_x(m^{1/b}u), T = 1 - a/c, \tag{9}$$

and Eq. (8) for the absolute value implies

$$\partial_t T m^{1/b} + \partial_x(|m|^{1/b}u) = 0. \tag{10}$$

Adding and subtracting Eq.(9) and Eq.(10) implies

$$\partial_t T(m^{1/b})_\pm + \partial_x((m^{1/b})_\pm u) = 0, (m^{1/b})_\pm = \frac{1}{2}(m^{1/b} \pm |m|^{1/b}). \tag{11}$$

Consequently, regions of positive and negative m are transported by the same velocity and their boundaries propagate so as to separately preserve the two integrals,

$$\int_{-\infty}^{\infty} (m^{1/b})_\pm dx. \tag{12}$$

The common transport velocity allows a transformation to Lagrangian coordinates $X_\pm$ defined by

$$dX_\pm = (m^{1/b})_\pm (dx - udt),$$

so that

$$\partial_t X_\pm + U\partial_x X_\pm = 0. \tag{13}$$

This formal transformation is not strictly defined where $(m^{1/b})_\pm$ vanishes. However, by Eq.(11), regions where $(m^{1/b})_\pm$ where vanishes do not propagate and do not contribute to the integrated value of $X_\pm = \int_{-\infty}^{x}(m^{1/b})_\pm(y,0)dy$. Hence, these regions may be identified and excluded initially. The formal inverse relation holding in the remaining regions

$$dx = (m^{1/b})_\pm^{-1} dX_\pm + udt \tag{14}$$

implies that

$$\frac{dx}{dt}|_{X_\pm} = u(x,t), \tag{15}$$

so the Lagrangian trajectories $x = x(X_\pm, t)$ of positive and negative integrated initial values of $X_\pm = \int_{-\infty}^{x}(m^{1/b})_\pm(y,0)dy$, dy are transported by the same velocity $u = g * m$.

3 Traveling wave solutions

For $b = 0$, Eq.(1) is Galilean invariant and its traveling wave solutions satisfy

$$(u(z) - c + a)m'(z) = 0, z = x - ct, \tag{16}$$

where prime $'$ denotes d/dz. Eq.(16) admits generalized functions $m'(z) \simeq \delta(z)$ matched by $u - c + a = 0$ at $z = 0$. The velocity u is given by the integral of the Green's function that relates m and $u = g * m$

$$u - c + a = (c-a)[\int g(y)dy]_0^z. \tag{17}$$

When $g(x) = e^{-|x|/\alpha}$ (the Green's function for the 1D Helmholtz operator),we have $m = u - \alpha^2 u_{xx}$. Consequently, the equation $m' = u' - \alpha^2 u''' = \pm 2\delta(z)$ with $u - c + a = 0$ at $z = 0$ is satisfied by

$$u - c + a = \pm(c-a)[\int e^{-|y|/\alpha}dy]_0^z = \pm(c-a)sgn(z)(1 - e^{-|x|/\alpha}). \tag{18}$$

This represents a rightward moving traveling wave that connects the left state $u - c + a = \pm(c - a)$ to the same two right states.

From what has been discussed above, we have the following theorem:

Theorem 1 *Assume $b = 0$, $u = g * m$, $m = u - \alpha^2 u_{xx}$, Eq.(1) has the solution $u = \pm(c - a)sgn(z)(1 - e^{-|x|/\alpha}) + c - a$.*

We define $p(x) = \frac{1}{2\alpha^2} e^{-|\frac{x}{\alpha^2}|}$,$x \in R$, then $(1 - \alpha^2\partial_x^2)^{-1} f = p * f$ for all $f \in L^2(R)$, where $*$ denotes convolution. Using this identity, we can rewrite Eq.(1) as the following nonlocal form

$$u_t + uu_x - au_x + \partial_x p * [\frac{3\alpha^2}{2} u_x^2 + 2au] = 0. \quad (19)$$

Remark 1 *(peakons) The symmetric connections $u = \pm(c-a)e^{-|x|/\alpha}$ with a jump in derivative at $z = 0$are the peakons, for which $m = u - \alpha^2 u_{xx}$ and $g(x) = e^{-|x|/\alpha}$.*

