

## Generalized Projective Synchronization Using Nonlinear Control Method

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**Abstract:**Based on symbolic computation system *Maple* and Lyapunov stability theory, the generalized projective synchronization problem of drive-response systems is investigated. The generalized projective synchronization of two identical chaotic systems is achieved via the nonlinear control, and the synchronization of two different chaotic systems is also achieved via the corresponding nonlinear control. To illustrate our results, numerical simulations are used to perform the process of the synchronization. The orbits of drive systems and orbits of the response systems are put in the same plot for understanding intuitively.

**Keywords:**generalized projective synchronization; Lyapunov theory; chaotic systems

### 1 Introduction

Since Pecora and Carroll [1] discovered the synchronization of the chaotic system, the synchronization problem in chaotic systems has been intensively and extensively studied in recent decades. It has been extensively exploited in various areas, such as secure communications, life science, and so on. Up to now, there exist many types of chaos synchronization in dynamical systems such as complete synchronization, partial synchronization, phase synchronization, lag synchronization, anticipated synchronization, generalized synchronization, etc[2-8]. In particular, amongst all kinds of chaos synchronization, projective synchronization reported by Mainieri and Rehacek [9] is one of the most noticeable ones that the drive and response vectors evolve in a proportional scale—the vectors become proportional. The early projective synchronization is usually observable only in a class of systems with partial-linearity [10], recently some researchers [11-14] have achieved control of the projective synchronization in more general class of chaotic systems including non-partially-linear systems.

In the area of generalized synchronization, Yang and Chua [15] realized generalized synchronization based on the linear transform method. Recently, Meng and Wang [16] proposed a new generalized synchronization algorithm based on nonlinear control, which expanded the application range of generalized synchronization.

In this paper, the generalized projective synchronization of two identical chaotic systems and two different chaotic systems is considered respectively. Using the nonlinear control theory, a novel generalized projective synchronization algorithm is proposed. Two identical Rössler systems are investigated to achieve the generalized projective synchronization by the algorithm presented by us. For illustrate the effectiveness of the algorithm to two different chaotic system, the chua circle and the disk dynamo model are chosen to make them reach the generalized projective synchronization. Numerical simulations are used to perform the process of the synchronization and we successfully put the orbits of drive systems and orbits of the response systems in the same plot for understanding intuitively.

The rest of this paper is organized as follows: Generalized projective synchronization scheme via nonlinear control is given in section 2; In section 3, generalized projective synchronization of two identical

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Rössler systems is realized via nonlinear control; In section 4, generalized projective synchronization between the chua circle and the disk dynamo model is obtained; Finally, some summary and conclusions are given in section 5.

## 2 Generalized projective synchronization scheme via nonlinear control

Firstly, we quote some notations which are used throughout this paper: For a vector  $x$ ,  $\|x\| = (x^T x)^{1/2}$ , where  $x^T$  denotes the transpose of the vector  $x$ . For a matrix  $A$ , let  $\|A\|$  indicate the norm of  $A$  induced by the Euclidean vector norm, i.e.,  $\|A\| = (\lambda_{max}(A^T A))^{1/2}$ , where  $\lambda(\bullet)$  represents the maximum eigenvalue of matrix  $\bullet$ .

Consider the following drive chaotic system:

$$\dot{x} = f(x), \quad (2.1)$$

where  $x = (x_1, x_2, \dots, x_n)^T \in R^n$  is the state vector of drive system,  $f(\bullet)$  is a continuous vector function.

The controlled response system is given by the following equation:

$$\dot{y} = Ay + Bg(y) + u, \quad (2.2)$$

where  $y = (y_1, y_2, \dots, y_m)^T \in R^m$  is the state vector of response system,  $A$  and  $B$  are system matrices with proper dimensions,  $g(\bullet)$  is a continuous vector function, and  $u$  is the controller. Usually, function  $g(\bullet)$  is globally Lipschitz continuous.

**Definition 1** Define the synchronization errors of systems (2.1) and (2.2) as

$$e(t) = x - my, \quad (2.3)$$

if the errors satisfy the following property,

$$\lim_{t \rightarrow \infty} \|e\| = \lim_{t \rightarrow \infty} \|x - my\| = 0$$

where  $m$  is a nonzero constant, then we say that there exist be the generalized projective synchronization between systems (2.1) and (2.2), and call  $m$  a "scaling factor".

