ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.8(2009) No.1,pp.45-49

Variation of Recurrent Fractal Interpolation Function

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(Received 30 April 2008, accepted 20 Feburary 2009)

Abstract: Based on the variation properties of continous functions, the variation properties of recurrent Fractal Interpolation(rFIF) are studied.And the rank of the variation of the general rFIF is estimated. Using the rank of the variation, the dimension theorem is proved without the condition that the connection matrix is irreducible.

Key words: recurrent fractal interpolation function; variation; connection matrix; dimension

1 Introduction

The fractal interpolation functions firstly proposed by Barnsley [1] not only open a new research field in the approximation theory of functions but also are applied in computer graphics. The unsmoothed curve can be described by the fractal interpolation functions due to the unique characteristic applied in unsmoothed-curve fitting, such as coastline, fracture surface etc.A lot of work has been devoted to the study of the fractals and the fractal interpolation functions, such as[1-8].

The paper first introduces the concept and properties of continuous function variation, and then investigates variation of recurrent fractal interpolation function (rFIF), at last figures out the dimension theorem of rFIF image based on the property of variation of rFIF.

Based on the investigation on continuous function variation [2-3], δ-variation has the properties as follows: where f, f_1, f_2 is continuous function for I and $V_{f,\delta}(I)$ is δ-variation of continuous function f.

Lemma 1 [3]: *(1)$V_{c_1 f+c_2,\delta}(I) = |c_1| \cdot V_{f,\delta}(I)$, for random constant c_1, c_2.*
(2)$V_{f_1,\delta}(I) - V_{f_2,\delta}(I) \le V_{f_1+f_2,\delta}(I) \le V_{f_1,\delta}(I) + V_{f_2,\delta}(I)$.

Lemma 2 [3]: *(1) If $f(t)$ is differentiable functions on I, then $0 \le V_{f,\delta}(I) \le 2\max\limits_{t\in I}|f'(t)| \cdot |I| \cdot \delta$.*
(2) If $f(t)$ is continuous function on I and $L(t) = at + b$, then

$$V_{f\circ L,\delta}(I) = \frac{1}{|a|} V_{f,|a|\delta}(L(I))$$

Lemma 3 [3]: *If $f(t)$ is continuous function on I, $0 = x_0 < x_1 < x_2 \cdots < x_N = I$ and $I_i = [x_{i-1}, x_i]$, then $\sum\limits_{i=1}^{N} V_{f,\delta}(I_i) \le V_{f,\delta}(I) \le \sum\limits_{i=1}^{N} V_{f,\delta}(I_i) + 2(N-1) \cdot V_f(I) \cdot \delta$.*

Recurrent FIF is developed based on FIF whereas rFIF is more flexible than FIF. Set $a = x_0 < x_1 < x_2 \cdots x_N = b$ being partition in closed interval I=[0, 1] where $N \ge 2$ and let $y_0, y_1, y_2, \cdots, y_N$ be random real number. For any $i = 1, 2, \cdots N$, set $I_i = [x_{i-1}, x_i]$, $D_i = \left[x_{l(i)}, x_{r(i)}\right]$, where $l(i), r(i) \in \{0, 1, \cdots N\}$ and $x_{r(i)} - x_{l(i)} > x_i - x_{i-1}$ and let $K_i = I_i \times R, K = I \times R$.

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Now define ω_i: $K \to K$ is the following transformation:

$$\omega_i = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_i(x) \\ F_i(x,y) \end{pmatrix} \quad (1)$$

where $L_i(x) = a_i x + b_i, F_i(x,y) = d_i y + p_i(x) \in P_r$, $p_i(x) \in P_r |d_i| < 1$, and let ω_i satisfy $\left(x_{l(i)}, y_{l(i)}\right)$ mapping to (x_{i-1}, y_{i-1}) and $\left(x_{r(i)}, y_{r(i)}\right)$ mapping to (x_i, y_i)(or $\left(x_{l(i)}, y_{l(i)}\right)$ mapping to (x_i, y_i) and $\left(x_{r(i)}, y_{r(i)}\right)$ mapping to(x_{i-1}, y_{i-1}) respectively). Therefore, the incidence matrix C= (C_{ij}) is determined, where if $I_j \subset D_i$, C_{ij}=1 otherwise C_{ij}=0.