Remark 2 *(ramps) The antisymmetric connections $u = \pm(c - a)sgn(z)(1 - e^{-|x|/\alpha})$ (with $u - c + a = \mp(c - a)$ connecting to $u - c + a = \pm(c - a)$), with no jump in derivative at $z = 0$, are the regularized shocks (cliffs). These propagate rightward but may face either leftward or rightward, because Eq.(1) in the absence of viscosity has no entropy condition that would distinguish between leftward and rightward facing solutions.*

Remark 3 *(cliffs) Eq.(1) also has ramp-like similarity solutions $u \simeq x/t$ when $g(x) = e^{-|x|/\alpha}$ for any b. These may emerge in the initial value problem for the peakon case of Eq.(1) and interact with the peakons and cliffs.*

For $b = 0$, the traveling wave Eq.(16) apparently has only the first integral for $m = u - \alpha^2 u_{xx}$

$$(u - c + a)(u - \alpha^2 u'') - \frac{u^2}{2} + \frac{\alpha^2}{2} u'^2 = K. \quad (20)$$

Thus, perhaps surprisingly, we have been unable to find a second integral for the traveling wave equation for peakons when $b = 0$.

Reversibility means that Eq.(1) is invariant under the transformation $u(x, t) \to -u(x, -t)$. Consequently, the rightward traveling waves have leftward moving counterparts under the symmetry $c - a \to -c + a$. The case of constant velocity $u = \pm(c - a)$ is also a solution.

For $b \neq 0$, the conservation law (9) for traveling waves becomes

$$((u - c + a)m^{1/b})' = 0, \quad (21)$$

which yields after one integration

$$(u - c + a)^b m = K, \quad (22)$$

where K is the first integral.

For $g(x) = e^{-|x|/\alpha}$, so $m = u - \alpha^2 u_{xx}$, Eq.(22) becomes

$$(u - c + a)^b (u - \alpha^2 u'') = K. \quad (23)$$

For $u - c + a \neq 0$ we rewrite Eq.(23) as

$$\alpha^2 u'' = u - K(u - c + a)^{-b} \quad (24)$$

and integrate again to give the second integral in two separate cases,

$$\alpha^2 u'^2 = \begin{cases} u^2 - \frac{2K}{1-b}(u - c + a)^{1-b} + 2H, b \neq 1 \\ u^2 - 2K log(u - c + a) + 2H, b = 1. \end{cases} \quad (25)$$

We shall rearrange this into quadratures:

$$\pm\frac{dz}{\alpha} = \frac{du}{[u^2 - \frac{2K}{1-b}(u - c + a)^{1-b} + 2H]^{\frac{1}{2}}} \quad \text{for} \quad b \neq 1, \quad (26)$$

and

$$\pm\frac{dz}{\alpha}=\frac{du}{[u^2-2Klog(u-c+a)+2H]^{\frac{1}{2}}}\quad \text{for}\quad b=1. \tag{27}$$

For $b=1$ and $K\neq 0$, the integral in Eq.(27) is transcendental.

For $K=0$, the two quadratures Eq.(26) and Eq.(27) are equal, independent of b , and elementary, thereby yielding the traveling wave solutions

$$e^{-|z|/\alpha}=\frac{u+\sqrt{u^2+2H}}{c-a+\sqrt{(c-a)^2+2H}}, \tag{28}$$

with $u-c+a=0$ at $z=0$.

For $H=0$, Eq.(28) recovers the peakon traveling wave.

For $H>0$, Eq.(28) gives a rightward moving traveling wave that is a continuous deformation of the peakon.

For $H>0$ and $c=a$, Eq.(28) gives stationary solutions of the form

$$\frac{u+\sqrt{u^2+2H}}{\sqrt{2H}}=e^{-|z|/\alpha}, \tag{29}$$

then

$$u=\frac{\sqrt{2H}}{2}e^{-|z|/\alpha}-\frac{\sqrt{2H}}{2}e^{|z|/\alpha}. \tag{30}$$

From what has been discussed above, we have the following theorem:

Theorem 2 *Assume* $b\neq 0$, $u=g*m$, $m=u-\alpha^2u_{xx}$, *and* $H>0, K=0$ *, Eq.(1) has the solution* $u=\frac{\sqrt{2H}}{2}e^{-|z|/\alpha}-\frac{\sqrt{2H}}{2}e^{|z|/\alpha}$.

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