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First Integral Method for the Improved Modified KdV Equation

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Abstract: New exact solutions of one important partial differential equations are obtained by using the first integral method. The efficiency of the method is demonstrated by applying it for the improved modified KdV (IMKdV) equation.

Key words: First integral method; IMKdV equation; Nonlinear partial differential equations

1 Introduction

The investigation of the exact solution of nonlinear partial differential equations play an important role in the study of the nonlinear science. The exact solutions of the nonlinear equations facilitates the verification of the numerical solvers and aids in the stability analysis of solutions. In the past decade, there has been significant progress in the development of these methods such as Hirota bilinear method [1-2], homogenous balance method [3], variational iteration method [4-9], the Riccati expansion method with constant coefficients[10], the modified extended tanh-function method [11-12], the generalized hyperbolic function [13-14], cosine-function method [15], and the variable separation method [16-17]. The study of numerical methods for the solution of partial differential equations (PDEs) has enjoyed an intense period of activity over the last 40 years from both theoretical and practical points of view. Improvements in numerical techniques, together with the rapid advance in computer technology, have meant that many of the PDEs arising from engineering and scientific applications, which were previously intractable, can now be routinely solved [18]. In finite difference methods differential operators are approximated and difference equations are solved. In the finite element method the continuous domain is represented as a collection of a finite number N of sub-domains known as elements. The collection of elements is called the finite element mesh. The differential equations for time dependent problems are approximated by the finite element method to obtain a set of ordinary differential equations (ODEs) in time. These differential equations are solved approximately by finite difference methods. In all finite difference and finite elements it is necessary to have boundary and initial conditions. However, the Adomian decomposition method, which has been developed by George Adomian [19], depends only on the initial conditions and obtains a solution in series which converges to the exact solution of the problem.

The aim of this paper is extended the first integral method proposed by Feng [20-23] to find the exact solutions of the the improved modified KdV equation [24-28].

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2 The first integral method

Consider the nonlinear PDE:

$$F(u, u_t, u_x, u_{xx}, u_{xt}, ...) = 0 \tag{1}$$

where $u(x,t)$is the solution of the equation (1). We use the transformations

$$u(x,t) = f(\eta),\ \eta = k\,x + \omega\ t \tag{2}$$

where k and ω are constants. Based on this we obtain

$$\frac{\partial}{\partial t}(.) = \omega\ \frac{d}{d\eta}(.)\,,\ \frac{\partial}{\partial x}(.) = k\frac{d}{d\eta}(.)\,,\frac{\partial^2}{\partial x^2}(.) = k^2\ \frac{d^2}{d\eta^2}(.)\,,\ ... \tag{3}$$

We use (3) to change the PDE (1) to ODE:

$$G(f,\ f_\eta, f_{\eta\eta}, ...) = 0 \tag{4}$$

Next, we introduce a new independent variables

$$X(\eta) = f(\eta),\ Y = f_\eta(\eta) \tag{5}$$

which leads a system of ODEs

$$\begin{aligned} X_\eta(\eta) &= Y(\eta), \\ Y_\eta(\eta) &= F_1(X(\eta), Y(\eta)) \end{aligned} \tag{6}$$

By the qualitative theory of ordinary differential equations [18], if we can find the integrals to (6) under the same conditions, then the general solutions to (6) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are. We will apply the Division Theorem to obtain one first integral to (6) which reduces (4) to a first order integrable ordinary differential equation. An exact solution to (1) is then obtained by solving this equation. Now, let us recall the Division Theorem:

Division Theorem: Suppose that $P(\omega, z)$, $Q(\omega, z)$ are polynomials in $C[\omega, z]$ and $P(\omega, z)$ is irreducible in $C[\omega, z]$. If $Q(\omega, z)$ vanishes at all zero points of $P(\omega, z)$, then there exists a polynomial $G(\omega, z)$ in $C[\omega, z]$ such that

$$Q[\omega, z] = P[\omega, z]\, G[\omega, z]$$

3 Applications

In this section, we discuss the IMkdV equation which is nonlinear PDE by using the first integral method described in section 2 in different four casaes. Consider the IMKdV equation in the form

$$u_t + u^2\, u_x + u_{xxx} - u_{xxt} = 0 \tag{7}$$

Using (2) and (3) Eq.(1) becomes

$$\frac{d^2 f(\eta)}{d\eta^2} = \frac{\omega}{(\omega - k)k^2} f(\eta) + \frac{3}{3(\omega - k)k} f^3(\eta) \tag{8}$$

