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Homotopy Perturbation Method for Solving Thomas-Fermi Equation Using Pade Approximants

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Abstract: In this paper, we apply a relatively new technique which is called the homotopy perturbation method (HPM) for solving Thomas-Fermi equation which plays an important role in mathematical physics. The initial slope of the Thomas-Fermi potential $y'(0)$ is computed by converting the obtained series solution into several diagonal Pade approximants. The analytical results are calculated in terms of convergent series with easily computable components. Numerical results reveal the complete reliability of the proposed scheme.

Keywords: homotopy perturbation method; Thomas Fermi equation; pade approximants

1 Introduction

Many problems in mathematical physics and astrophysics can be modeled by the Thomas-Fermi equation which is of the form:

$$y''(x) = \frac{y^{3/2}}{x^{1/2}}. \tag{1}$$

The Thomas-Fermi equation has been used to model the effective nuclear charge in heavy atoms and to study the potentials and charge densities of atoms $[1-6, 17, 21, 22]$. Equation (1) has been the focus of many researchers; several techniques including variational approach, -expansion, Green's function and decomposition have been applied to solve the Thomas-Fermi model, see $[1-8, 11, 21, 22]$ and the references therein. Most of these presented methods have the inbuilt deficiencies like huge computational work and calculation of the so-called Adomian's polynomials. He $[8-10]$ developed the homotopy perturbation method (HPM) for solving linear, nonlinear, initial and boundary value problems by merging the standard homotopy and the perturbation. The homotopy perturbation method (HPM) was formulated by taking full advantage of the standard homotopy and perturbation methods. The HPM has been applied to a wide class of functional equations; see $[9-10, 12-20]$ and the references therein. In this method the solution is given in an infinite series usually converging to an accurate solution, see $8-10, 12-20]$. Inspired and motivated by the ongoing research in this area, we apply the homotopy perturbation method (HPM) for solving Thomas-Fermi equation. We also introduce an essential modification in the initial value which makes the application of the proposed method easier and improves the efficiency of the algorithm. The proposed mathematical technique is free from linearization, discretization, transformation or restrictive assumptions. Moreover, diagonal Pade approximants have been used to make the work more concise and to get a better understanding of the solution behavior. The combination of the series solution and Pade approximants is very effective and reliable since the power series in isolation are never useful to solve boundary value problems, see $[5, 12-20, 22]$ and the references therein.

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2 Homotopy Perturbation Method

To explain and to convey the idea of the homotopy perturbation method, we consider a general equation of the type,

$$L\left(u\right) = 0, \tag{2}$$

where L is any integral or differential operator. We define a convex homotopy $H(u,p)$ by

$$H\left(u,p\right) = \left(1-p\right)F\left(u\right) + pL\left(u\right), \tag{3}$$

where $F(u)$ is a functional operator with known solutions v_0, which can be obtained easily. It is clear that, for

$$H\left(u,p\right) = 0, \tag{4}$$

we have

$$H\left(u,0\right) = F\left(u\right), \quad H\left(u,1\right) = L\left(u\right).$$

This shows that $H\left(u,p\right)$continuously traces an implicitly defined curve from a starting point $H(v0,0)$ to a solution function $H(f,1)$. The embedding parameter monotonically increases from zero to unit as the trivial problem $F(u) = 0$ is continuously deforms the original problem $L(u) = 0$. The embedding parameter $p \in (0,1]$ can be considered as an expanding parameter $[9-15, 18-26]$. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter, $[9-15]$ to obtain

$$u = \sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots. \tag{5}$$

If $p \to 1$, then (5) corresponds to (3) and becomes the approximate solution of the form,

$$f = \lim_{p\to 1} u = \sum_{i=0}^{\infty} u_i = u_0 + u_1 + u_2 + \cdots. \tag{6}$$

It is well known that series (6) is convergent for most of the cases and also the rate of convergence is dependent on $L(u)$; see $[9-15]$. We assume that (6) has a unique solution. The comparisons of like powers of p give solutions of various orders.

3 Numerical Applications

In this section, we use the homotopy perturbation method (HTM) for solving Thomas-Fermi equation. We also introduce a slight modification in the selection of initial value which makes the application of the proposed algorithm simpler and improves its efficiency. To make the work more concise and for the better understanding of the solution behavior the series solution is replaced by the powerful diagonal Pade approximants.

Consider the Thomas-Fermi Eq. (1)

$$y''\left(x\right) = \frac{y^{3/2}}{x^{1/2}}$$

with boundary conditions

$$y\left(0\right) = 1, \qquad \lim_{x\to\infty} y\left(x\right) = 0. \tag{7}$$

Applying the homotopy perturbation method

$$y_0 + y_1 + y_2 + \cdots = y_0\left(x\right) + p\int_0^x \int_0^x x^{-1/2}\left(y_0 + py_1 + p^2 y_2 + \cdots\right)^{3/2} dx dx. \tag{8}$$

