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Subdirect Sums of H-matrices

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Abstract: This paper concerns the problem of the k-subdirect sum of two H-matrices. Some sufficient conditions are given such that the k-subdirect sum of two H-matrices is an H-matrix. Our main results generalize the one provided by Bru, Pedroche and Szyld [Subdirect sums of M-matrices and of their inverse, *Electron. J. Linear Algebra*, 13,162-174(2005)]. Furthermore, the subdirect sum of block strictly diagonally dominant (BSDD) matrices is also researched. Examples are given to illustrate the theoretical results presented.

Keywords: k-subdirect sum; H-matrix; block strictly diagonally dominant matrix; overlapping block

1 Introduction

Subdirect sums of matrices are generalizations of the usual sum of matrices. They were introduced by Fallat and Johnson [1], where many of their properties were analyzed. Foe example, they studied that the subdirect sum of positive definite matrices, or of symmetric M-matrices, is a positive definite matrix or symmetric M-matrix, respectively. Following, Bru, Pedroche and Szyld mainly studied sufficient conditions for the subdirect sum of M-matrices to be an M-matrix. Afterwards, in [5] sufficient conditions are given so that the subdirect sum of S-strictly diagonally dominant (S-SDD) matrices is an S-SDD matrix. The case of the subdirect sum of strictly diagonally dominant (SDD) matrices is also treated.

In fact, S-SDD matrices and M-matrices are contained into the class of H-matrices. When differential equations [11–14] are discreted, linear system $Ax = b$ can arise from many practical problems, where A is often an M-matrix or an H-matrix. Properties of H-matrices were widely studied by many authors, e.g., see [2], [3] and [7]. We recall the definition and some results of H-matrices.

Given a matrix $A = (a_{ij}) \in R^{n\times n}$, the *comparison matrix* of A, $\mu(A) = [\alpha_{ij}]$, is defined by

$$\alpha_{ij} = \begin{cases} |a_{ij}|, \; i = j, \\ -|a_{ij}|, \; i \neq j, \end{cases}$$

if $\mu(A)$ is an M-matrix, then A is called an H-matrix.

If there exists a positive vector $v = (v_1, v_2, \ldots, v_n)$ such that

$$|a_{ii}|v_i > \sum_{i\neq j} |a_{ij}|v_j, \; i, j = 1, 2, \ldots, n,$$

then A is called to be an H-matrix. The two definitions of H-matrix are equality. Every principal submatrix of an H-matrix A is an H-matrix.

All through the paper, we denote $e = [1, 1, \ldots, 1]^T$ and $|A| = (|a_{ij}|)$.

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In this paper, we will study the subdirect sum of H-matrices and present sufficient conditions to justify that the k-subdirect sum of H-matrices belongs to the same class. Our main results extend the ones provided by Bru, Pedroche and Szyld [4]. Additionally, we also consider the case of the subdirect sum of block strictly diagonally dominant matrices. Examples are given to illustrate the theoretical results presented.

2 Subdirect sums of H-matrices

We begin to introduce the definition of subdirect sum of matrices. Suppose A and B are partitioned into 2×2 blocks as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in R^{n\times n}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \in R^{m\times m}, \tag{1}$$

where A_{22} and B_{11} are square matrices of order k, k is an integer such that $1 \le k \le \min(n,m)$. Following [1], we call the following square matrix of order $N = n + m - k$,

$$H = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22}+B_{11} & B_{12} \\ 0 & B_{21} & B_{22} \end{bmatrix} \tag{2}$$

the *k-subdirect sum* of A and B and we denote it by $H = A \oplus_k B$.

As is well known that subdirect sum of two M-matrices, $A \oplus_k B$, may not be an M-matrix when k=1. Correspondingly, it does not necessarily hold even for the more simpler H-matrices. The following example can illustrate it.

Example 2.1 *Given two H-matrices*

$$A = \begin{bmatrix} 4 & 1 & \vdots & -1 \\ -1 & 3 & \vdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \vdots & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0.4 & \vdots & 1 & 0.7 \\ 0.5 & \vdots & 0.3 & 1 \end{bmatrix}.$$

However,

$$H = A \oplus_1 B = \begin{bmatrix} 4 & 1 & -1 & 0 & 0 \\ -1 & 3 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 1 & 0.7 \\ 0 & 0 & 0.5 & 0.3 & 1 \end{bmatrix}$$

is not an H-matrix.

We begin with discussing the first result of the subdirect sum of H-matrices when $k = 1$.