Let $\tilde{H} = H(K_1) \times H(K_2) \times \cdots H(K_N)$, where $H(K_i)$ is a set composed by all non-vacuous subsets of K_i. For $\forall A = (A_1, A_2, \cdots, A_N) \in \tilde{H}$, define:

$$W \begin{pmatrix} A_1 \\ \vdots \\ A_N \end{pmatrix} \doteq \begin{pmatrix} C_{11}\omega_1 & \cdots & C_{1N}\omega_1 \\ \vdots & \ddots & \vdots \\ C_{N1}\omega_N & \cdots & C_{NN}\omega_N \end{pmatrix} \begin{pmatrix} A_1 \\ \vdots \\ A_N \end{pmatrix} \doteq \begin{pmatrix} \bigcup_{j=1}^{N} C_{1j}\omega_1(A_j) \\ \vdots \\ \bigcup_{j=1}^{N} C_{Nj}\omega_N(A_j) \end{pmatrix} \quad (2)$$

where for any $i, j = 1, 2, \cdots N$, set $C_{ij}\omega_i(A_j) = \begin{cases} \omega_i(A_j), if\ C_{ij} = 1 \\ \emptyset,\ if\ C_{ij} = 0 \end{cases}$. Thus W is a mapping from $\tilde{H}$ to $\tilde{H}$. And for any continuous function $f(t)$ defined on I, let $G_i(f) = \{(x, f(x)) | x \in I_i\}$. It can be proved that there must be certain continuous function f to let $(G_1(f), G_2(f) \cdots G_N(f))$ be the fixed point of W showed in (1) and (2) and f satisfies $f(x_i) = y_i$ $(i = 1, 2, \cdots N)$ which is named as rFIF determined by $\{\omega_i, i = 1, 2, \cdots N\}$.

2 Analysis on variation

Recurrent FIF is a type of special continuous function and has properties of variation as follows:

Theorem 4 *For $i = 1, 2, \cdots N$, let $I(i) = \{j : C_{ij} > 0\}$, thus there exist $\alpha_i > 0$ and $\beta_i > 0$ for $0 < \varepsilon < 1$ so that $|d_i a_i| \sum_{j \in I(i)} V_{f,\frac{\delta}{a_i}}(I_j) - \alpha_i \delta \le V_{f,\delta}(I_i) \le |d_i a_i| \sum_{j \in I(i)} V_{f,\frac{\delta}{a_i}}(I_j) + \beta_i \delta$.*

Proof. Since the sufficient and necessary condition for continuous function f being the fixed point of W is $f(t) = d_i f\left(L_i^{-1}(t)\right) + p_i\left(L_i^{-1}(t)\right)$ for any$t \in I_i$, where$L_i^{-1}(t) = \frac{t}{a_i} + \frac{b_i}{a_i}$. Based on Lemma 1, 2 and 3, the following formula can be derived:

$$|d_i| V_{f \circ L_i^{-1},\delta}(I_i) - V_{p \circ L_i^{-1},\delta}(I_i) \le V_{f,\delta}(I_i) \le |d_i| V_{f \circ L_i^{-1},\delta}(I_i) + V_{p \circ L_i^{-1},\delta}(I_i) \le V_{f,\delta}(I_i)$$

$$|d_i| a_i V_{f,\frac{\delta}{a_i}}(D_i) - V_{p,\frac{\delta}{a_i}}(D_i) \le V_{f,\delta}(I_i) \le |d_i| a_i V_{f,\frac{\delta}{a_i}}(D_i) + V_{p,\frac{\delta}{a_i}}(D_i)$$

$$|d_i| a_i V_{f,\frac{\delta}{a_i}}(D_i) - 2\max\left|p'_i(t)\right| \cdot |D_i| \cdot \delta \le V_{f,\delta}(I_i) \le |d_i| a_i V_{f,\frac{\delta}{a_i}}(D_i) + 2\max\left|p'_i(t)\right| \cdot |D_i| \cdot \delta$$

$$|d_i| a_i V_{f,\frac{\delta}{a_i}}(D_i) - 2\max\left|p'_i(t)\right| \cdot |I| \cdot \delta \le V_{f,\delta}(I_i) \le |d_i| a_i V_{f,\frac{\delta}{a_i}}(D_i) + 2\max\left|p'_i(t)\right| \cdot |I| \cdot \delta$$

