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A New Two-step Method for Solving Nonlinear Equations

Jishe Feng *
Department of Mathematics, Longdong University
Qingyang,Gansu,745000,P.R.China

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Abstract: We suggest a new two-step iterative method for solving nonlinear equations. The new iterative method has quadratic convergence. Some numerical experiments illustrate that the new method can compete with Basto method.[M. Basto, V. Semiao, F. L. Calheiros, A new iterative method to compute nonlinear equations, Appl. Math. Comput. 173(2006)468-483.].

Keywords: Nonlinear equation; Iterative method; Adomian decomposition method

1 Introduction

Solving nonlinear equations is one of the most important problems in numerical analysis. Much attention has given to develop several iterative methods for solving nonlinear equations, see[1-4] and the references therein. Chun[1] and Basto[2] have proposed and studied several methods for nonlinear equations with higher order convergence by using the decomposition technique of Adomian[3,5]. They have to used the higher order differential derivatives which is a serious drawback. To overcome this drawback, we suggest and analyze a method for solving nonlinear equations, which does not been involved the higher order derivative of the function.

Adomian's technique applied to nonlinear equation, consist in transforming the equation $f(x) = 0$ into the canonical form

$$x = x_0 + F(x) \tag{1}$$

where x_0 is a constant and F a nonlinear function. The solution is given in a series form,

$$x = \sum_{i=0}^{\infty} x_i \tag{2}$$

If the initial value x_0 is given, the rest of the terms $x_n, n \geq 1$, will be settled by a recursive relation. The nonlinear function $F(x)$ is decomposed into a particular series of polynomials

$$F(x) = \sum_{i=0}^{\infty} A_i \tag{3}$$

where A_i is the so-called Adomian polynomials. For each n, the polynomial A_n depends only on $x_{0,}x_{1,\cdots,}x_n$.

Substituting (2) and (3) into (1), one obtains

$$\sum_{i=0}^{\infty} x_i = x_0 + \sum_{n=1}^{\infty} A_i \tag{4}$$

To determine the components x_n, one employs the recursive relation

$$x_{n+1} = A_n \tag{5}$$

*Corresponding author. *E-mail address*: gsfjs6567@126.com

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Basto[2] using the following Adomian decomposition method to (1), is equivalent to determining the sequence $(S_n)_n = (x_1 + x_2 + \cdots x_n)_n$ by using the iterative scheme

$$S_0 = 0 \tag{6}$$
$$S_{n+1} = F(x_0 + S_n)$$

This scheme may be associated with the functional equation

$$S = F(x_0 + S) \tag{7}$$

To improve the order of convergence of the sequence $(S_n)_n$, instead of solving (7) Basto[2] using the following equation for the solution S^* of (7):

$$S = G(x_0 + S) = \frac{-F'(x_0 + S^*)S + F(x_0 + S)}{1 - F'(x_0 + S^*)} \tag{8}$$

This is the key technique of Basto[2].

2 Iterative method and convergence analysis

Considering the nonlinear equation $f(x) = 0$, and writing $f(x + h)$ in Taylor's series expansion about x, one obtains

$$f(x + h) = f(x) + hf'(x) + g(h) \tag{9}$$

$$g(h) = f(x + h) - f(x) - hf'(x) \tag{10}$$

Supposing $f\prime(x) \neq 0$, one searches for a value of h such that

$$f(x + h) = 0, \ \ i.e. \ \ f(x) + hf'(x) + g(h) = 0 \tag{11}$$

This is equivalent to finding the following h

$$h = -\frac{f(x)}{f'(x)} - \frac{g(h)}{f'(x)} \tag{12}$$

equation (12) can be rewritten in the following form

$$h = c + N(h) \tag{13}$$

where

$$c = -\frac{f(x)}{f'(x)} \tag{14}$$

and

$$N(h) = -\frac{g(h)}{f'(x)} = -\frac{f(x + h) - f(x) - f'(x)h}{f'(x)} \tag{15}$$

Here c is a constant and $N(h)$ is a nonlinear function. When applying Adomian's method to (13), we use the technique of Basto [2] and obtain

$$S = \frac{-N'(c + S^*)S + N(c + S)}{1 - N'(c + S^*)} \tag{16}$$

when x is sufficiently close to the real solution of $f(x) = 0$, $S^* \approx 0$. Thus (16) converts to

$$S = \frac{-N'(c)S + N(c + S)}{1 - N'(c)} \tag{17}$$

Applying the Adomian method to (13), one obtain

$$A_0 = N(h_0) = N(c) = \frac{N(c)}{1 - N'(c)} = -\frac{f(x + c)}{2f'(x) - f'(x + c)} \tag{18}$$

Now we construct the iterative method. For $h \approx h_0 = -\frac{f(x)}{f'(x)}$, obtains $h + x \approx x - \frac{f(x)}{f'(x)}$,which yields Newton method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

For $k = 1$, one obtains $h \approx h_0 + N(h_0), h + x \approx x + h_0 + N(h_0)$, which suggests the following two-step iterative method

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= y_n - \frac{f(y_n)}{2f'(x_n) - f'(y_n)} \end{aligned} \tag{19}$$

Comparing with the method in [2], Eq.(19) only use the first derivative.