The given initial values admits the use of $y_0(x) = 1 + Bx$,where $B = y'(0)$, but we use the modified approach and take $y_0(x) = 1$. Comparing the co-efficient of like powers of p

$$
\begin{aligned}
p^{(0)} &: \ y_0(x) = 1, \\
p^{(1)} &: \ y_1(x) = Bx + \frac{4}{3}x^{3/2}, \\
p^{(2)} &: \ y_2(x) = \frac{2}{5}Bx^{5/2} + \frac{1}{3}x^3, \\
p^{(3)} &: \ y_3(x) = \frac{3}{70}Bx^{7/2} + \frac{2}{15}Bx^4 + \frac{2}{27}x^{9/2} + \frac{1}{3}x^3, \\
p^{(4)} &: \ y_4(x) = -\frac{1}{252}B^3x^{9/2} + \frac{1}{175}B^2x^5 + \frac{3}{70}B^2x^{7/2} + \frac{2}{27}x^{9/2}, \\
p^{(5)} &: \ y_5(x) = \frac{1}{1056}B^4x^{11/2} + \frac{4}{1575}B^3x^6 + \frac{557}{100100}B^2x^{13/2} + \frac{4}{693}Bx^7 \\
&\quad + \frac{101}{52650}x^{15/2}, \\
p^{(6)} &: \ y_6(x) = -\frac{3}{9152}B^5x^{13/2} - \frac{29}{24255}B^4x^7 - \frac{512}{351000}B^3x^{15/2} - \frac{46}{45045}B^2x^8 \\
&\quad - \frac{113}{1178100}Bx^{17/2} + \frac{23}{473850}x^9, \\
&\quad \vdots
\end{aligned}
$$

The series solution is given as

$$
\begin{aligned}
y(x) \ = \ & 1 + Bx + \frac{4}{3}x^{3/2} + \frac{2}{5}Bx^{5/2} + \frac{1}{3}x^3 + \frac{3}{70}Bx^{7/2} + \frac{2}{15}Bx^4 + \frac{2}{27}x^{9/2} \\
& - \frac{1}{252}B^3x^{9/2} + \frac{1}{175}B^2x^5 + \frac{3}{70}B^2x^{7/2} + \frac{2}{27}x\frac{1}{1056}B^4x^{11/2} + \frac{4}{1575}B^3x^6 \\
& + \frac{557}{100100}B^2x^{13/2} + \frac{4}{693}Bx^7 + \frac{101}{52650}x^{15/2}\frac{1}{1056}B^4x^{11/2} + \frac{4}{1575}B^3x^6 \\
& + \frac{557}{100100}B^2x^{13/2} + \frac{4}{693}Bx^7 + \frac{101}{52650}x^{15/2} - \frac{3}{9152}B^5x^{13/2} - \frac{29}{24255}B^4x^7 \\
& - \frac{512}{351000}B^3x^{15/2} - \frac{46}{45045}B^2x^8 - \frac{113}{1178100}Bx^{17/2} + \frac{23}{473850}x^9 + \cdots,
\end{aligned}
$$

Setting $x^{1/2} = t$ the series solution is obtained as

$$
\begin{aligned}
y(x) \ = \ & 1 + Bt + \frac{4}{3}t^3 + \frac{2}{5}Bt^5 + \frac{1}{3}t^t + \frac{3}{70}Bt^7 + \frac{2}{15}Bt^8 + \frac{2}{27}t^9 - \frac{1}{252}B^3t^9 \\
& + \frac{1}{175}B^2t^{10} + \frac{3}{70}B^2t^7 + \frac{2}{27}t^9 + \frac{1}{1056}B^4t^{11} + \frac{4}{1575}B^3t^{12} \\
& + \frac{557}{100100}B^2t^{13} + \frac{4}{693}Bt^{14} + \frac{101}{52650}t^{15} + \frac{1}{1056}B^4t^{11} + \frac{4}{1575}B^3t^{12} \\
& + \frac{557}{100100}B^2t^{13} + \frac{4}{693}Bt^{14} + \frac{101}{52650}t^{15} - \frac{3}{9152}B^5t^{13} - \frac{29}{24255}B^4t^{14} \\
& - \frac{512}{351000}B^3t^{15} - \frac{46}{45045}B^2t^{16} - \frac{113}{1178100}Bt^{17} + \frac{23}{473850}t^{18} + \cdots,
\end{aligned}
$$

Table 1: Pade approximants and initial slopes $[5, 29]$

Pade apparoximants	Initial slope $y'(0)$	Error (%)
$[2/2]$	-1.211413729	23.71
$[4/4]$	-1.550525919	2.36
$[7/7]$	-1.586021037	$12.9x10^{-2}$
$[8/8]$	-1.588076820	$3.66x10^{-4}$
$[10/10]$	-1.588076779	$3.64x10^{-4}$

4 Conclusion

In this paper, we applied the homotopy perturbation method (HPM) for solving Thomas-Fermi equation. Moreover, the powerful diagonal Pade approximants are applied to get a better understanding of the solution behavior. The proposed technique is employed without any linearization, transformation, discretization or restrictive assumptions. It may be concluded that the suggested algorithm is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems. The fact the HPM solves nonlinear problems without using Adomian's polynomials is a clear advantage of this technique over the decomposition method.

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