Theorem 2.1 *Let the block matrix*

$$H = \begin{bmatrix} H_{11} & H_{12} & 0 \\ h_{21} & h_{22} & h_{23} \\ 0 & H_{32} & H_{33} \end{bmatrix},$$

where H_{11} and H_{33} are square matrices of order $n-1$ and $m-1$, respectively, and with h_{22} a number. Then H is an H-matrix if and only if h_{22} may be written as $h_{22} = a_{22} + b_{11}$ so that

$$A = \begin{bmatrix} H_{11} & H_{12} \\ h_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & h_{23} \\ H_{32} & H_{33} \end{bmatrix}$$

are both H-matrices.

Proof. First suppose H is an H-matrix, then there exists a positive diagonal matrix D =diag(D_1, d, D_2) where $D_1 \in R^{(n-1)\times(n-1)}$ and $D_2 \in R^{(m-1)\times(m-1)}$ are diagonal matrices, such that HD is an SDD matrix, that is,

$$HD = \begin{bmatrix} H_{11}D_1 & H_{12}d & 0 \\ h_{21}D_1 & h_{22}d & h_{23}D_2 \\ 0 & H_{32}d & H_{33}D_2 \end{bmatrix},$$

since $|h_{22}|d = |a_{22} + b_{11}|d > |h_{21}|D_1 e + |h_{23}|D_2 e$, then there exists $\varepsilon > 0$, such that

$$|a_{22} + b_{11}|d = |h_{21}|D_1 e + |h_{23}|D_2 e + \varepsilon.$$

Let

$$|a_{22}| = \frac{1}{d}(|h_{21}|D_1 e + \frac{\varepsilon}{2}), \quad |b_{11}| = \frac{1}{d}(|h_{23}|D_2 e + \frac{\varepsilon}{2}).$$

Then

$$A' = \begin{bmatrix} H_{11}D_1 & H_{12}d \\ h_{21}D_1 & a_{22}d \end{bmatrix}, \ B' = \begin{bmatrix} b_{11}d & h_{23}D_2 \\ H_{32}d & H_{33}D_2 \end{bmatrix}$$

are two SDD matrices. Furthermore,

$$A' = \begin{bmatrix} H_{11} & H_{12} \\ h_{21} & a_{22} \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & d \end{bmatrix} = AD', \ B' = \begin{bmatrix} b_{11} & h_{23} \\ H_{32} & H_{33} \end{bmatrix} \begin{bmatrix} d & 0 \\ 0 & D_2 \end{bmatrix} = BD'',$$

where

$$D' = \begin{bmatrix} D_1 & 0 \\ 0 & d \end{bmatrix}, \ D'' = \begin{bmatrix} d & 0 \\ 0 & D_2 \end{bmatrix}.$$

Thus we obtain that A and B are two H-matrices. Conversely, suppose

$$A = \begin{bmatrix} H_{11} & H_{12} \\ h_{21} & a_{22} \end{bmatrix}, \ B = \begin{bmatrix} b_{11} & h_{23} \\ H_{32} & H_{33} \end{bmatrix}$$

are two H-matrices. By definition, $H = A \oplus_1 B$ is given by

$$H = \begin{bmatrix} H_{11} & H_{12} & 0 \\ h_{21} & a_{22} + b_{11} & h_{23} \\ 0 & H_{32} & H_{33} \end{bmatrix},$$

with $h_{22} = a_{22} + b_{11}$, the comparison matrix of H

$$\mu(H) = \begin{bmatrix} \mu(H_{11}) & -|H_{12}| & 0 \\ -|h_{21}| & |a_{22} + b_{11}| & -|h_{23}| \\ 0 & -|H_{32}| & \mu(H_{33}) \end{bmatrix},$$

where $\mu(H_{11})$ and $\mu(H_{33})$ denote the comparison matrices of H_{11} and H_{33}, in view of the result of [1], $\mu(H)$ is an M-matrix, so H is an H-matrix. ■

Nevertheless, subdirect sum of two H-matrices may not be an H-matrix when $k \geq 2$, the following example illustrates it.