Theorem 1 *Let $\alpha \in I$ be a zero of a sufficiently differentiable function $f : I \to R$ for an open interval I. If x_0 is sufficiently close to α, then the two-step iterative method (19) has quadratic convergence.*

Proof. Since f is sufficiently differentiable, by expanding $f(x_n)$ and $f\prime(x_n)$ about α, one obtains

$$f(x_n) = f'(\alpha)(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + O(e_n^5)) \tag{20}$$

$$f'(x_n) = f'(\alpha)(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + O(e_n^4)) \tag{21}$$

where $e_n = x_n - \alpha, c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}$ for $k = 2, 3, \cdots$. From (20) and (21), one obtains

$$-\frac{f(x_n)}{f'(x_n)} = -e_n + c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 - (7c_2c_3 - 3c_4 - 4c_2^3)e_n^4 + O(e_n^5) \tag{22}$$

$$y_n = \alpha + c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 - (7c_2c_3 - 3c_4 - 4c_2^3)e_n^4 + O(e_n^5) \tag{23}$$

expanding $f(y_n)$ about α and using (23), one obtains

$$f(y_n) = f'(\alpha)(c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 - (7c_2c_3 - 3c_4 - 5c_2^3)e_n^4 + O(e_n^5)) \tag{24}$$

From (24) and (21), one obtains

$$f'(y_n) = f'(\alpha)(2c_2e_n - 6(c_2^2 - c_3)e_n^2 - 4(7c_2c_3 - 3c_4 - 5c_2^3)e_n^3 + O(e_n^4)) \tag{25}$$

$$2f'(x_n) - f'(y_n) = f'(\alpha)(2 + 2c_2e_n + 6c_2^2e_n^2 + (28c_2c_3 - 4c_4 - 20c_2^3)e_n^3 + O(e_n^4)) \tag{26}$$

Using (24) and (26), one obtains

$$\frac{f(y_n)}{2f'(x_n) - f'(y_n)} = \frac{c_2}{2}e_n^2 + (c_3 - \frac{c_2^2}{2})e_n^3 + \frac{-9c_2c_3 + 3c_4 + 5c_2^3}{2}e_n^4 + O(e_n^5) \tag{27}$$

From (23) and (27), one obtains

$$x_{n+1} = \alpha + \frac{c_2}{2}e_n^2 + (c_3 - \frac{3c_2^2}{2})e_n^3 + \cdots \tag{28}$$

which implies that the two-step iterative method (19) has quadratic convergence. ■

3 Numerical experiments

We present some examples to illustrate the efficiency of the iterative method proposed in this paper. We compare the method (19)[denoted by New method] with Newton-Raphson method, Adomian method, Rabolian method, Abbasbandy method, Basto method[2].

We use the following test equations, which are the same as in [2]:

$$\begin{aligned} f_1(x) &= x^3 + 4x^2 + 8x + 8 = 0, x_0 = -1 \\ f_2(x) &= x - 2 - e^{-x} = 0, x_0 = 2 \\ f_3(x) &= x^2 - (1-x)^5 = 0, x_0 = 0.2 \\ f_4(x) &= e^x - 3x^2 = 0, x_0 = 0 \end{aligned}$$

Numerical computations have been carried out by using the software Maple9.5. The results are presented in Tables 1-5.

Table 1: Number of iterations and solution obtained for the different methods

Method	$f_1(x) = x^3 + 4x^2 + 8x + 8 = 0, x_0 = -1$	
	Number iterations	Obtained solution
Newton-Raphson	1	-2.000000000
Adomian		Slow convergence
Babolian		Divergence
Abbasbandy	2	-2.003987741
Basto	3	-2.000100903
New method	1	-2

The exact solution prospected is $x = -2$.

Table 2:Number of iterations and solution obtained for the different methods

Method	$f_2(x) = x - 2 - e^{-x} = 0, x_0 = 2$	
	Number iterations	Obtained solution
Newton-Raphson	3	2.120028239
Adomian	6	2.120003306
Babolian	4	2.120016168
Abbasbandy	2	2.120028239
Basto	2	2.120028239
New method	2	2.120028239

The exact solution prospected is $x = 2.120028239$.

Table 3: Number of iterations and solution obtained for the different methods

Method	$f_3(x) = x^2 - (1-x)^5 = 0, x_0 = 0.2$	
	Number iterations	Obtained solution
Newton-Raphson	3	0.345953774
Adomian	10	0.340622225
Babolian	5	0.346021366
Abbasbandy	2	0.345954646
Basto	2	0.3459522189
New method	3	0.3459548158

The exact solution prospected is $x =$0.345954816.

Table 4: Number of iterations and solution obtained for the different methods

Method	$f_4(x) = e^x - 3x^2 = 0, x_0 = 0.5$	
	Number iterations	Obtained solution
Newton-Raphson	4	0.910007662
Abbasbandy	4	0.910007573
Basto	3	0.910007573
New method	4	0.9100075727

The exact solution prospected is x =0.910007573.

Table 5: Number of iterations and solution obtained for the different methods

Method	$f_4(x) = e^x - 3x^2 = 0, x_0 = 0$	
	Number iterations	Obtained solution
Newton-Raphson	5	-0.458962274
Abbasbandy	5	-0.458964191
Basto	2	-0.458992962
New method	5	-0.4589622675

The exact solution prospected is x =-0.458962268.

The number of iterations of Basto method's are less than the new method in Table 3 and Table 5, but the solutions computed by the new method are more exact.

4 Conclusion

We suggest and analyze a new two-step iterative methods for solving nonlinear equations. The new iterative method has quadratic convergence, some numerical experiments illustrate the performance of the new method is better than Basto method which has third convergence.

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