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Numerical Solution of Singularly Perturbed Problems

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Abstract: In this paper, numerical solution of singularly perturbed boundary value problems are given by using finite element method. Collocation method is applied with quadratic and cubic B-spline base functions over the geometrically graded mesh of the solution domain.

Keywords: Collocation; graded mesh; spline; singularly perturbed

1 Introduction

In this paper we consider the solution of the singularly perturbed problems

$$-\varepsilon u'' + p(x)u' + q(x)u = f(x), \ \ 0 \le x \le 1 \tag{1}$$

with boundary conditions

$$u\,(0) = \lambda \text{ and } u\,(1) = \beta \tag{2}$$

where $\varepsilon > 0$ is a small parameter, $p(x)$, $q(x)$, $f(x)$ are smooth, bounded functions and λ, β are finite constants. The possibility of the thin layers at the boundaries of the domain makes the standard numerical discretization of the problem to be unreliable. Since the thickness of these layers depends on ε, unbounded oscillations may be observed in approximate solutions that are obtained with the usual numerical methods for Eq.(1) when $\varepsilon \to 0$.

These problems are used widely in many areas such as chemical reactor theory, optimal control, quantum mechanics, reaction-diffusion process, aerodynamics, etc. Many authors have studied on this problem and tried to overcome the above-mentioned difficulties. D.J.Fyfe [5] used cubic splines on equal and unequal intervals and compared the results. He observed that very little advantage is gained by using unequal intervals. G.Beckett and J.A.Mackenzie [1] gave a pth order Galerkin finite element method on a non-uniform grid. In their study the grid is constructed by equidistributing a strictly positive monitor function. After the appropriate selection of the monitor function parameters they proved that the numerical solution is insensitive to the size of the singular perturbation parameter and achieves the optimal rate of convergence with respect to the mesh density. Employing coordinate stretching a Galerkin-spectral method is applied to the singularly perturbed boundary value problems by W.Liu and T.Tang [7]. M.K.Kadalbajoo and K.C.Patidar [6] gave some difference schemes using spline in tension. They showed that these methods are second-order accurate. S.C.S.Rao and M.Kumar [10] presented a cubic B-spline collocation method. In that study, they decomposed the solution domain into three non-overlapping subdomains and solved the differential equation on these domains. Recently, Tirmizi et.al. have proposed a non polynomial spline method [12] in which the quartic splines have been employed as interpolation functions.

The definitions of B-splines over the geomertically graded mesh was given in reference [2]. Dag and Sahin have used these B-splines [3] for the numerical solutions of Burgers' equation to reduce the error near the boundaries. In this article, we used the finite element method with the quadratic and the cubic B-splines. After giving the expressions of the mentioned B-splines over the geometrically graded mesh we applied the

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collocation method to Eq.(1). To be able to use the quadratic B-splines in the collocation method, the setting $-u' = v$ gives a first order system of equations for Eq.(1). This system can be solved by employing the quadratic B-spline collocation method. Numerical results are illustrated for some test problems.

Briefly, outline is as follows. In Section 2, numerical methods are given. Numerical experiments are carried out for two test problems and errors of those methods are compared in Section 3. Finally conclusion is given in last section.

2 B-spline Collocation Methods

For numerical purpose, let us divide the solution domain $[0, 1]$ into subintervals by the knots x_m such that

$$0 = x_0 < x_1 < \cdots < x_N = 1$$

where $x_{m+1} = x_m + h_m$ and h_m is the size of interval $[x_m, x_{m+1}]$ with relation $h_m = \sigma h_{m-1}$. Here σ is mesh ratio constant.