Example 2.2 *Given two H-matrices*

$$A = \begin{bmatrix} 2 & \vdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & \vdots & 5 & 0 \\ 1 & \vdots & 9 & 5 \end{bmatrix}, B = \begin{bmatrix} 5 & -9 & \vdots & 1 \\ 0 & 5 & \vdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \vdots & 2 \end{bmatrix}.$$

The matrix $H = A \oplus_2 B$ *is*

$$H = A \oplus_2 B = \begin{bmatrix} 2 & 1 & 1 & 0 \\ -1 & 10 & -9 & 1 \\ 1 & 9 & 10 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

In fact

$$\mu(H)^{-1} \approx \begin{bmatrix} 0.25 & -0.06 & -0.06 & -0.06 \\ -0.23 & -0.03 & -0.08 & -0.06 \\ -0.26 & -0.08 & -0.03 & -0.06 \\ -0.26 & -0.06 & -0.06 & 0.44 \end{bmatrix}$$

is not an M*-matrix and therefore* H *is not an* H*-matrix.*

Now we advance a sufficient condition to analyze the k-subdirect sums of two H-matrices when $k \geq 2$.

Theorem 2.2 *Let* $A \in R^{n\times n}$, $B \in R^{m\times m}$ *be two* H*-matrices, which are partitioned as in (1),* k *be an integer such that* $2 \leq k \leq \min(n, m)$. *There exist* $v_1 > 0 \in R^{(n-k)\times 1}$, $v_2 > 0 \in R^{k\times 1}$, $v_1' > 0 \in R^{k\times 1}$ *and* $v_2' > 0 \in R^{(m-k)\times 1}$ *such that*

$$\begin{cases} \mu(A_{11})v_1 - |A_{12}|v_2 > 0, \\ \mu(A_{22})v_2 - |A_{21}|v_1 > 0, \end{cases} \begin{cases} \mu(B_{11})v_1' - |B_{12}|v_2' > 0, \\ \mu(B_{22})v_2' - |B_{21}|v_1' > 0. \end{cases} \tag{3}$$

Let $C = A_{22} + B_{11}$ *be an* H*-matrix and*

$$v_\varepsilon = \mu(C)^{-1}(|A_{21}|v_1 + |B_{12}|v_2' + \varepsilon), \forall\, \varepsilon > 0 \in R^{k\times 1}. \tag{4}$$

Then, if $v_\varepsilon \leq v_2$, $v_\varepsilon \leq v_1'$, *the* k*-subdirect sum* $H = A \oplus_k B$ *in (2) is an* H*-matrix.*

Proof. We will show that there exists a positive vector $v = (v_1, v_2, \ldots, v_N)$ such that

$$|h_{ii}|v_i > \sum_{i\neq j} |h_{ij}|v_j,\ i = 1, 2, \ldots, N.$$

Note that from (3) we can get

$$\begin{cases} \mu(A_{11})v_1 - |A_{12}|v_2 > 0, \\ \mu(B_{22})v_2' - |B_{21}|v_1' > 0. \end{cases}$$

Obviously, comparison matrix of H

$$\mu(A \oplus_k B) = \begin{bmatrix} \mu(A_{11}) & -|A_{12}| & 0 \\ -|A_{21}| & \mu(C) & -|B_{12}| \\ 0 & -|B_{21}| & \mu(B_{22}) \end{bmatrix}.$$

Taking the positive vector $V = [v_1, v_\varepsilon, v_2']^T$ it is a routine to show that $\mu(A \oplus_k B)V > 0_{N\times 1}$ and therefore $\mu(A \oplus_k B)$ is an M-matrix and hence $A \oplus_k B$ is an H-matrix. ■

Note that A, B are H-matrices and therefore the positive vectors (v_1, v_2) and (v_1', v_2') of (3) always exist.

Remark 2.1 *When* $k = m = n$*(i.e.,we have a usual sum of matrices* A *and* B*) this theorem is useless.*

Corollary 2.1 *Let* A *and* B *be two* H*-matrices partitioned as in (1) and* $A_{22} + B_{11}$ *be an* H*-matrix. If* A_{12} *and* B_{21} *are zero matrices, then for all* $\varepsilon > 0$ $A \oplus_k B$ *is an* H*-matrix.*

Corollary 2.2 *Let* A *and* B *be two* H*-matrices partitioned as in (1) where* $A_{21} = 0$. *There exists positive vector* $V = [v_1', v_2']^T$ *where* $v_1' > 0 \in R^{k\times 1}$ *and* $v_2' > 0 \in R^{(m-k)\times 1}$ *such that* $\mu(B)V > 0$. *Let* $A_{22} + B_{11}$ *be an* H*-matrix and* $v_\varepsilon = \mu(C)^{-1}(|B_{12}|v_2' + \varepsilon)$, $\forall \varepsilon > 0 \in R^{k\times 1}$. *If* $v_\varepsilon \leq v_1'$, *then the* k*-subdirect sum* $H = A \oplus_k B$ *is an* H*-matrix.*

The following example [4, Example 2.5] can illustrate that Theorem 2.2 is more general than Theorem 2.3 of [4].

