

Synchronization of the Non-delayed and Delayed Coupling Complex Network Via Adaptive Controlling

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Abstract: This paper studies the synchronization of a general complex dynamical network with non-delayed and delayed coupling. What is more, the coupling matrix is not necessary symmetric or irreducible. Adaptive controllers have been designed to make the dynamical network synchronize. Analytical results show that the states of the dynamical network with non-delayed and delayed coupling can globally asymptotically synchronize onto a desired orbit under the designed controllers. A numerical example is given to demonstrate the validity of the proposed method, in which the famous Lü system is chosen as the node of the network.

Keywords: Synchronization; Adaptive control; complex dynamical networks; non-delayed and delayed coupling

1 Introduction

One of the interesting and significant phenomena in complex dynamic networks is the synchronization of all dynamic nodes. Network synchronization has many applications in various fields such as the synchronous information exchange in the internet and the synchronous transfer of digital or analog signals in the communication networks [1-3]. Synchronization problem of complex networks has been extensively investigated during the past two decades [4-19]. In these researches above, most of the authors use the method of adding controllers to all the nodes to make the complex networks get synchronization. This method is valid for almost all complex networks including random, regular, small-world, and scale-free structure. Especial, Li *et al.* introduced a linear state feedback controller to synchronize a complex network to a desired orbit [15]; Li and Chen showed several robust adaptive controllers for complex dynamical network with unknown but bounded nonlinear couplings [17]; Zhou *et al.* investigated adaptive synchronization of uncertain complex dynamical networks [20].

It is noticed that most of the studies on synchronization of dynamical network have been performed under some implicit assumptions that there exists the information communication of nodes via the edges only at time t [4,8-10,16] or at time $t - \tau$ [6,7,10,12-14]. However, in many circumstances, this simplification does not match satisfactorily the peculiarities of real networks: there exists the information communication of nodes not only at time t but also at time $t - \tau$. In effect, this phenomenon exists commonly in our real world, for example, in the stock market, decision-making of single trader is influenced by that of others at time t as well as at time $t - \tau$. Whereas the synchronization of both delay-coupled and non-delay coupled complex dynamical network have almost been ignored in the literatures.

Motivated by the above discussions, in this letter, we will investigate the synchronization of the networks with both delayed and non-delayed coupling. A simple controller is designed to make the nodes of network synchronized. Our network model contains the previous network model with only non-delayed coupling or only delayed coupling as a special case. In particular, the coupling matrix is not necessary symmetric or

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irreducible. In this paper, adaptive method is employed to realize the synchronization of complex network and efficient controller is designed which ensures the states of each node to reach the desired manifold. Numerical examples are also provided to demonstrate the effectiveness of the theory.

The rest of the Letter is organized as following. The network model is introduced and some necessary lemmas, hypotheses are given in Sec. 2. The synchronization of the complex networks with non-delayed and delayed coupling is discussed in Sec. 3. Examples and their simulations are obtained in Sec. 4. Finally, the Letter is concluded in Sec. 5.

2 Model and preliminaries

Now we consider the dynamical complex networks consisting of N identical nodes that are n -dimensional dynamical units. The model with non-delayed and delayed coupling can be described as

$$\dot{x}_i(t) = Gx_i(t) + h(x_i(t), t) + c \sum_{j=1}^N a_{ij} \Gamma x_j(t) + c \sum_{j=1}^N b_{ij} \Gamma x_j(t - \tau), \quad i = 1, 2, \dots, N, \quad (2.1)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \dots, x_{in}(t))^T \in R^n$ is the state vector of node i , Gx_i is the linear part of the node dynamical with $G \in R^{n \times n}$ and $h : \Omega \times R^+ \rightarrow R^n$ is a continuously differential nonlinear function, c is the coupling strength, $\tau \geq 0$ is the coupled delay, $A = (a_{ij})$, $B = (b_{ij}) \in R^{N \times N}$ are the coupling matrices with zero-sum rows, which represent the coupling strength and the underlying topology for non-delayed configuration and delayed one at time t respectively, $a_{ij} \geq 0$, $b_{ij} \geq 0$ for $i \neq j$, a_{ij} , b_{ij} are defined as follows: if there is a connection from node j to node i ($i \neq j$), $a_{ij} > 0$, $b_{ij} > 0$, otherwise $a_{ij} = 0$, $b_{ij} = 0$ ($i \neq j$) for $i, j = 1, 2, \dots, N$, and $\Gamma \in R^{n \times n}$ is a positive definite diagonal matrix which describes the individual couplings between node i and node j .