To construct the geometrically graded mesh, determination of the first element size h_0 is necessary. Since

$$h_0 + h_1 + \cdots + h_{N-1} = 1$$

it is easy to write

$$h_0 = \frac{1}{1 + \sigma + \sigma^2 + \cdots + \sigma^{N-1}}.$$

This partition will be uniform if the mesh ratio σ is taken as unity. To obtain finer mesh at the left boundary, σ must be chosen as $\sigma > 1$. On the other hand, to make the mesh size smaller at the right boundary, σ must be chosen as $\sigma < 1$. The mentioned selection of σ will be done by experimentally in Section 3

2.1 Quadratic B-spline collocation method (QM)

The expression of the quadratic B-splines over the geometrically graded mesh may be given in the following form [2]:

$$\begin{array}{l} Q_{m-1} \\ Q_m \\ Q_{m+1} \end{array} = \frac{1}{h_m^2} \left\{ \begin{array}{l} (h_m - \xi)^2 \sigma, \\ h_m^2 + 2h_m \sigma \xi - (1+\sigma)\xi^2, \\ \xi^2 \end{array} \right. \tag{3}$$

where $\xi = x - x_m$ and $0 \leq \xi \leq h_m$. A quadratic B-spline covers 3 elements. Any quadratic B-spline Q_m and its derivatives vanish outside of the interval $[x_{m-1}, x_{m+2}]$ and therefore an element is covered by 3 successive quadratic B-splines. The set of the quadratic B-splines $\{Q_{-1}, Q_0, ..., Q_N\}$ forms a basis [9] for the functions defined on the solution domain. Thence, an approximation u_N to the analytical solution u can be written as

$$u_N = \sum_{m=-1}^{N} \delta_m Q_m \tag{4}$$

where δ_m are unknown parameters. By the substitution of the value of Q_m at the knots x_m in Eq.(4), the nodal value u_m and its derivative u'_m are expressed in terms of δ_m by

$$\begin{aligned} u_m = u(x_m) &= \sigma\delta_{m-1} + \delta_m, \\ u'_m = u'(x_m) &= \frac{2\sigma}{h_m}(\delta_m - \delta_{m-1}). \end{aligned} \tag{5}$$

To obtain smooth solutions using quadratic B-splines, the differential equation should have at most first order derivatives. But the Eq.(1) has second order derivative. Taking $-u' = v$ in Eq.(1) we can turn Eq.(1) into a system including first order differential equations such that

$$\begin{aligned} \varepsilon v' - p(x)v + q(x)u = f(x), \\ v + u' = 0 \end{aligned} \tag{6}$$

hence the quadratic B-splines become applicable for the numerical solution of system (6).

When we apply the collocation method to system (6), the functions u, v and their derivatives are replaced with their equals obtained from Eq.(5). This substitution yields the following system:

$$\begin{aligned} \varepsilon\frac{2\sigma}{h_m}(\gamma_m-\gamma_{m-1})-p\left(x_m\right)\left(\sigma\gamma_{m-1}+\gamma_m\right)+q(x_m)\left(\sigma\delta_{m-1}+\delta_m\right) &= f\left(x_m\right) \\ \left(\sigma\gamma_{m-1}+\gamma_m\right)+\frac{2\sigma}{h_m}(\delta_m-\delta_{m-1}) &= 0 \end{aligned}$$

With necessary operations, this system can be written in matrix form as

$$\mathbf{AX}=\mathbf{F} \tag{7}$$

where

$$\mathbf{A}=\begin{bmatrix} q_0\sigma & \alpha_{01} & q_0 & \alpha_{02} & & & & & & \\ -\alpha_{03} & \sigma & \alpha_{03} & 1 & & & & & & \\ & & q_1\sigma & \alpha_{11} & q_1 & \alpha_{12} & & & & \\ & & -\alpha_{13} & \sigma & \alpha_{13} & 1 & & & & \\ & & & & \ddots & \ddots & \ddots & & & \\ & & & & & & q_N\sigma & \alpha_{N1} & q_N & \alpha_{N2} \\ & & & & & & -\alpha_{N3} & \sigma & \alpha_{N3} & 1 \end{bmatrix},$$

$$\mathbf{X}=\left[\delta_{-1},\gamma_{-1},\delta_0,\gamma_0,\cdots,\delta_N,\gamma_N\right]^T,\mathbf{F}=\left[f_0,0,f_1,0,\cdots,f_N,0\right]^T$$

and

$$\begin{aligned} p_m &= p(x_m),\ q_m=q(x_m),\ f_m=f(x_m),\ m=0,1,\cdots,N \\ \alpha_{m1} &= -p_m\sigma-\varepsilon\frac{2\sigma}{h_m},\ \alpha_{m2}=-p_m\sigma+\varepsilon\frac{2\sigma}{h_m},\ \alpha_{m3}=\frac{2\sigma}{h_m}. \end{aligned}$$