Using (5) we get

$$\overset{\bullet}{X}(\eta) = Y(\eta) \tag{9a}$$

$$\overset{\bullet}{Y}(\eta) = \frac{\omega}{(\omega - k)k^2} X(\eta) + \frac{3}{3(\omega - k)k} X^3(\eta) \tag{9b}$$

According to the first integral method, we suppose that $X(\eta)$ and $Y(\eta)$ are the nontrivial solutions of (9), and $q(X,Y) = \sum\limits_{i=0}^{m} a_i(X)\,Y^i = 0$ is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$q[X(\eta),Y(\eta)] = \sum_{i=0}^{m} a_i(X)\,Y^i = 0 \tag{10}$$

where $a_i(X)(i = 0,\ 1,\ \ldots,\ m)$ are polynomials of X and $a_m(X) \neq 0$. Eq.(10) is called the first integral to (9). Now we take four different cases for m.

Case I: Assuming that $m = 1$ in Eq.(10), due to the Division Theorem, there exists a polynomial $g(X) + h(X)\,Y$ in the complex domain $C[\omega, z]$ such that

$$\frac{dq}{d\eta} = \frac{\partial q}{\partial X}\frac{\partial X}{\partial \eta} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \eta} = (g(X) + h(X)\,Y)\sum_{i=0}^{1} a_i(X)\,Y^i \tag{11}$$

By equating the coefficients of Y^i $(i = 2,1,0)$ on both sides of Eq.(11), we have

$$\dot{a}_1(X) = h(X)\,a_1(X) \tag{12a}$$

$$\dot{a}_0(X) = \ g(X)\,a_1(X) + h(X)\,a_0(X) \tag{12b}$$

$$a_1(X)\,(\frac{\omega}{(\omega-k)k^2}X + \frac{3}{3(\omega-k)k}X^3) = g(X)\,a_0(X) \tag{12c}$$

Since $a_1(X)$ is a polynomial of X, from (12a), we deduce that $a_1(X)$ is a constant and $h(X) = 0$. For simplicity, we take $a_1(X) = 1$, and balancing the degrees of $g(X),\ \ a_1(X)$ and $a_0(X)$, we conclude that deg $g(X) = 1$ only. Now we discuss this case : if deg $g(X) = 1$, suppose that $g(X) = A_1\,X + B_0$, such that $A_1 \neq 0$ then we find $a_1(X)$, and $a_0(X)$.

$$a_0(X) = A_0 + B_0\,X\, + \frac{A_1\,X^2}{2} \tag{13}$$

where A_0 is arbitrary integration constant.

Substituting $a_0(X)$, $a_1(X)$ and $g(X)$ in (12c) and setting all the coefficients of powers X to be zero. Then we obtain a system of nonlinear algebraic equations

$$-A_0\,B_0 = 0,\ -A_0 A_1 - B_0^2 + \frac{\omega}{k^2(\omega-k)} = 0,\ -3A_1 B_0 = 0,\ -3A_1^2 + \frac{2}{k(\omega-k)} = 0 \tag{14}$$

solving the last algebraic equations , we obtain

$$A_0 = \frac{2+3A_1^2k^2}{2\,A_1k^2},\ \omega = \frac{2+3A_1^2k^2}{2\,A_1k^2},\ B_0 = 0 \tag{15}$$

Using (1) in (10), we obtain

$$Y = -\frac{2+3A_1^2k^2(1+X^2)}{2\,A_1k^2} \tag{16}$$

Combining (2) with (9), we obtain the exact solution to Eq.(8) as follows:

$$f(\eta) = -\frac{\sqrt{2+3A_1^2k^2}\ \tan(\sqrt{2+3A_1^2k^2}\,(\eta+\eta_0))}{A_1\,k} \tag{17}$$

where η_0 is an arbitrary integration constant. Then the exact solution to (7) can be written as

$$u(x,t) = -\frac{\sqrt{2+3A_1^2k^2}\ \tan(\frac{\sqrt{2+3A_1^2k^2}\,(\frac{2\,t}{3A_1^2k^2}+k(x+t)+\eta_0)}{2\,k})}{A_1\,k} \tag{18}$$

This is a new solution for the IMKdV equation.