Therefore, $|d_i| a_i \sum_{j \in I(i)} V_{f,\frac{\delta}{a_i}}(I_j) - \alpha_i \cdot \delta \le V_{f,\delta}(I_i) \le |d_i| a_i \sum_{j \in I(i)} V_{f,\frac{\delta}{a_i}}(I_j) + \beta_i \cdot \delta$ 3

There into $\alpha_i = 2\max|p'_i(t)| \cdot |I|, \beta_i = 2\max|p'_i(t)| \cdot |I| + 2(N-1)V_f(I)$. ■

Theorem 5 [9] *For non-negative matrix $Q = CS(1)$, set V being the corresponding directed graph of Q, V^i being the nontrivial strong connected branch of V and Q_i being the sub-matrix of Q corresponding to Vi and assume $\lambda_i = \rho(Q_i) > 1$, thus $\lim_{\delta \to 0} V_{f,\delta}(I_j)\delta^{-1} = \infty, \forall j = i(1), i(2), \ldots, i(t_i)$.*

Theorem 6 *Set f being the transformation $\{\omega_i, i = 1, 2, \cdots N\}$ and incidence matrix C. Let $S(d)$ be diagonal matrixdiag $\left\{|d_1| \cdot a_1^{d-1}, \cdots |d_N| \cdot a_N^{d-1}\right\}$. Set $Q = CS(1)$ and V corresponding to directed graph of Q with V^i being the nontrivial strong connected branch of V. For $i = 1, 2, \cdots r$, set $C_i Q_i$ and $S_i(d)$ is* *respectively corresponding to sub-matrix of C_iQ_i and $S_i(d)$. If V^i is nontrivial, there is $B_1, B_2, C > 0$ to make $B_1 \sum_{j=1}^{N} \delta^{2-D_j} \le V_{f,\delta}(I) \le B_2 \sum_{j=1}^{N} \delta^{2-D_j} + C$ valid. (The concept of non-negative matrix, directed graph and strong connected branch refers to [10])*

Proof. (1) Suppose $D > 1$, let $\lambda = \rho(Q_u)$, so that $\lambda = \rho(Q_u) = \rho(C_uS_u(1)) > 1$ can be derived based on Perron-Frobenius theorem due to $\rho(C_uS_u(D)) = \rho(C_uS_u(d_u)) = 1$. Let $v = \left(v_{u(1)}, \cdots v_{u(tu)}\right)$ be the strict positive eigenvector of Q_u corresponding to eigenvalue λ and $w = \left(w_{u(1)}, \cdots w_{u(tu)}\right)$ be the strict positive eigenvector of $C_uS_u(D)$ corresponding to eigenvalue 1. Select suitable $\alpha > 0$ to satisfy$\alpha \cdot v_j > \alpha_j$, where $j = u(1), u(2), \cdots, u(tu)$, the following formula is attained by (3):

$$V_{f,\delta}(I_j) \ge |d_j|\, a_j \sum_{c_{ij}=1} V_{f,\frac{\delta}{a_i}}(I_k) - \alpha \cdot \delta v_j \tag{3}$$

By theorem 3, there is $\delta_0 > 0$ to make

$$V_{f,\delta}(I_j) \ge \gamma\delta^{2-D} w_j + {}^{\alpha\delta v_j}\!/\!_{(\lambda-1)} \tag{4}$$

valid for all $0 < \delta < \delta_0$ and $j = u(1), u(2), \cdots, u(tu)$, where $a^* = \min\{|a_i|, i = 1, 2, \cdots N\}$. If $aa^*\delta_0 \le \delta \le a^*\delta_0$, $a^*\delta_0 \le {}^{\delta}\!/\!_{|a_i|} \le \delta_0$. Therefore

$$\begin{aligned} V_{f,\delta}(I_j) &\ge |d_j|\, a_j \sum_{k\in I(j)\cap\{u(1),\cdots u(tu)\}} V_{f,\frac{\delta}{a_j}}(I_k) - \alpha \cdot \delta v_j \\ &\ge |d_j|\, a_j \sum_{k\in I(j)\cap\{u(1),\cdots u(tu)\}} \left[\gamma\left(\frac{\delta}{a_j}\right)^{2-D} w_j + \alpha v_j\left(\tfrac{\delta}{a_j}\right)\Big/(\lambda-1)\right] - \alpha\cdot\delta v_j \\ &\ge \gamma\delta^{2-D} w_j + {}^{\alpha\delta v_j}\!/\!_{(\lambda-1)} \end{aligned}$$