Matrix system (7) has $2N+2$ equations and $2N+4$ unknowns. In order to solve this system, the numbers of equations and unknowns must be equal. From the boundary conditions (2) and Eq.(5) it is easy to write

$$\delta_{-1}=\frac{\lambda-\delta_0}{\sigma},\ \ \delta_N=\beta-\sigma\delta_{N-1}.$$

Using these equalities, δ_{-1} and δ_N can be eliminated from the system and then matrix equation (7) can be solved with Thomas algorithm. Substituting the obtained parameters δ_m in Eq.(5), the numerical solution is found at the knots x_m.

2.2 Cubic B-spline collocation method (CM)

The expression of the graded cubic B-splines may be written [2] as

$$\begin{matrix} Q_{m-1} \\ Q_m \\ Q_{m+1} \\ Q_{m+2} \end{matrix} = \frac{1}{h_m^3}\left\{\begin{array}{l} (h_m-\xi)^3\sigma^3 \\ -\sigma\left(\sigma^2+\sigma+1\right)(h_m-\xi)^2-6\sigma h_m(h_m-\xi)-3\sigma^2h_m^2 \\ -\left(\sigma^2+\sigma+1\right)\xi^3+3\sigma^2h_m\xi^2+3\sigma h_m^2\xi+h_m^3 \\ \xi^3. \end{array}\right. \tag{8}$$

over the above mesh where $\xi=x-x_m$ and $0\leq\xi\leq h_m$. A cubic B-spline ϕ_m covers four successive finite elements and vanish outside the interval $[x_{m-2},x_{m+2}]$. Therefore a typical element $[x_m,x_{m+1}]$ is covered by four cubic B-splines $\phi_{m-1},\phi_m,\phi_{m+1}$ and ϕ_{m+2}. The set of those B-splines $(\phi_{-1},\phi_0,...,\phi_{N+1})$ is a basis for piecewise polynomials over the solution domain $[a,b]$. Using this property, a numerical approximation to the analytical solution u can be constructed as follows:

$$u_N=\sum_{m=-1}^{N+1}\delta_m\phi_m \tag{9}$$

where the parameters δ_m are unknowns. The graded cubic B-splines have the second order continuity. So the approximate function has also the continuity order two. The nodal value u_m and its space derivatives u'_m, u''_m have the following expressions at the knots x_m :

$$\begin{aligned} u_m &= \sigma^3\delta_{m-1} + 2\sigma\left(\sigma+1\right)\delta_m + \delta_{m+1}, \\ h_m u'_m &= 3\sigma\left[\delta_{m+1} + \left(\sigma^2-1\right)\delta_m - \sigma^2\delta_{m-1}\right], \\ h_m^2 u''_m &= 6\sigma^2\left[\sigma\delta_{m-1} - \left(\sigma+1\right)\delta_m + \delta_{m+1}\right]. \end{aligned} \tag{10}$$

The application of the collocation method is based on the substitution of the expressions (10) in the differential equation (1). This gives the following equation system:

$$\begin{aligned} &-\varepsilon\tfrac{6\sigma^2}{h_m^2}\left[\sigma\delta_{m-1} - \left(\sigma+1\right)\delta_m + \delta_{m+1}\right] + p(x_m)\tfrac{3\sigma}{h_m}\left[\delta_{m+1} + \left(\sigma^2-1\right)\delta_m - \sigma^2\delta_{m-1}\right] \\ &+q(x_m)\left[\sigma^3\delta_{m-1} + 2\sigma\left(\sigma+1\right)\delta_m + \delta_{m+1}\right] = f(x_m), \qquad m = 0...N \end{aligned}$$