Case II: Assuming that $m = 2$ in Eq.(10), due to the Division Theorem, there exists a polynomial $g(X) + h(X)\,Y$ in the complex domain $C[X, Y]$ such that

$$\frac{dq}{d\eta} = \frac{\partial q}{\partial X}\frac{\partial X}{\partial \eta} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \eta} = (g(X) + h(X)\,Y)\sum_{i=0}^{2} a_i(X)\,Y^i \tag{19}$$

By equating the coefficients of $Y^i\,(i = 3,\ 2,\ 1,\ 0)$ on both sides of Eq.(19), we have

$$\overset{\bullet}{a}_2(X) = h(X)\,a_2(X) \tag{20a}$$

$$\overset{\bullet}{a}_1(X) = g(X)\,a_2(X) + h(X)\,a_1(X) \tag{20b}$$

$$\overset{\bullet}{a}_0(X) = -2a_2\,(\frac{\omega}{(\omega-k)k^2}X + \frac{3}{3(\omega-k)k}X^3) + g(X)\,a_1(X) + h(X)a_0(X) \tag{20c}$$

$$a_1(X)\,(\frac{\omega}{(\omega-k)k^2}X + \frac{3}{3(\omega-k)k}X^3) = g(X)\,a_0(X) \tag{20d}$$

Since $a_2(X)$ is a polynomial of X, from (20a), we deduce that $a_2(X)$ is a constant and $h(X) = 0$. For simplicity, we take $a_2(X) = 1$, and balancing the degrees of $g(X)$, $a_1(X)$ and $a_0(X)$, we conclude that deg $g(X) = 1$ only. Now we discuss this case. Suppose that $g(X) = A_1\,X + B_0$then we find $a_1(X)$ and $a_0(X)$.

$$a_1(X) = \frac{A_1}{2}X^2 + B_0\,X + A_0 \tag{21}$$

$$a_0(X) = d + A_0\,B_0X + \tfrac{1}{2}\,A_0A_1\,X^3 + \tfrac{(4+3A_1^2k(k-\omega))}{24\,k(k-\omega)}X^4 + \tfrac{A_0A_1k^2(k-\omega)+B_0^2k^2(k-\omega)+2\omega}{2k^2(k-\omega)}X^2 \tag{22}$$

where $A_0,\ d$ are arbitrary integration constants.

Substituting $a_0(X)$, $a_1(X)$, $a_2(X)$ and $g(X)$ in the equation (20d) and setting all the coefficients of powers X to be zero. Then we obtain a system of nonlinear algebraic equations

$$\begin{array}{l} -B_0\,d = 0,\ -A_0B_0^2 - A_1d + \frac{A_0\omega}{k^2(\omega-k)} = 0, \\ \frac{B_0(-4\omega+3A_0A_1k^2(\omega-k)+B_0^2k^2(\omega-k))}{k^2(\omega-k)} = 0, \\ \frac{A_0k(-2-3A_1^2k(\omega-k)-3A_1(2B_0^2k^2(\omega-k)+3\omega)}{k^2(\omega-k)} = 0, \\ -\frac{5A_1(8+3A_1^2k(\omega-k)}{k(\omega-k)} = 0 \end{array} \tag{23}$$

Solving it, we obtain

$$d = \frac{64 + 48\,A_1^2k^2 + 9A_1^4k^4}{16\,A_1^2k^4},\ A_0 = \frac{8 + 3A_1^2k^2}{2A_1k^2},\ \omega = \frac{8 + 3A_1^2k^2}{3A_1k^2},\ B_0 = 0 \tag{24}$$