**Lemma 1 (Barbalat lemma [17])** If  $f(t)$  is uniformly continuous, and  $\lim_{t \rightarrow \infty} \int_0^t |f(\tau)| d\tau$  is bounded, then  $f(t) \rightarrow 0$  when  $t \rightarrow \infty$ .

**Theorem 2** For  $g(y)$  of the response chaotic system (2.2) satisfy the Lipschitz continuous condition, if the controller  $u$  is designed as

$$u = \frac{1}{m}[f(x) - Ax + \epsilon(x - my) - Bg(x)] + \frac{1}{m}B[g(my) - mg(y)], \quad (2.4)$$

where  $\epsilon = \text{diag}(\epsilon_1, \epsilon_2, \dots, \epsilon_m)$ , and satisfies

$$\min(\epsilon_i) > (L \|B\| + \|A\|); \quad (2.5)$$

then the generalized projective synchronization of systems (2.1) and (2.2) will be obtained.

**Proof.** The synchronization errors are defined as Eq.(2.3)

$$e(t) = x - my,$$

then with Eq.(2.4), the error dynamical system can be described as

$$\dot{e} = \dot{x} - m\dot{y} = Ae - \epsilon e + B[g(x) - g(my)]. \quad (2.6)$$

Let a Lyapunov error function

$$V(t) = \frac{1}{2} e^T e = \frac{1}{2} \| e \|^2. \quad (2.7)$$

it is easy to know that  $V(t)$  is a non-negative function.

Evaluating the time derivative of  $V(t)$  along the trajectory of Eq.(2.7) gives

$$\begin{aligned} \dot{V}(t) &= e^T A e - e^T \epsilon e + e^T B [g(x) - g(my)] \\ &\leq \| A \| \| e \|^2 - \min(\epsilon_i) \| e \|^2 + \| e \| \| B \| \| g(x) - g(my) \| \\ &\leq \| A \| \| e \|^2 - \min(\epsilon_i) \| e \|^2 + L \| B \| \| e \|^2 \\ &= (\| A \| + L \| B \| - \min(\epsilon_i)) \| e \|^2. \end{aligned} \quad (2.8)$$

It is obvious that  $\dot{V}(t) \leq 0$ . Therefore,  $V(t)$  is uniformly continuous. Let  $K = \min(\epsilon_i) - (L \| B \| + \| A \|)$ , then  $\dot{V}(t) \leq -K \| e \|^2$ , where  $K > 0$ . Thus, the following can be drawn:

$$V(t) \leq V(0)e^{-2Kt}. \quad (2.9)$$

From Eq.(2.9), we can know that  $\lim_{t \rightarrow \infty} \int_0^t V(t) dt$  is bounded. Moreover,  $V(t)$  is uniformly continuous. According to the Barbalat lemma, we can get  $\lim_{t \rightarrow \infty} V(t) = 0$ . Namely,  $\lim_{t \rightarrow \infty} \| e(t) \| = 0$ . Therefore, the error system (2.6) is asymptotically stable. Namely, the drive system (2.1) and the response system (2.2) can asymptotically achieve the generalized projective synchronization. This completes the proof of the theorem. ■

**Remark 3** In paper [16], the generalized synchronization via nonlinear control is investigated. Here we apply the nonlinear control method to the generalized projective synchronization. To realize the generalized projective synchronization of chaotic system, a nonlinear control algorithm scheme is proposed in this paper. For the character of generalized projective synchronization, we successfully simulate the orbits of both drive system and response system in the same plot to observe intuitively.

In the following, to demonstrate the validity of the proposed scheme, the scheme is used to achieve the synchronization of two identical and different chaotic systems respectively.