After some operations this system may be converted into matrix form as

$$\mathbf{AX} = \mathbf{F} \tag{11}$$

where

$$\mathbf{A} = \begin{bmatrix} \theta_{01} & \theta_{02} & \theta_{03} & & & \\ & \theta_{11} & \theta_{12} & \theta_{13} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \theta_{N1} & \theta_{N2} & \theta_{N3} \end{bmatrix},$$

$$\mathbf{X} = \left[\delta_{-1}, \delta_0, \cdots, \delta_{N+1}\right]^T, \qquad \mathbf{F} = \left[f_0, f_1, \cdots, f_N\right]^T$$

and

$$\theta_{m1} = -\varepsilon\frac{6\sigma^3}{h_m^2} - p_m\frac{3\sigma^3}{h_m} + q_m\sigma^3, \theta_{m2} = \varepsilon\frac{6\sigma^2\left(\sigma+1\right)}{h_m^2} + p_m\frac{3\sigma\left(\sigma^2-1\right)}{h_m} + 2q_m\sigma\left(\sigma+1\right),$$

$$\theta_{m3} = -\varepsilon\frac{6\sigma^2}{h_m^2} - p_m\frac{3\sigma}{h_m} + q_m, p_m = p(x_m),\ q_m = q(x_m),\ f_m = f(x_m),\ m = 0, ..., N.$$

There are $N+1$ equations in $N+3$ unknowns.The boundary conditions (2) enable us to eliminate the unknowns $\delta_{-1}, \delta_{N+1}$ from the system. Using Eq.(2) together with Eq.(10) the boundary parameters can be found as

$$\begin{aligned} \delta_{-1} &= \frac{-2\left(\sigma+1\right)}{\sigma^2}\delta_0 - \frac{1}{\sigma^3}\delta_1 + \frac{1}{\sigma^3}u_0, \\ \delta_{N+1} &= -\sigma^3\delta_{N-1} - 2\sigma\left(\sigma+1\right)\delta_N + u_N. \end{aligned}$$

With the replacement of these equalities in (11), the system returns a solvable tridiagonal band matrix system in dimension $(N+1)\times(N+1)$. A way of solving this type systems is using Thomas algorithm. By the substitution of the obtained parameters δ_m in Eq.(10) the solutions are computed at the knots x_m.

3 Numerical Experiments

We have tested the accuracy of the numerical methods on two examples. Errors are measured with the norm

$$L_\infty = \left|u - u_N\right|_\infty = \max_j\left|u_j - \left(u_N\right)_j\right|.$$

Since the boundary layers are at the right boundary in both examples, in order to minimize the error, we have searched the interval $(0, 1)$ for the best choice of the mesh ratio σ. Solution profiles are illustrated in Figs. 1-4 for the first example and in Figs. 5-8 for the second example. These figures are graphed for $N = 20$ and two different ε. In order to see the success of the numerical methods more clear, exact solutions and obtained results are illustrated together in all figures. In all figures, continuous line is used for the exact solutions and the lines$\cdots$o$\cdots$o$\cdots$, $\cdots$+$\cdots$+$\cdots$ are used for QM and CM respectively. Using uniform mesh leads to oscillations, seen in Figs. 1, 3, 5 and 7, in solution profiles because of the boundary layer. As observed from Figs. 2, 4, 6 and 8, after the best choice of the mesh ratio σ, these oscillations disappear. Using various ε and N, calculated numerical errors are tabulated and compared in Table 1 and Table 2 for the first and the second examples respectively.

Example 1 Our first example is

$$-\varepsilon u'' + u' = \exp(x), \, u(0) = u(1) = 0$$

with the exact solution

$$u(x) = \frac{1}{1-\varepsilon}\left[\exp(x) - \frac{1-\exp(1-1/\varepsilon) + (\exp(1)-1)\exp\left((x-1)/\varepsilon\right)}{1-\exp(-1/\varepsilon)}\right].$$

taken from [8].