Since G is a given constant matrix, then there exists a nonnegative constant β satisfying $\|G\|_2 \leq \beta$.

Let $C([-\tau, 0], R^n)$ be the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^n with the norm $\|\phi\| = \sup_{-\tau \leq \theta \leq 0} \|\phi(\theta)\|$. For the functional differential equation (2.1), its initial conditions are given by $x_i(t) = \varphi_i(t) \in C([-\tau, 0], R^n)$. We always assume that (2.1) has a unique solution with respect to initial conditions.

Our goal is to design the adaptive controller $u_i(t)$ to synchronize the network (2.1) onto a given evolution $s(t)$, i.e.,

$$\lim_{t \rightarrow \infty} \|x_i(t) - s(t)\| = 0, \quad i = 1, 2, \dots, N,$$

where $s(t)$ is a synchronous solution of the network (2.1).

In the letter, we have the following hypothesis:

Hypothesis 1 (H1): Suppose that $h(x_i(t), t)$ is Lipschitz continuous. That is, there exists a Lipschitz constant μ satisfying $\|h(x_i(t), t) - h(s, t)\|_2 \leq \mu \|e_i\|_2$ for $1 \leq i \leq N$.

3 Adaptive global synchronization with delayed and non-delayed coupling

In order to achieve the global synchronization, we introduce adaptive control strategy to nodes in the network (2.1). Then the controlled network is given by

$$\dot{x}_i(t) = Gx_i(t) + h(x_i(t), t) + c \sum_{j=1}^N a_{ij} \Gamma x_j(t) + c \sum_{j=1}^N b_{ij} \Gamma x_j(t - \tau) + u_i(t), \quad i = 1, 2, \dots, N, \quad (3.1)$$

where

$$u_i(t) = -p_i(t) (x_i(t) - s(t)) \quad (i = 1, 2, \dots, N), \quad (3.2)$$

are the adaptive controllers, $p_i(t)$ are time-delaying gains. To guarantee negative feedback, the adaptive gains are designed as:

$$\dot{p}_i(t) = q_i (x_i(t) - s(t))^T (x_i(t) - s(t)), \quad i = 1, 2, \dots, N, \quad (3.3)$$

where q_i is a positive constant to be determined.

Define

$$e_i(t) = x_i(t) - s(t), e_i(t - \tau) = x_i(t - \tau) - s(t - \tau).$$

Then we have the following dynamical error equations:

$$\dot{e}_i(t) = Ge_i(t) + h(x_i(t), t) - h(s, t) + c \sum_{j=1}^N a_{ij}\Gamma e_j(t) + c \sum_{j=1}^N b_{ij}\Gamma e_j(t - \tau) + u_i(t) \quad (3.4)$$

It is obvious that the stability of the synchronous solution $x_i(t)$ of the network (3.1) is equivalent to the stability of the errors $e_i(t)$ for all $i = 1, 2, \dots, N$ about its zero solution. And we have the following result.

Theorem 1 *The complex dynamical network (3.1) globally asymptotically synchronizes with the desired evolution $s(t)$ under the adaptive controllers (3.2) and the adaptive gains (3.3), where $p_i(0)$ and q_i are arbitrary positive constants.*

Proof. Selecting a Lyapunov-Krasovskii function as

$$V(t) = \frac{1}{2} \sum_{i=1}^N e_i^T(t) e_i(t) + \frac{1}{2} \sum_{i=1}^N \frac{(p_i - p)^2}{q_i} + \sum_{j=1}^N \int_t^{t-\tau} (e_j(s))^T Q_j (e_j(s)) ds$$

where p is a positive constant to be determined.