### 3 Generalized projective synchronization of two identical Rössler systems via nonlinear control

Consider the Rössler system

$$\begin{cases} \dot{x}_1(t) = -x_2(t) - x_3(t), \\ \dot{x}_2(t) = x_1(t) + \alpha x_2(t), \\ \dot{x}_3(t) = \beta + x_3(t)(x_1(t) - \gamma), \end{cases} \quad (3.1)$$

and the controlled response system is defined as following

$$\begin{cases} \dot{y}_1(t) = -y_2(t) - y_3(t) + u_1(t), \\ \dot{y}_2(t) = y_1(t) + \alpha y_2(t) + u_2(t), \\ \dot{y}_3(t) = \beta + y_3(t)(y_1(t) - \gamma) + u_3(t), \end{cases} \quad (3.2)$$

where  $u = (u_1, u_2, u_3)^T$  is the controller,  $\alpha = \beta = 0.2$ , and  $\gamma = 5.7$ .

Rewrite system (3.2) in the form of Eq. (2.2), where

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0.2 & 0 \\ 0 & 0 & -5.7 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, g(y) = \begin{pmatrix} 0 \\ 0 \\ y_1 y_3 + 0.2 \end{pmatrix}.$$

We can easily get that  $\| A \| = 5.7897$  and  $\| B \| = 1$ . Here we choose  $L = 1$  and select  $\epsilon$  as

$$\epsilon = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

It is easy to verify that Eq.(2.5) is satisfied. That is to say Eq.(2.9) is satisfied for the error function  $e(t)$ . Based on the Barbalat lemma, we can get that the two identical Rössler systems obtain generalized projective synchronization.

The controller  $u$  can be got from Eq.(2.4). Here we let  $m = 2$  and choose the initial values of the drive system and the response system as  $(x_1(0), x_2(0), x_3(0)) = (1.5, 2.0, 3.0)$  and  $(y_1(0), y_2(0), y_3(0)) = (0.1, 0.5, 0.2)$ , respectively. Fig.1 shows the numerical simulation of the error  $e$  of the two identical Rössler systems. Obviously,  $e_1, e_2$  and  $e_3$  converge to zero finally after the controller is activated. Fig.2 reveals the numerical global synchronization between them with different initial values as mentioned above. Fig.3 gives out the simulation orbit of the variables of the drive system, and the simulation orbits of the response system after synchronization. From the Fig.3, we can easily observe the ratio of the amplitudes of the two systems tends to a constant scaling factor "m".

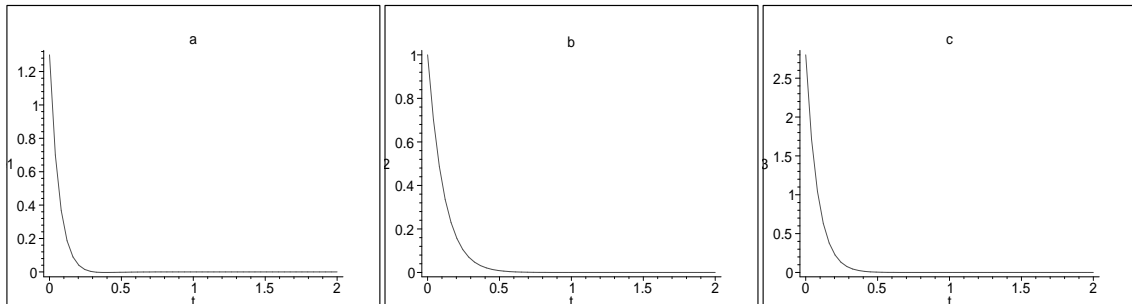


Fig. 1: (a) denotes the orbit of error function  $e_1$ ; (b) denotes the orbit of error function  $e_2$ ; (c) denotes the orbit of error function  $e_3$ .

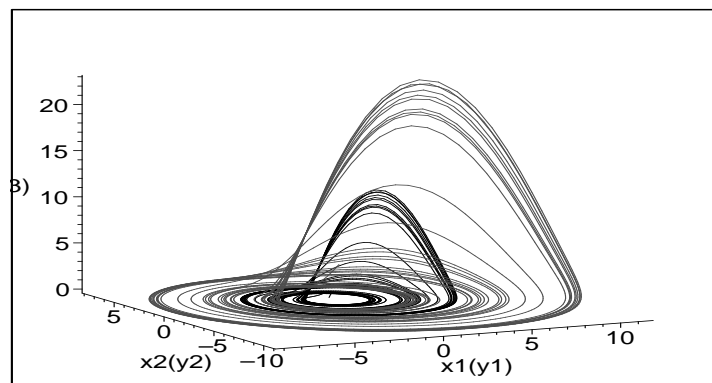


Fig. 2: the dark one denotes for the response system, and the other one denotes for the drive system.