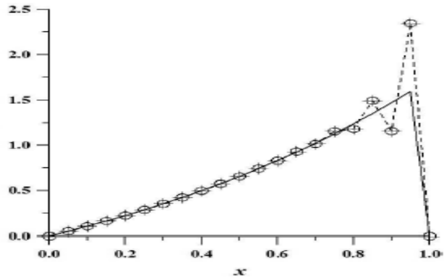

Figure 1: Uniform mesh for $\varepsilon = 0.01$

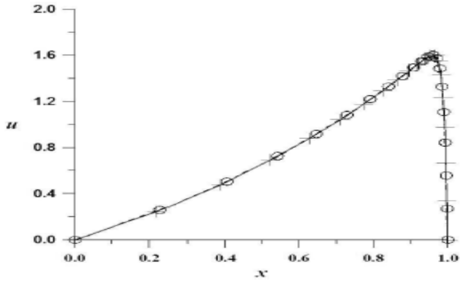

Figure 2: Graded mesh for $\varepsilon = 0.01$

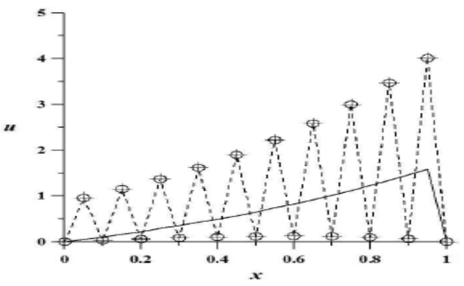

Figure 3: Uniform mesh for $\varepsilon = 0.001$

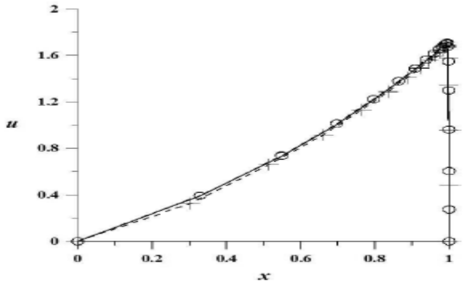

Figure 4: Graded mesh for $\varepsilon = 0.001$

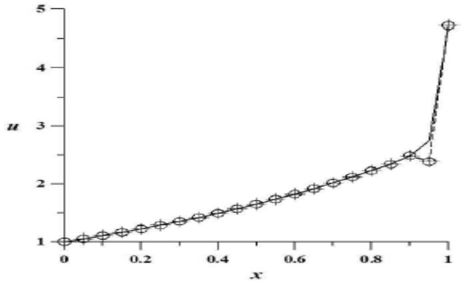

Figure 5: Uniform mesh for $\varepsilon = 0.01$

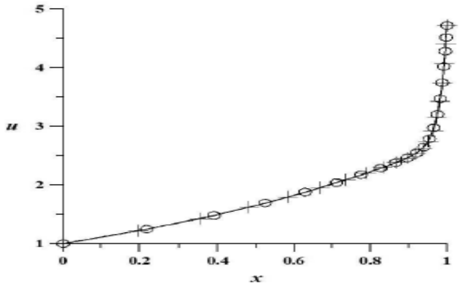

Figure 6: Graded mesh for $\varepsilon = 0.01$

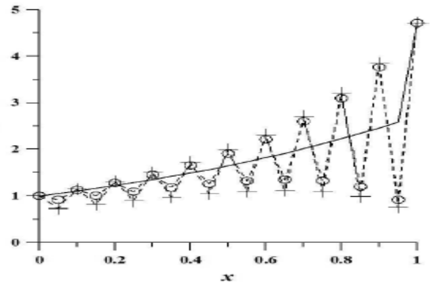

Figure 7: Uniform mesh for $\varepsilon = 0.001$

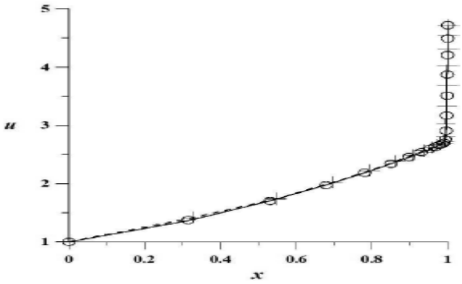

Figure 8: Graded mesh for $\varepsilon = 0.001$

Table 1: Numerical errors for Example 1

	ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
QM	$1/4$	0.14E-2	0.36E-3	0.91E-4	0.23E-4	0.65E-5	0.15E-5
CM		0.33E-2	0.81E-3	0.20E-3	0.51E-4	0.13E-4	0.35E-5
[6]		0.45E-3	0.11E-3	0.28E-4	0.70E-5	0.18E-5	0.44E-6
QM	$1/8$	0.43E-2	0.11E-2	0.27E-3	0.66E-4	0.17E-4	0.45E-5
CM		0.82E-2	0.21E-2	0.52E-3	0.13E-3	0.32E-4	0.85E-5
[6]		0.66E-3	0.16E-3	0.41E-4	0.10E-4	0.26E-5	0.64E-6
QM	$1/16$	0.68E-2	0.17E-2	0.43E-3	0.11E-3	0.27E-4	0.74E-5
CM		0.14E-1	0.35E-2	0.87E-3	0.22E-3	0.54E-4	0.14E-4
[6]		0.81E-3	0.21E-3	0.51E-4	0.13E-4	0.32E-5	0.80E-6