Differentiating the function $V(t)$ by using the inequality $2x^T y \leq x^T Q x + y^T Q^{-1} y$ ([7]), we have

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^N e_i^T(t) \left[\frac{G + G^T}{2} e_i(t) + h(x_i(t), t) - h(s, t) + c \sum_{j=1}^N a_{ij}\Gamma e_j(t) + c \sum_{j=1}^N b_{ij}\Gamma e_j(t - \tau) \right] \\ &\quad - \sum_{i=1}^N p_i e_i^T(t) e_i(t) + \sum_{i=1}^N (p_i - p) \|e_i(t)\|^2 + \sum_{j=1}^N \left[(e_j(t))^T Q_j e_j(t) - (e_j(t - \tau))^T Q_j (e_j(t - \tau)) \right] \\ &\leq \sum_{i=1}^N (\beta + \mu - p) e_i^T(t) e_i(t) + c \sum_{i=1}^N \sum_{j=1}^N e_i^T(t) a_{ij}\Gamma e_j(t) + c \sum_{i=1}^N \sum_{j=1}^N e_i^T(t) b_{ij}\Gamma e_j(t - \tau) \\ &\quad + \sum_{j=1}^N (e_j(t))^T Q_j e_j(t) - \sum_{j=1}^N (e_j(t - \tau))^T Q_j e_j(t - \tau) \\ &\leq \sum_{i=1}^N (\beta + \mu - p) e_j^T(t) e_j(t) + c \sum_{j=1}^N e_j^T(t) \gamma_j A^s e_j(t) + c \sum_{j=1}^N e_j^T(t) \gamma_j B e_j(t - \tau) \\ &\quad + \sum_{j=1}^N (e_j(t))^T Q_j e_j(t) - \sum_{j=1}^N (e_j(t - \tau))^T Q_j e_j(t - \tau) \\ &= \sum_{j=1}^N (e_j^T(t), e_j^T(t - \tau)) \begin{bmatrix} (\beta + \mu - p) I_N + Q_j + c\gamma_j A^s & \frac{1}{2} c\gamma_j B \\ \frac{1}{2} c\gamma_j B^T & -Q_j \end{bmatrix} (e_j(t), e_j(t - \tau))^T, \end{aligned}$$

where $\Gamma = \text{diag} \{ \gamma_1, \gamma_2, \dots, \gamma_N \}$, $x_j(t) = (x_{j1}(t), x_{j2}(t), \dots, x_{jn}(t))^T$.

From the Schur complements ([21]),

$$\begin{bmatrix} (\beta + \mu - p) I_N + Q_j + c\gamma_j A^s & \frac{1}{2} c\gamma_j B \\ \frac{1}{2} c\gamma_j B^T & -Q_j \end{bmatrix} < 0$$

is equivalent to

$$(\beta + \mu - p) I_N + Q_j + c\gamma_j A^s + \frac{(c\gamma_j)^2}{4} B^T Q_j^{-1} B < 0.$$

Taking suitable positive parameters p, c, γ_j, Q_j such that

$$(\beta + \mu - p) I_N + Q_j + c\gamma_j A^S + \frac{(c\gamma_j)^2}{4} B^T Q_j^{-1} B < 0.$$

So we have $\dot{V}(t) < 0$. From Lemma 1, we know the systems (3.1) is globally asymptotically synchronizes with the desired evolution $s(t)$ under the adaptive controllers (3.2) and the adaptive gains (3.3). This completes the proof of the theorem. ■

4 Numerical results

In this section, a simple example is to explain the effectiveness of the proposed network synchronization criteria. We assume that the controlled network (3.1) consists of three identical Lü systems. Here, a single Lü circuit, as the desired orbit, is described by

$$\dot{s}_1 = r_1(s_2 - s_1), \quad \dot{s}_2 = r_3 s_2 - s_1 s_3, \quad \dot{s}_3 = -r_2 s_3 + s_1 s_2,$$

where $r_1 = 36, \quad r_2 = 3, \quad r_3 = 20$.

The node dynamics is then given by

$$\begin{aligned} \dot{x}_i(t) &= \begin{pmatrix} -r_1 & 0 & 0 \\ 0 & r_3 & 0 \\ 0 & 0 & r_2 \end{pmatrix} \begin{pmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{pmatrix} + \begin{pmatrix} 0 \\ -x_{i1}x_{i3} \\ x_{i1}x_{i2} \end{pmatrix} + c \sum_{j=1}^3 a_{ij} \Gamma x_j(t) + c \sum_{j=1}^3 b_{ij} \Gamma x_j(t - \tau) - p_i(t) e_i(t) \\ &= Gx_i(t) + h(x_i(t), t) + c \sum_{j=1}^3 a_{ij} \Gamma x_j(t) + c \sum_{j=1}^3 b_{ij} \Gamma x_j(t - \tau) - p_i(t) e_i(t) \end{aligned}$$

and

$$\dot{p}_i(t) = q_i \|e_i(t)\|^2, \quad i = 1, 2, 3.$$

Obviously, $\beta = \|G\|_2 \approx 52.9843$, and

$$h(x_i(t), t) - h(s, t) = \begin{pmatrix} 0 \\ -x_{i1}x_{i3} + s_1 s_3 \\ x_{i1}x_{i2} - s_1 s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -x_{i3}e_{i1} - s_1 e_{i3} \\ x_{i2}e_{i1} + s_1 e_{i2} \end{pmatrix}, \quad i = 1, 2, 3.$$