Example 2.3 *([4, Example 2.5]) The two H-matrices*

$$A = \begin{bmatrix} 5 & \vdots & -1/2 & -1/3 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & \vdots & 4 & -2 \\ -1 & \vdots & -6 & 10 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 & \vdots & -1/3 \\ -3 & 9 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -2 & -1/2 & \vdots & 6 \end{bmatrix}$$

and positive vectors

$$V_1 = \begin{bmatrix} 1 \\ \cdots \\ 1 \\ 1 \end{bmatrix}, V_2 = \begin{bmatrix} 2.5 \\ 1 \\ \cdots \\ 1 \end{bmatrix}$$

satisfy $\mu(A)V_1 > 0$, $\mu(B)V_2 > 0$. Computing vector v_ε from (4) we obtain $v_\varepsilon \approx \begin{bmatrix} 0.887 \\ 0.525 \end{bmatrix}$, with $\varepsilon = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then v_ε satisfies the conditions of Theorem 2.2. Therefore the 2-subdirect sum

$$H = A \oplus_2 B = \begin{bmatrix} 5 & -1/2 & -1/3 & 0 \\ -1 & 5 & -4 & -1/3 \\ -1 & -9 & 19 & 0 \\ 0 & -2 & -1/2 & 6 \end{bmatrix}$$

is an M-matrix, i.e., is an H-matrix in accordance with Theorem 2.2. However, Theorem 2.3 of [4] can not be sure whether the example is an M-matrix or not.

Example 2.4 *The following nonnegative H-matrices*

$$A = \begin{bmatrix} 3 & \vdots & 2 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1/2 & \vdots & 2 & 3 \\ 1 & \vdots & 1 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & \vdots & 1/3 \\ 3 & 9 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 2 & 1/2 & \vdots & 6 \end{bmatrix}$$

and positive vectors

$$V_1 = \begin{bmatrix} 1.8 \\ \cdots \\ 2 \\ 1 \end{bmatrix}, V_2 = \begin{bmatrix} 2.5 \\ 1 \\ \cdots \\ 1 \end{bmatrix}$$

satisfy $\mu(A)V_1 > 0$, $\mu(B)V_2 > 0$. Computing vector v_ε from (4) we obtain $v_\varepsilon \approx \begin{bmatrix} 1.349 \\ 0.556 \end{bmatrix}$, with $\varepsilon = \begin{bmatrix} 1/30 \\ 1/30 \end{bmatrix}$. Then v_ε satisfies the conditions of Theorem 2.2. Therefore the 2-subdirect sum

$$H = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 1/2 & 3 & 5 & 1/3 \\ 1 & 4 & 13 & 0 \\ 0 & 2 & 1/2 & 6 \end{bmatrix}$$

is a nonnegative H-matrix in accordance with Theorem 2.2.

In the special case of A and B reducible H-matrices, the results of Theorem 2.2 are easy to be established. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \in R^{n\times n},\ B = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix} \in R^{m\times m}, \tag{5}$$

with A_{22} and B_{11} are square matrices of order k.

Theorem 2.3 *Let A and B be reducible H-matrices partitioned as in (5) and $A_{22} + B_{11}$ be an H-matrix. Then $H = A \oplus_k B$ is a block upper triangular H-matrix.*

Proof. It is well known that if every diagonal block of block upper triangular matrix H is an H-matrix, then H is an H-matrix. Since A_{11}, B_{22} and $A_{22} + B_{11}$ are H-matrices, then H is a block upper triangular H-matrix. ■

In some applications, such as in domain decomposition [8], [9] and [10] matrices A and B partitioned as in (1) arise with a common block, i.e., $A_{22} = B_{11}$. Bru, Pedroche and Szyld [4] introduced that it does not ensure $H = A \oplus_k B$ to be an M-matrix when $A_{22} = B_{11}$. Equivalently, the subdirect sum $H = A \oplus_k B$ given by two H-matrices partitioned as in (1) with a common block may not be an H-matrix.