QM	$1/32$	0.96E-2	0.24E-2	0.60E-3	0.15E-3	0.38E-4	0.95E-5
CM		0.20E-1	0.51E-2	0.13E-2	0.32E-3	0.80E-4	0.20E-4
[6]		0.89E-3	0.23E-3	0.58E-4	0.15E-4	0.36E-5	0.91E-6
QM	$1/64$	0.13E-1	0.32E-2	0.81E-3	0.20E-3	0.51E-4	0.13E-4
CM		0.27E-1	0.70E-2	0.17E-2	0.44E-3	0.11E-3	0.27E-4
[6]		0.91E-3	0.24E-3	0.62E-4	0.16E-4	0.39E-5	0.98E-6
QM	$1/128$	0.17E-1	0.42E-2	0.11E-2	0.26E-3	0.66E-4	0.17E-4
CM		0.32E-1	0.90E-2	0.23E-2	0.57E-3	0.14E-3	0.36E-4
[6]		0.80E-3	0.24E-3	0.64E-4	0.16E-4	0.41E-5	0.10E-5
QM	$1/256$	0.21E-1	0.53E-2	0.13E-2	0.33E-3	0.83E-4	0.21E-4
CM		0.28E-1	0.11E-1	0.29E-2	0.72E-3	0.18E-3	0.45E-4
[6]		0.68E-3	0.21E-3	0.61E-4	0.16E-4	0.42E-5	0.11E-5
QM	$1/512$	0.25E-1	0.65E-2	0.16E-2	0.41E-3	0.10E-3	0.25E-4
CM		0.34E-1	0.14E-1	0.35E-2	0.88E-3	0.22E-3	0.55E-4
[6]		0.60E-3	0.18E-3	0.53E-4	0.15E-4	0.42E-5	0.11E-5
QM	$1/1000$	0.31E-1	0.77E-2	0.19E-2	0.48E-3	0.12E-3	0.30E-4
CM		0.42E-1	0.16E-1	0.42E-2	0.10E-2	0.26E-3	0.65E-4
[6]		0.55E-3	0.16E-3	0.46E-4	0.14E-4	0.39E-5	0.10E-5

Example 2 For the second example let us consider the differential equation taken from [4]

$$-\varepsilon u'' + \frac{1}{x+1}u' + \frac{1}{x+2}u = f(x), u(0) = 1 + 2^{-1/\varepsilon},\ u(1) = \exp(1) + 2$$

where

$$f(x) = \left(-\varepsilon + \frac{1}{x+1} + \frac{1}{x+2}\right)\exp(x) + \frac{1}{x+2}2^{-1/\varepsilon}\left(x+1\right)^{(1+1/\varepsilon)}.$$