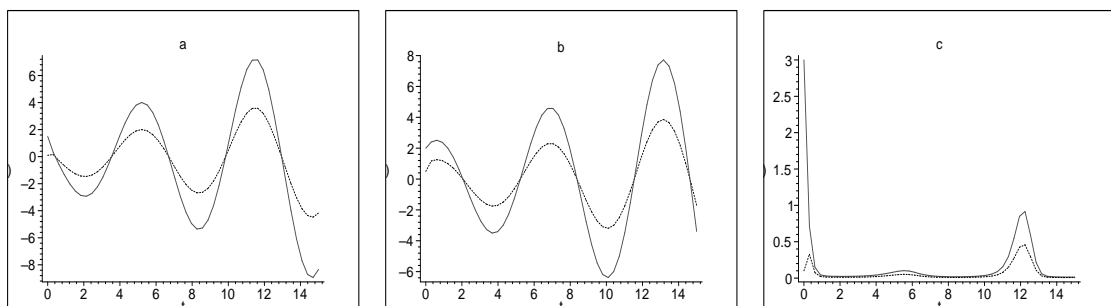


Fig. 3: (a) the orbits of  $x_1$  and  $y_1$ : the real line denotes  $x_1$  and the broken line denotes  $y_1$ ; (b) the orbits of  $x_2$  and  $y_2$ : the real line denotes  $x_2$  and the broken line denotes  $y_2$ ; (c) the orbits of  $x_3$  and  $y_3$ : the real line denotes  $x_3$  and the broken line denotes  $y_3$ .

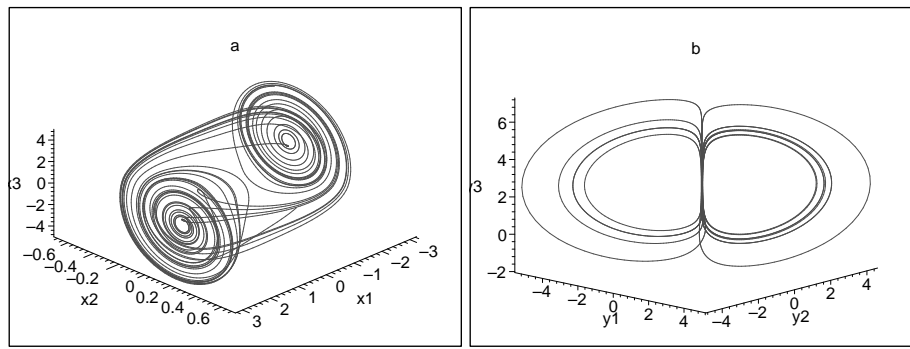


Fig. 4: (a) the attractor of the chua's circle; (b) the attractor of disk dynamo model without the controller.

#### 4 Generalized projective synchronization between the chua circle and the disk dynamo model

In the following, we use nonlinear control method to synchronize the chua's circle and the disk dynamo model to the fixed scaling factor  $m$ .

The drive system (chua's circle) is described as follows:

$$\begin{cases} \dot{x}_1 = \alpha(x_2 - x_1 - f(x_1)), \\ \dot{x}_2 = x_1 - x_2 + x_3, \\ \dot{x}_3 = -\beta x_2, \end{cases} \quad (4.1)$$

where

$$f(x_1) = \begin{cases} bx_1 + a - b, & \text{if } x_1 \geq 1, \\ bx_1 - a + b, & \text{if } x_1 \leq -1, \\ ax_1, & \text{otherwise,} \end{cases}$$

And the response system (disk dynamo model) is introduced as below

$$\begin{cases} \dot{y}_1 = -2y_1 + y_2y_3 + u_1, \\ \dot{y}_2 = -2y_2 + y_1(y_3 - 1) + u_2, \\ \dot{y}_3 = 1 - y_1y_2 + u_3, \end{cases} \quad (4.2)$$

The two above attractors are shown in Fig.4.