Example 2.5 *Given the H-matrices*

$$A = \begin{bmatrix} 4 & \vdots & 3 & 3 \\ \cdots & \cdots & \cdots & \cdots \\ 4 & \vdots & 7 & 1 \\ 0.5 & \vdots & 2 & 4 \end{bmatrix},\ B = \begin{bmatrix} 7 & 1 & \vdots & 1 \\ 2 & 4 & \vdots & 4 \\ \cdots & \cdots & \cdots & \cdots \\ 9 & 2 & \vdots & 6 \end{bmatrix}.$$

We have that the 2-subdirect sum

$$H = A \oplus_2 B = \begin{bmatrix} 4 & 3 & 3 & 0 \\ 4 & 14 & 2 & 1 \\ 0.5 & 4 & 8 & 4 \\ 0 & 9 & 2 & 6 \end{bmatrix}$$

is not an H-matrix. In fact

$$\mu(H)^{-1} \approx \begin{bmatrix} -7.32 & 2.82 & 1.15 & 2.14 \\ -3.62 & 1.42 & 0.98 & 1.04 \\ -6.48 & 2.34 & 0.55 & 1.82 \\ -7.59 & 2.91 & 1.66 & 2.33 \end{bmatrix}$$

is not nonnegative matrices.

Here, we continue to discuss that A and B to be principal submatrices of a given H-matrix H such that they have a common block with all positive (or negative) diagonal entries.

Theorem 2.4 *Let H be an H-matrix of order $n + m - k$, and let A and B be two overlapping principal submatrices of order k. Then the k-subdirect sum $C = A \oplus_k B$ is an H-matrix.*

Proof. Let

$$H = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix}$$

be an H-matrix with H_{22} square matrix of order $k \geq 1$, then there exists a positive diagonal matrix $D = \mathrm{diag}(x_{n-k}I_{n-k}, x_k I_k, x_{m-k}I_{m-k})$ such that HD is an SDD matrix. Obviously,

$$C = A \oplus_k B = \begin{bmatrix} H_{11} & H_{12} & 0 \\ H_{21} & 2H_{22} & H_{23} \\ 0 & H_{32} & H_{33} \end{bmatrix}$$

is an H-matrix. Indeed, we still set $D = \mathrm{diag}(x_{n-k}I_{n-k}, x_kI_k, x_{m-k}I_{m-k})$. ■

We give the following example to illustrate the result of Theorem 2.4.

Example 2.6 *Consider an H-matrix*

$$H = \begin{bmatrix} 1 & \vdots & 0.3 & -0.4 & \vdots & 0.5 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0.9 & \vdots & 1.6 & 0.4 & \vdots & 0.7 \\ 0.1 & \vdots & 0.4 & 1.3 & \vdots & 0.4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -0.1 & \vdots & -0.9 & 0.1 & \vdots & 2 \end{bmatrix}.$$

Taking $D = \mathrm{diag}(1, 1, 0.7, 0.6)$, HD is an SDD matrix. The 2-subdirect sum

$$C = A \oplus_2 B = \begin{bmatrix} 1 & \vdots & 0.3 & -0.4 & \vdots & 0.5 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0.9 & \vdots & 3.2 & 0.8 & \vdots & 0.7 \\ 0.1 & \vdots & 0.8 & 2.6 & \vdots & 0.4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -0.1 & \vdots & -0.9 & 0.1 & \vdots & 2 \end{bmatrix}$$

is an H-matrix according to Theorem 2.4. In fact, let $D = \mathrm{diag}(1, 1, 0.7, 0.6)$, CD is an SDD matrix.

3 Subdirect sum of block strictly diagonally dominant matrices

Bru, Pedroche and Szyld [5] show that the k-subdirect sum of strictly diagonally dominant (SDD) matrices is an SDD matrix. In this section, we consider the problem of the subdirect sums of block strictly diagonally dominant (BSDD) matrices.

Let $A = (A_{ij}) \in R^{N\times N}$, $B = (B_{ij}) \in R^{N\times N}$ be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{bmatrix}, \ B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1m} \\ B_{21} & B_{22} & \cdots & B_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mm} \end{bmatrix}. \tag{6}$$

Then its block comparison matrix $\mu_b(A) = (b_{ij})$ is defined by

$$b_{ij} = \begin{cases} ||A_{ij}^{-1}||^{-1}, \ i = j, \\ -||A_{ij}||, \ i \neq j, \end{cases}$$