This is derived by (3), (4). So (4) is valid for all $j = u(1), u(2)\cdots u(tu)$ and $aa^*\delta_0 \le \delta \le a^*\delta_0$. By induction (4) is confirmed to be also valid for$a^n a^*\delta_0 \le \delta \le \delta_0$, $n = 1, 2, \cdots$.

On the other side, let the maximum position of elements in V be U.

(a) For element V_i with random position being 1, if V_i is nontrivial, V_i must belong to a certain nontrivial strong connected branch $V^u = \left\{V_{u(1)}, \cdots, V_{u(tu)}\right\}$.

And $I(j) = I(j) \cap \{u(1), \cdots u(tu)\}$ for any $j = u(1), u(2)\cdots u(tu)$ due to$p(V_i) = 1$. For any $\varepsilon > 0$, let λ be spectral radius of $C_uS_u(D+\varepsilon)$, thus $\lambda = \rho(C_uS_u(D+\varepsilon)) < \rho(C_uS_u(du)) = 1$ due to P-F theorem. Let $x = \left(x_{u(1)}, \cdots x_{u(tu)}\right)$ and $\left(y_{u(1)}, \cdots y_{u(tu)}\right)$ be the positive eigenvector of $C_uS_u(D)$ respectively corresponding to eigenvalue 1 and λ. Based on theorem 2 and $D \ge 1$, there exists $\beta > 0$ to make

$$\begin{aligned} V_{f,\delta}(I_j) &\le |d_j|\, a_j \sum_{k\in I(j)} V_{f,\frac{\delta}{a_j}}(I_k) + \beta\cdot\delta^{2-(D+\varepsilon)} y_j \\ &= |d_j|\, a_j \sum_{k\in I(j)\cap\{u(1),\cdots u(tu)\}} V_{f,\frac{\delta}{a_j}}(I_k) + \beta\cdot\delta^{2-(D+\varepsilon)} y_j \end{aligned} \tag{5}$$

$j = u(1), u(2)\cdots u(tu)$ valid for $0 < \delta \le 1$. Select enough large γ to let δ satisfy $a^* \le \delta \le 1$, so that

$$V_{f,\delta}(I_j) \le \gamma\delta^{2-du} x_j + \beta\delta^{2-(D+\varepsilon)} y_j (1-\lambda)^{-1} \tag{6}$$

Therefore (6) is confirmed to be valid for $0 < \delta \le 1$ based on (5).

(b) Assume (5) being valid for any element V_m $(1 \le s \le U-1)$ whose position is less than s. Select a V_i with position s+1. If V_i is nontrivial and assume V_i not belonging to any connected branches, $p(V_j) \le s$ for $j \in I(i)$. Therefore

$$\overline{\dim} A_i = \max_{\{j:j\in I(i)\}} \overline{\dim} W_i(A_j) \le \max_{\{j:j\in I(i)\}} \overline{\dim} A_j \le \max_{\{j:p(V_j)\le s\}} \overline{\dim} A_j \le D.$$

Assume that V_i is nontrivial and belongs to a connected branch $V^u = \left\{V_{u(1)}, \cdots, V_{u(tu)}\right\}$. Set $x = \left(x_{u(1)}, \cdots x_{u(tu)}\right)$ to be the positive eigenvector of $C_u S_u\left(du\right)$ corresponding to eigenvalue 1. From theorem 3,

$$V_{f,\delta}\left(I_j\right) \le \left|d_j\right| a_j \sum_{k\in I(j)\cap\{u(1),\cdots u(tu)\}} V_{f,\frac{\delta}{a_j}}\left(I_k\right) + \left|d_j\right| a_j \sum_{\{k:p(V_k)\le s\}} V_{f,\frac{\delta}{a_j}}\left(I_k\right) + \beta_2\delta \tag{7}$$