Using (24) in (10), we obtain two equal roots for Y

$$Y = \frac{-8 - 3A_1^2k^2 - A_1^2k^2X^2}{4A_1k^2}, \tag{25}$$

Combining (25) with (9), we obtain the exact solution to Eq.(8) as follows:

$$f(\eta) = -\frac{\sqrt{8 + 3A_1^2k^2}\ \tan(\frac{\sqrt{8+3A_1^2k^2}(\eta+\eta_0)}{4\,k})}{A_1k} \tag{26}$$

where η_0 is an arbitrary integration constant. Then the exact solution to (7) can be written as

$$u(x,t) = -\frac{\sqrt{8 + 3A_1^2k^2}\ \tan(\frac{\sqrt{8+3A_1^2k^2}(\frac{8\,t}{3A_1^2k}+k(x+t)+\eta_0)}{4\,k})}{A_1k} \tag{27}$$

This is a new solution for the IMKdV equation.

Case III: Assuming that $m = 3$ in equation (10), due to the Division Theorem, there exists a polynomial $g(X) + h(X)\,Y$ in the complex domain $C[X, Y]$ such that

$$\frac{dq}{d\eta} = \frac{\partial q}{\partial X}\frac{\partial X}{\partial \eta} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \eta} = (g(X) + h(X)\,Y)\sum_{i=0}^{3} a_i(X)\,Y^i \tag{28}$$

By equating the coefficients of Y^i $(i = 4, 3, 2, 1, 0)$ on both sides of Eq.(28), we have

$$\overset{\bullet}{a}_3(X) = h(X)\,a_3(X) \tag{29a}$$

$$\overset{\bullet}{a}_2(X) = g(X)\,a_3(X) + h(X)\,a_2(X) \tag{29b}$$

$$\overset{\bullet}{a}_1(X) = -3a_3\,(\frac{\omega}{(\omega - k)k^2}X + \frac{3}{3(\omega - k)k}X^3) + g(X)\,a_2(X) + h(X)a_1(X) \tag{29c}$$

$$\overset{\bullet}{a}_0(X) = -2a_2(\frac{\omega}{(\omega - k)k^2}X + \frac{3}{3(\omega - k)k}X^3) + g(X)\,a_1(X) + h(X)a_0(X) \tag{29d}$$

$$a_1(X)\,(\frac{\omega}{(\omega - k)k^2}X + \frac{3}{3(\omega - k)k}X^3) = g(X)\,a_0(X) \tag{29e}$$

Since $a_3(X)$ is a polynomial of X, from (29a), we deduce that $a_2(X)$ is a constant and $h(X) = 0$. For simplicity, we take $a_3(X) = 1$, and balancing the degrees of $g(X), a_2(X),\ a_1(X)$ and $a_0(X)$, we conclude that deg $g(X) = 1$ only. Now we discuss this case. Suppose that $g(X) = A_1\,X + B_0$ then we find $a_1(X)$ and $a_0(X)$.

$$a_2(X) = \frac{A_1}{2}X^2 + B_0\,X + A_0, \tag{30}$$

$$a_1(X) = A_0 + B_0B_1X + \tfrac{1}{2}\,B_0A_1\,X^3 + \tfrac{(2+A_1^2k(k-\omega))}{8\,k(k-\omega)}X^4 + \tfrac{A_1B_1k^2(k-\omega)+B_0^2k^2(k-\omega)+3\omega}{2k^2(k-\omega)}X^2, \tag{31}$$

$$a_0(X) = \frac{\begin{array}{l}720dk^2(k-\omega) + X(360A_0k^2(2B_0 + A_1X)(k-\omega) + X(120B_0^3k^2X(k-\omega)\\ +180B_0^2k^2(2B_1 + A_1X^2)(k-\omega) + 6B_0X(22kX^2 + 60A_1B_1k^2(k-\omega)\\ +15A_1^2k^2X^2(k-\omega) + 140\omega) + 5(6B_1(4kX^2 + 3A_1^2k^3X^2 + 24\omega - 3A_1^2k^2X^2\omega)\\ +A_1X^2(14kX^2 + 3A_1^2k^3X^2 + 90\omega - 3A_1^2k^2X^2\omega))))\end{array}}{720\,k^2(k-\omega)} \tag{32}$$

where $A_0,\ d$ and B_1 are arbitrary integration constants.