The exact solution of this problem is given by

$$u(x) = \exp(x) + 2^{-1/\varepsilon}\left(x+1\right)^{(1+1/\varepsilon)}.$$

Table 2:Numerical errors for Example 2

	ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
QM	$1/4$	0.83E-5	0.23E-5	0.15E-5	0.42E-6	0.45E-6	0.48E-6
CM		0.39E-3	0.99E-4	0.25E-4	0.62E-5	0.15E-5	0.39E-6
[6]		0.12E-3	0.30E-4	0.75E-5	0.19E-5	0.47E-6	0.12E-6
QM	$1/8$	0.12E-3	0.30E-4	0.81E-5	0.23E-5	0.47E-6	0.46E-6
CM		0.24E-2	0.60E-3	0.15E-3	0.37E-4	0.93E-5	0.23E-5
[6]		0.52E-3	0.13E-3	0.33E-4	0.82E-5	0.20E-5	0.51E-6
QM	$1/16$	0.90E-3	0.22E-3	0.54E-4	0.14E-4	0.34E-5	0.14E-5
CM		0.68E-2	0.17E-2	0.43E-3	0.11E-3	0.27E-4	0.66E-5
[6]		0.18E-2	0.44E-3	0.11E-3	0.28E-4	0.70E-5	0.17E-5
QM	$1/32$	0.27E-2	0.67E-3	0.17E-3	0.42E-4	0.10E-4	0.35E-5
CM		0.13E-1	0.33E-2	0.83E-3	0.21E-3	0.52E-4	0.13E-4
[6]		0.45E-2	0.11E-2	0.27E-3	0.67E-4	0.17E-4	0.42E-5
QM	$1/64$	0.50E-2	0.13E-2	0.32E-3	0.79E-4	0.20E-4	0.55E-5
CM		0.21E-1	0.53E-2	0.13E-2	0.33E-3	0.83E-4	0.21E-4
[6]		0.82E-2	0.25E-2	0.60E-3	0.15E-3	0.37E-4	0.92E-5
QM	$1/128$	0.76E-2	0.19E-2	0.48E-3	0.12E-3	0.30E-4	0.75E-5
CM		0.29E-1	0.77E-2	0.19E-2	0.48E-3	0.12E-3	0.30E-4
[6]		0.80E-2	0.43E-2	0.13E-2	0.31E-3	0.77E-4	0.19E-4
QM	$1/256$	0.11E-1	0.26E-2	0.65E-3	0.16E-3	0.41E-4	0.10E-4
CM		0.38E-1	0.10E-1	0.26E-2	0.64E-3	0.16E-3	0.40E-4
[6]		0.45E-2	0.42E-2	0.22E-2	0.66E-3	0.16E-3	0.39E-4
QM	$1/512$	0.13E-1	0.33E-2	0.83E-3	0.21E-3	0.52E-4	0.13E-4
CM		0.52E-1	0.13E-1	0.33E-2	0.82E-3	0.21E-3	0.51E-4
[6]		0.19E-2	0.24E-2	0.22E-2	0.11E-2	0.33E-3	0.80E-4
QM	$1/1000$	0.13E-1	0.40E-2	0.10E-2	0.25E-3	0.63E-4	0.16E-4
CM		0.44E-1	0.16E-1	0.40E-2	0.10E-2	0.25E-3	0.63E-4
[6]		0.62E-3	0.12E-2	0.13E-2	0.11E-2	0.56E-3	0.16E-3

4 Conclusion

Quadratic and cubic B-spline algorithms are applied to singularly perturbed problems. Difficulties arised from the modelling of the boundary layers in numerical methods are tried to overcome by using B-splines over the geometrically graded mesh. Simplicity of the adaptation of B-splines and obtaining acceptable solutions can be noted as advantages of given numerical methods. Consequently, in getting the numerical solution of the differential equations having boundary layers, B-spline collocation methods over the geometrically graded mesh are advisable.

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