For $\varepsilon > 0$, set λ being spectral radium of $C_u S_u\left(D+\varepsilon\right)$, so that $\lambda = \rho\left(C_u S_u\left(D+\varepsilon\right)\right)$ $< \rho\left(C_u S_u\left(du\right)\right) = 1$ due to P-F theorem. Let $\left(y_{u(1)}, \cdots y_{u(tu)}\right)$ be the positive eigenvector of $C_u S_u\left(du\right)$ corresponding to eigenvalue λ. By deduction and (7), for $\beta' > 0$,

$$V_{f,\delta}\left(I_j\right) \le \left|d_j\right| a_j \sum_{k\in I(j)\cap\{u(1),\cdots u(tu)\}} V_{f,\frac{\delta}{a_j}}\left(I_k\right) + \beta'\cdot\delta^{2-(D+\varepsilon)} y_j \tag{8}$$

where $j = u\left(1\right), u\left(2\right)\cdots u\left(tu\right)$. For $0 < \delta \le 1$, select suitable γ to let

$$V_{f,\delta}\left(I_j\right) \le \gamma\delta^{2-du} x_j + \beta\delta^{2-(D+\varepsilon)} y_j\left(1-\lambda\right)^{-1} \tag{9}$$

be valid for $a^* \le \delta \le 1$. Thus (9) is valid for $0 < \delta \le 1$ based on (8). Further to get

$$\gamma\delta^{2-D} w_j + \alpha\delta v_j\left(\lambda-1\right)^{-1} \le V_{f,\delta}\left(I_j\right) \le \gamma\delta^{2-D} x_j + \beta\delta^{2-(D+\varepsilon)} y_j\left(1-\lambda\right)^{-1},$$

$$\gamma w_j + \alpha\delta^{1-D} v_j\left(\lambda-1\right)^{-1} \le \delta^{D-2} V_{f,\delta}\left(I_j\right) \le \gamma x_j + \beta\delta^{-\varepsilon} y_j\left(1-\lambda\right)^{-1} \quad .$$

Thus there exists $B_j, B_j' > 0$ to let $B_j \le \delta^{D-2} V_{f,\delta}\left(I_j\right) \le B_j'$ be valid. According to Lemma 3,

$$\sum_{j=1}^{N} \delta^{2-D_j} B_j \le V_{f,\delta}\left(I\right) \le \sum_{j=1}^{N} \delta^{2-D_j} B_j' + 2\left(N-1\right)\cdot V_f\left(I\right)\cdot\delta.$$

Let $B_1 = \min B_j, B_2 = \max B_j', j \in (1, 2, \cdots N) C = 2\left(N-1\right)\cdot V_f\left(I\right)\cdot\delta$, then

$$B_1 \sum_{j=1}^{N} \delta^{2-D_j} \le V_{f,\delta}\left(I\right) \le B_2 \sum_{j=1}^{N} \delta^{2-D_j} + C.$$

■

3 Conclusion

The box dimension of continuous functions can be directly resolved by variation of continuous functions. The formula refers to [2]. Thus according to the properties of rFIF variation proofed above, the dimension theorem of rFIF is showed as follows:

Theorem 7 *Let f be rFIF determined by $\{\omega_i, i = 1, 2, \cdots N\}$ and incidence matrix C, marked as $G = Graph\left(f\right)$. Let $S\left(d\right)$ be diagonal matrixdiag $\left\{\left|d_1\right|\cdot a_1^{d-1}, \cdots \left|d_N\right|\cdot a_N^{d-1}\right\}$. Set $Q = CS\left(1\right)$ and V corresponding to the directed graph of Q with V^i being the nontrivial strong connected branch of V. For $i = 1, 2, \cdots r$, let C_i, Q_i and $S_i\left(d\right)$ be respectively corresponding to the sub-matrix ofC, Q and $S\left(d\right)$ of V^i and d_i is the only real number satisfying $\rho\left(C_i S_i\left(d_i\right)\right) = 1$, then* $\dim_B G = \max\left\{d_1, d_2, \cdots d_m, 1\right\}$.

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