Substituting $a_0(X)$, $a_1(X)$, $a_2(X)$, $a_3(X)$ and $g(X)$ in the Eq.(29e) and setting all the coefficients of powers X to be zero. Then we obtain a system of nonlinear algebraic equations and solving it we get

$$d = 0,\ A_0 = 0,\ B_1 = \frac{3A_1}{2} + \frac{1}{A_1k^2}, \omega = k + \frac{2}{3A_1^2k},\ B_0 = 0 \tag{33a}$$

$$\begin{aligned} d &= \frac{8 + 36A_1^2k^2 + 54A_1^4k^4 + 27A_1^6k^6}{6A_1^3k^6}, B_1 = \frac{2 + 3A_1^2k^2}{2A_1k^2},\\ \omega &= \frac{2 + 3A_1^2k^2}{3A_1k^2}, B_0 = 0,\ A_0 = \frac{4 + 12A_1^2k^2 + 9A_1^4k^4}{3A_1^2k^4}\end{aligned} \tag{33b}$$

$$\begin{aligned} d &= \frac{216 + 108A_1^2k^2 + 18A_1^4k^4 + A_1^6k^6}{8A_1^3k^6},\ A_0 = \frac{3}{4}(A_1^2 + \frac{36}{A_1^2k^4} + \frac{12}{k^2}),\\ B_1 &= \frac{3(6 + A_1^2k^2)}{2A_1k^2},\ \omega = \frac{6 + A_1^2k^2}{A_1k^2},\ B_0 = 0\end{aligned} \tag{33c}$$

$$d = \frac{A_0(2 + 3A_1^2k^2)}{2A_1k^2},\ B_1 = \frac{2 + 3A_1^2k^2}{2A_1k^2}, \omega = \frac{2 + 3A_1^2k^2}{3A_1k^2}, B_0 = 0 \tag{33d}$$

Using (33a) in (10), we obtain two equal roots for Y

$$Y = \frac{-2 - 3A_1^2k^2 - A_1^2k^2X^2}{2A_1k^2} \tag{34a}$$

$$Y = -\frac{i\sqrt{-4X^2 - 6A_1^2k^2X^2 - A_1^2k^2X^4}}{2\,k} \tag{34b}$$

$$Y = \frac{i\sqrt{-4X^2 - 6A_1^2k^2X^2 - A_1^2k^2X^4}}{2\,k} \tag{34c}$$

Combining (34a) with (9), we obtain the exact solution to (7) can be written as

$$u(x,t) = -\frac{\sqrt{2+3A_1^2k^2}\,\tan(\frac{\sqrt{2+3A_1^2k^2}(\frac{2\,t}{3A_1^2k}+k(x+t)+\eta_0)}{2k})}{A_1k} \tag{35a}$$

also, from (34b) - (34c) we obtained new solution for the IMKdV equation

$$u(x,t) = -\frac{4\,i\,e^{\sqrt{4+6A_1^2k^2}(\frac{\frac{2\,t}{3A_1^2k}+k(x+t)+\eta_0)}{2k})}\sqrt{4+6A_1^2k^2}}{e^{\sqrt{4+6A_1^2k^2}(\frac{\frac{2\,t}{3A_1^2k}+k(x+t)+\eta_0)}{k})} + 4A_1^2k^2} \tag{35b}$$

$$u(x,t) = -\frac{4\,i\,e^{\sqrt{4+6A_1^2k^2}(\frac{\frac{2\,t}{3A_1^2k}+k(x+t)+\eta_0)}{2k})}\sqrt{4+6A_1^2k^2}}{1 + 4A_1^2e^{\sqrt{4+6A_1^2k^2}(\frac{\frac{2\,t}{3A_1^2k}+k(x+t)+\eta_0)}{k})}k^2} \tag{35c}$$

by the same method we get the following solution corresponding to (33b) – (33d),

$$u(x,t) = -\frac{\sqrt{2+3A_1^2k^2}\,\tan(\frac{\sqrt{2+3A_1^2k^2}(\frac{2\,t}{3A_1^2k}+k(x+t)+\eta_0)}{2k})}{A_1k} \tag{36}$$

$$u(x,t) = -\frac{\sqrt{3}\sqrt{6+A_1^2k^2}\,\tan(\frac{\sqrt{2+\frac{A_1^2k^2}{3}}(\frac{6\,t}{A_1^2k}+k(x+t)+\eta_0)}{2k})}{A_1k} \tag{37}$$