(here $||\cdot||$ is some multiplicative matrix norm with $||I||$=1). Then we can reformulate the definition of block diagonal dominance due to Varga and Feingold [6]. A is said to be block strictly diagonally dominant if its block comparison matrix exists and $\mu_b(A)$ is strictly diagonally dominant. We continue to partition (6) as follows

$$A = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}, \ B = \begin{bmatrix} \widetilde{B}_{11} & \widetilde{B}_{12} \\ \widetilde{B}_{21} & \widetilde{B}_{22} \end{bmatrix}, \tag{7}$$

where $\widetilde{A}_{22}$ and $\widetilde{B}_{11}$ are block square matrices of order k with corresponding partition. Then

$$\widetilde{H} = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} & 0 \\ \widetilde{A}_{21} & \widetilde{A}_{22} + \widetilde{B}_{11} & \widetilde{B}_{12} \\ 0 & \widetilde{B}_{21} & \widetilde{B}_{22} \end{bmatrix} \tag{8}$$

is called the k-subdirect sum of block matrices with corresponding partition. Note that $\mu_b(\widetilde{H}) \in R^{3\times 3}$

Generally, the k-subdirect sum of BSDD matrices is not a BSDD matrix.

Example 3.1 *The block matrices*

$$A = \begin{bmatrix} 1 & 0 & \vdots & 1/4 & 5/8 \\ 0 & 1 & \vdots & 1/20 & 29/40 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -5/12 & 1/3 & \vdots & 1 & 2 \\ -1/6 & 2/3 & \vdots & -2 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 2 & -3 & \vdots & 4 & 0 \\ 3 & 2 & \vdots & 0 & 4 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 4 & 0 & \vdots & 2 & -3 \\ 0 & 4 & \vdots & 3 & 2 \end{bmatrix}$$

are two BSDD matrices. However

$$\widetilde{H} = \begin{bmatrix} 1 & 0 & \vdots & 1/4 & 5/8 & \vdots & 0 & 0 \\ 0 & 1 & \vdots & 1/20 & 29/40 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -5/12 & 1/3 & \vdots & 3 & -1 & \vdots & 4 & 0 \\ -1/6 & 2/3 & \vdots & 1 & 3 & \vdots & 0 & 4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & 4 & 0 & \vdots & 2 & -3 \\ 0 & 0 & \vdots & 0 & 4 & \vdots & 3 & 2 \end{bmatrix}$$

is not a BSDD matrix, since we have

$$\mu_b(\widetilde{H}) = \begin{bmatrix} 1 & -7/8 & 0 \\ -5/6 & 5/6 & -4 \\ 0 & -4 & 13/5 \end{bmatrix}$$

is not strictly diagonally dominant (using the ∞ norm).

In the following we will present sufficient conditions to ensure that $\widetilde{H} = A \oplus_k B$ is a BSDD matrix.

Theorem 3.1 *Let $A = (A_{ij}) \in R^{N\times N}$, $B = (B_{ij}) \in R^{N\times N}$ be block strictly diagonally dominant matrices partitioned as (7), let $\widetilde{A}_{22} + \widetilde{B}_{11}$ be nonsingular. If*

$$\|(\widetilde{A}_{22} + \widetilde{B}_{11})^{-1}\| \leq \frac{1}{\|\widetilde{A}_{22}^{-1}\|^{-1} + \|\widetilde{B}_{11}^{-1}\|^{-1}}, \tag{9}$$

then $\widetilde{H} = A \oplus_k B$ in (8) is block strictly diagonally dominant.

Proof. Since $\mu_b(A)$ and $\mu_b(B)$ are strictly diagonally dominant, namely,

$$\begin{cases} \|\widetilde{A}_{11}^{-1}\|^{-1} > \|\widetilde{A}_{12}\|, \\ \|\widetilde{A}_{22}^{-1}\|^{-1} > \|\widetilde{A}_{21}\|, \end{cases} \quad \begin{cases} \|\widetilde{B}_{11}^{-1}\|^{-1} > \|\widetilde{B}_{12}\|, \\ \|\widetilde{B}_{22}^{-1}\|^{-1} > \|\widetilde{B}_{21}\|. \end{cases} \tag{10}$$

According to inequality (9), we obtain

$$\begin{aligned} \|(\widetilde{A}_{22} + \widetilde{B}_{11})^{-1}\|^{-1} &\geq \|\widetilde{A}_{22}^{-1}\|^{-1} + \|\widetilde{B}_{11}^{-1}\|^{-1} \\ &\geq \|\widetilde{A}_{21}\| + \|\widetilde{B}_{12}\|. \end{aligned} \tag{11}$$

Finally, from (10) and (11), we conclude that $\widetilde{H} = A \oplus_k B$ is block strictly diagonally dominant. ■

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