It is well known that Lü attractor is bounded. Here we suppose that all nodes are running in the given bounded region. Our theoretical and numerical analyses show that there exist constants $M_1 = 25, \quad M_2 = 30$ and $M_3 = 45$ satisfying $\|x_{ij}\|, \|s_j\| \leq M_j$ for $1 \leq i \leq 3$ and $1 \leq j \leq 3$ ([22, 23]). Thus one gets

$$\begin{aligned} \|h(x_i(t), t) - h(s, t)\|_2 &\leq \sqrt{(-x_{i3}e_{i1} - s_1 e_{i3})^2 + (x_{i2}e_{i1} + s_1 e_{i2})^2} \\ &\leq \sqrt{2M_1^2 + M_2^2 + M_3^2} \|e_i(t)\|^2 \\ &\approx 64.6142 \|e_i(t)\|^2 \end{aligned}$$

Let $\beta = 64.6142$. The coupling matrix is

$$A = \begin{bmatrix} -1 & 1/2 & 1/2 \\ 1/3 & -1 & 2/3 \\ 2/3 & 1/3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1/4 & 3/4 \\ 1/2 & -1 & 1/2 \\ 1/3 & 2/3 & -1 \end{bmatrix}$$

We take $\Gamma = \text{diag}\{100, 100, 100\}, c = 0.2, Q_j = \text{diag}\{0.1, 0.2, 0.3\}, p_i(0) = 1, q_i = 100$ and sufficient large p such that

$$(\beta + \mu - p) I_N + Q_j + c\gamma_j A^S + \frac{(c\gamma_j)^2}{4} B^T Q_j^{-1} B < 0,$$

Then we can clearly see that the coupled network with delayed and non-delayed coupling can globally synchronize to the desired orbit, and the simulation results are plotted in Fig. 1.

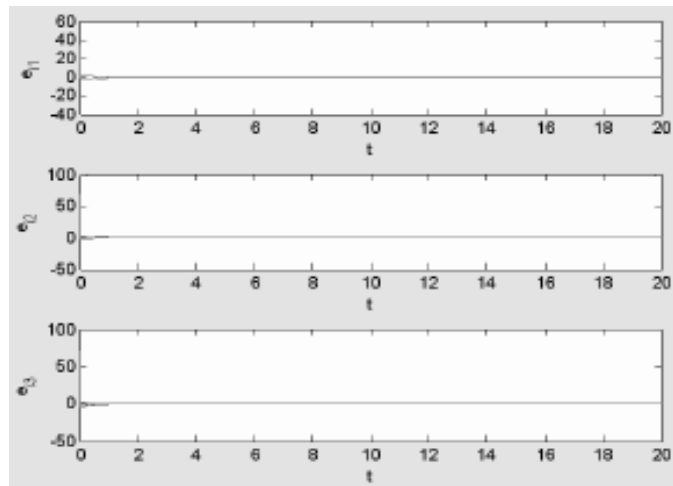


Figure 1: Synchronization errors e_{i1}, e_{i2}, e_{i3} of network (3.1) with $\tau = 0.5 (i = 1, 2, 3)$

Remark 1 It can be easily verified that the dynamical network will globally asymptotically converge to the desired evolution $s(t)$ under the adaptive controllers (3.2) and the adaptive gains (3.3) when $B = 0$ or $A = 0$. That is a special case of our network model (2.1).

5 Conclusion

The synchronization of a general dynamical network with non-delayed and delayed coupling is studied in this paper. An adaptive controller has been designed to ensure the global asymptotical synchronization with a desired orbit. By construction of Lyapunov-Krasovskii function, we obtain the analytical results that global synchronization of the controlled networks can be achieved under the designed adaptive controllers. The Lü system is chosen as the node of dynamical network in the numerical simulation, and the example shows the effectiveness of our method.

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