Case IV: Assuming that $m = 4$ in Eq.(10), due to the Division Theorem, there exists a polynomial $g(X) + h(X)\,Y$ in the complex domain $C[X, Y]$ such that

$$\frac{dq}{d\eta} = \frac{\partial\,q}{\partial\,X}\frac{\partial\,X}{\partial\,\eta} + \frac{\partial\,q}{\partial\,Y}\frac{\partial\,Y}{\partial\,\eta} = (g(X) + h(X)\,Y)\sum_{i=0}^{4} a_i(X)\,Y^i \tag{38}$$

By equating the coefficients of Y^i $(i = 5, 4, 3, 2, 1, 0)$ on both sides of Eq.(28), we have

$$\overset{\bullet}{a}_4(X) = h(X)\,a_4(X) \tag{39a}$$

$$\overset{\bullet}{a}_3(X) = \,g(X)\,a_4(X) + h(X)\,a_3(X) \tag{39b}$$

$$\overset{\bullet}{a}_2(X) = -4a_4\,(\frac{\omega}{(\omega-k)k^2}X + \frac{3}{3(\omega-k)k}X^3) + g(X)\,a_3(X) + h(X)a_2(X) \tag{39c}$$

$$\overset{\bullet}{a}_1(X) = -3a_3\,(\frac{\omega}{(\omega-k)k^2}X + \frac{3}{3(\omega-k)k}X^3) + g(X)\,a_2(X) + h(X)a_1(X) \tag{39d}$$

$$\overset{\bullet}{a}_0(X) = -2a_2\,(\frac{\omega}{(\omega-k)k^2}X + \frac{3}{3(\omega-k)k}X^3) + g(X)\,a_1(X) + h(X)a_0(X) \tag{39e}$$

$$a_1(X)\,(\frac{\omega}{(\omega-k)k^2}X + \frac{3}{3(\omega-k)k}X^3) = g(X)\,a_0(X) \tag{39f}$$

by the same method in the last cases we obtain the following solutions

$$u(x,t)=-\frac{\sqrt{8+3A_1^2k^2}\,\tan(\frac{\sqrt{8+3A_1^2k^2}(\frac{8\,t}{3A_1^2k}+k(x+t)+\eta_0)}{4\,k})}{A_1k} \tag{40}$$

$$u(x,t)=-\frac{4\,i\,e^{\frac{\sqrt{16+6A_1^2k^2}(\frac{8\,t}{3A_1^2k}+k(x+t)+\eta_0))}{4k}}\sqrt{16+6A_1^2k^2}}{e^{\frac{\sqrt{16+6A_1^2k^2}(\frac{8\,t}{3A_1^2k}+k(x+t)+\eta_0))}{2k}}+4A_1^2k^2} \tag{41}$$

$$u(x,t)=-\frac{4\,i\,e^{\frac{\sqrt{16+6A_1^2k^2}(\frac{8\,t}{3A_1^2k}+k(x+t)+\eta_0)}{4k}}\sqrt{16+6A_1^2k^2}}{1+4A_1^2e^{\frac{\sqrt{16+6A_1^2k^2}(\frac{8\,t}{3A_1^2k}+k(x+t)+\eta_0))}{2k}}k^2} \tag{42}$$

$$u(x,t)=-\frac{\sqrt{32+3A_1^2k^2}\,\tan(\frac{\sqrt{32+3A_1^2k^2}\,(\frac{32\,t}{3A_1^2k^2}+k(x+t)+\eta_0)}{8\,k})}{A_1\,k} \tag{43}$$

We have no need of discussion for the case $m\geq 5$ due to the fact that the polynomial equation with the degree greater than or equal to 5 in generally not solvable.

4 Conclusion

In this paper, the first integral method was applied successfully for solving nonlinear partial differential equations which is the IMKdV equation has been solved exactly. The first integral method described here is not efficient only but also has the merit of being widely applicable. Thus, we deduced that the proposed method can be extended to solve many nonlinear partial differential equations problems which are arising in the theory of solitons and other areas.

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