Rewrite system (4.2) in the form of Eq. (2.2), where

$$A = \begin{pmatrix} 0 & -2 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, g(y) = \begin{pmatrix} y_2y_3 \\ y_1y_3 \\ 1 - y_1y_2 \end{pmatrix}.$$

We can easily get that  $\| A \| = 2.236$  and  $\| B \| = 1$ . Here we choose  $L = 1$  and select  $\epsilon$  as

$$\epsilon = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

It is easy to verify that Eq.(2.5) is satisfied.

The controller  $u$  can be got from Eq.(2.4). Set  $\alpha = 10, \beta = 15.68, a = -1.2768$ , and  $b = -0.6888$ . We let  $m = 1/2$  and choose the initial values of the drive system and the response system as  $(x_1(0), x_2(0), x_3(0)) = (0.1, 0.5, 0.2)$  and  $(y_1(0), y_2(0), y_3(0)) = (0.2, 0.3, 0.5)$ , respectively. Fig.5 reveals the numerical global synchronization between the two different chaotic systems with different initial values as mentioned above. Fig.6 gives out the simulation orbit of the variables of the drive system, and the simulation orbits of the response system after synchronization. From the Fig.6, we can also easily observe the ratio of the amplitudes of the two systems tends to a constant scaling factor "  $m = 1/2$ ". At the end, Fig.7 shows the numerical simulation of the error  $e$  of the two different chaotic systems-the chua's circle and the disk dynamo model. Obviously,  $e_1, e_2$  and  $e_3$  converge to zero finally after the controller is activated.

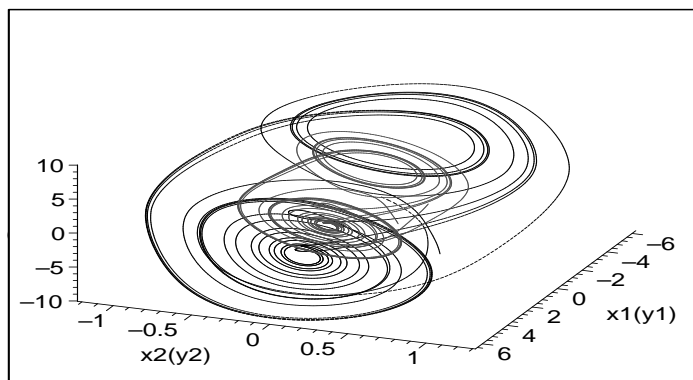


Fig. 5: the dark one denotes for the response system, and the other one denotes for the drive system.

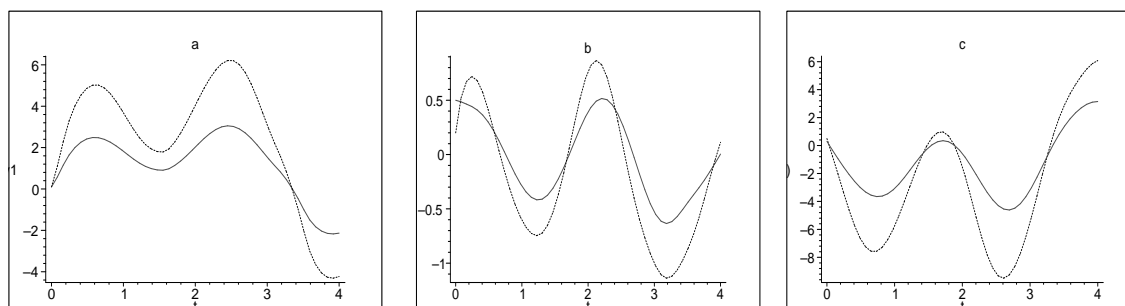


Fig. 6: (a) the orbits of  $x_1$  and  $y_1$ : the real line denotes  $x_1$  and the broken line denotes  $y_1$ ; (b) the orbits of  $x_2$  and  $y_2$ : the real line denotes  $x_2$  and the broken line denotes  $y_2$ ; (c) the orbits of  $x_3$  and  $y_3$ : the real line denotes  $x_3$  and the broken line denotes  $y_3$ .

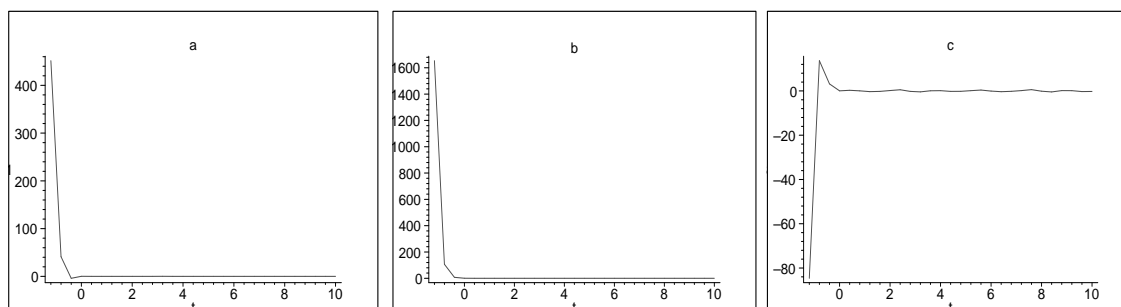


Fig. 7: (a) denotes the orbit of error function  $e_1$ ; (b) denotes the orbit of error function  $e_2$ ; (c) denotes the orbit of error function  $e_3$ .

## 5 Summary and conclusions

Based on symbolic computation system *Maple* and Lyapunov stability theory, we propose a scheme to realize the generalized projective synchronization via nonlinear control between two identical chaotic systems and two different chaotic systems, respectively. Numerical simulations are used to perform the process of the synchronization and successfully put the orbits of drive systems and orbits of the response systems in the same plot for understanding intuitively. With the aid of symbolic-numeric computation, the scheme can be used for other chaotic systems and hyperchaotic systems.

## References

- [1] L M Pecora, T L Carroll: Synchronization in Chaotic systems. *Phys. Rev. Lett.* 64:821-824(1990)

- [2] S Boccaletti, L M Pecora, A Pelaez: Unifying framework for synchronization of coupled dynamical systems. *Phys. Rev. E.* 63:066219(2001)
- [3] G Chen, X Dong: From Chaos to Order. World Scientific, Singapore. (1998)
- [4] X D Wang, L X Tian, L Q Yu: Linear feedback controlling and synchronization of the Chens chaotic system. *International Journal of Nonlinear Science.* 2:43-49(2006)
- [5] X S Yang: A framework for synchronization theory. *Chaos, Solitons and Fractals.* 11:1365-1368(2000)
- [6] Z Y Yan: A new scheme to generalized (lag, anticipated, and complete) synchronization in chaotic and hyperchaotic systems. *Chaos.* 15:013101-013110(2005)
- [7] D C Lu, A C Wang, X D Tian: Control and synchronization of a new hyperchaotic system with unknown parameters. *International Journal of Nonlinear Science.* 6:224-229(2008)
- [8] L X Tian, G g Dong: Predictive control of sudden occurrence of chaos. *International Journal of Nonlinear Science.* 5:99-105(2008)
- [9] R Mainieri, J Rehacek: Projective synchronization in three-dimensional chaotic systems. *Phys. Rev. Lett.* 82:3042-3045(1999)
- [10] D Xu, Z Li: Controlled projective synchronization in nonpartially-linear chaotic systems. *Int J Bifurcat Chaos.* 12:1395-1402(2002)
- [11] H Chen, M Sun: generalized projective synchronization of the energy resource system. *International Journal of Nonlinear Science.* 2:166-170(2006)
- [12] G H Li: Modified projective synchronization of chaotic systems. *Chaos, Solitons and Fractals.* 32:1786-1790(2007)
- [13] X Li, Y Chen, Z B Li: Function projective synchronization of discrete-time chaotic systems. *Z. Naturforsch.* 63a:7-14(2008)
- [14] J Yan, C Li: Generalized projective synchronization of a unified chaotic system. *Chaos, Solitons and Fractals.* 26:1119-1124(2005)
- [15] T Yang, L O Chua: Generalized synchronization of chaos via linear transformations. *Int. J. Bifurcation Chaos Appl. Sci. Eng.* 9:215-219(1999)
- [16] J Meng, X Y Wang: Generalized synchronization via nonlinear control. *Chaos.* 18:023108(2008)
- [17] K Gopalsamy: Stability and Oscillations in Delay Differnetial Equations of Population Dynamics. Kluwer Academic, Dordrecht, (1992)