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# Exact Solution for High-Order Integro-Differential Equations by NHPM

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**Abstract:**In this letter, an analytic technique, namely the New Homotopy Perturbation Method (NHPM) is applied for solving high-order integro-differential equations. To illustrate the ability and reliability of the method, two examples are given, revealing its effectiveness and simplicity.

**Keywords:**High-order integro-differential equations, New homotopy perturbation method

## 1 Introduction

In every phenomenon in real life, there are many parameters and variables related to each other under the law imperious on that phenomenon. When the relations between the parameters and variables are presented in mathematical language we usually derive a mathematical model of the problem, which may be an equation, a differential equation, an integral equation, a system of integral equations and etc. To solve these equations, homotopy perturbation method has been proposed by He in 1998 [1-4]. Recently a great deal of interest has been focused on the applications of the homotopy perturbation method, well addressed in [5-16]. In this method the solution is considered as the summation of an infinite series which usually converges rapidly to the exact solutions. This method has been used to solve wide variety mathematical problems. In this paper, we propose a new modification of homotopy perturbation method to solve high-order integro-differential equations. These equations have attracted much attention in a variety of applied sciences. In section 2, a new homotopy perturbation method (NHPM) is used for solving high-order integro-differential equations. To illustrate and show the efficiency of the method two examples are presented in section 3. A conclusion is given in section 4.

## 2 Application of NHPM to high-order integro-differential equations

To illustrate the basic ideas of this method, let us consider the following integro-differential equation

$$y^{(n)}(x)+f(x)y(x)+\int_0^x k(x,t)y(t)dt=g(x), \tag{1}$$

with initial conditions

$$y(0)=\alpha_0, y'(0)=\alpha_1,\ldots,y^{(n)}(0)=\alpha_n. \tag{2}$$

For solving Eq. (1) by NHPM we construct the following homotopy

$$(1-p)(Y^{(n)}(x)-y_0(x))+p(Y^{(n)}(x)+f(x)Y(x)+\int_0^x k(x,t)Y(t)dt-g(x))=0, \tag{3}$$

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or equivalently,

$$Y^{(n)}(x) = y_0(x) - p(y_0(x) + f(x)Y(x) + \int_0^x k(x,t)Y(t)dt - g(x)). \tag{4}$$

Applying the inverse operator, $L^{-1} = \int_0^x \int_0^\eta \cdots \int_0^\tau (\,\cdot\,)\, d\xi \cdots d\mu\, d\eta$ to both sides of equation (4), we obtain

$$\begin{array}{l} Y(x) = Y(0) + xY'(0) + \frac{x^2}{2!}Y''(0) + \cdots + \frac{x^n}{n!}Y^{(n)}(0) \\ \quad + \int_0^x \int_0^\eta \cdots \int_0^\tau (y_0(\xi) - p(y_0(\xi) + f(\xi)Y(\xi) + \int_a^x k(\xi,t)Y(t)dt - g(\xi)))\, d\xi \cdots d\mu\, d\eta, \end{array} \tag{5}$$

where $Y(0) = \alpha_0, Y'(0) = \alpha_1, \ldots, Y^{(n)}(0) = \alpha_n$.

Suppose the solution of Eq. (5) have the following form

$$Y(x) = Y_0(x) + pY_1(x) + p^2Y_2(x) + \cdots, \tag{6}$$

where $Y_i(x)$, $i = 1, 2, 3, \ldots$ are functions which should be determined.

Now suppose that the initial approximation to the solutions $y_0(x)$ has the form

$$y_0(x) = \sum_{j=0}^{\infty} \alpha_j P_j(x), \tag{7}$$

where $\alpha_j$ are unknown coefficients, $P_0(x), P_1(x), P_2(x), \ldots$ are specific functions.

Substituting (6) into (5) and equating the coefficients of $p$ with the same power leads to

$$\begin{array}{l} Y(x) = Y(0) + xY'(0) + \frac{x^2}{2!}Y''(0) + \cdots + \frac{x^n}{n!}Y^{(n)}(0) \\ \quad + \int_0^x \int_0^\eta \cdots \int_0^\tau (y_0(\xi) - p(y_0(\xi) + f(\xi)Y(\xi) + \int_a^x k(\xi,t)Y(t)dt - g(\xi))\, d\xi \cdots d\mu\, d\eta \\ p^0 : Y_0(x) = \alpha_0 + \alpha_1 x + \alpha_2 \frac{x^2}{2!} + \cdots + \alpha_n \frac{x^n}{n!} + \int_0^x \int_0^\eta \cdots \int_0^\tau y_0(\xi)\, d\xi \cdots d\mu\, d\eta, \\ p^1 : Y_1(x) = -\int_0^x \int_0^\eta \cdots \int_0^\tau (y_0(\xi) + f(\xi)Y_0(\xi) + \int_a^x k(\xi,t)Y_0(t)dt)\, d\xi \cdots d\mu\, d\eta, \\ p^2 : Y_2(x) = -\int_0^x \int_0^\eta \cdots \int_0^\tau (f(\xi)Y_1(\xi) + \int_a^x k(\xi,t)Y_1(t)dt)\, d\xi \cdots d\mu\, d\eta, \\ p^3 : Y_3(x) = -\int_0^x \int_0^\eta \cdots \int_0^\tau (f(\xi)Y_2(\xi) + \int_a^x k(\xi,t)Y_2(t)dt)\, d\xi \cdots d\mu\, d\eta, \\ \vdots \\ p^j : Y_{i,j}(x) = -\int_0^x \int_0^\eta \cdots \int_0^\tau (f(\xi)Y_{j-1}(\xi) + \int_a^x k(\xi,t)Y_{j-1}(t)dt)\, d\xi \cdots d\mu\, d\eta, \\ \vdots \end{array} \tag{8}$$

Now if these equations be solved in a way that $Y_1(x) = 0$ , then equations (8) result in $Y_2(x) = Y_3(x) = \cdots = 0$ , therefore the exact solution can be obtained by using

$$y(x) = Y_0(x) = \alpha_0 + \alpha_1 x + \alpha_2 \frac{x^2}{2!} + \cdots + \alpha_n \frac{x^n}{n!} + \int_0^x \int_0^\eta \cdots \int_0^\tau y_0(\xi)\, d\xi \cdots d\mu\, d\eta. \tag{9}$$

It is worthwhile to note that if $Y_1(x)$ is analytic at $x = x_0$ , then their Taylor series

$$Y_1(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \tag{10}$$

can be used in Eqs. (8), where $a_0, a_1, a_2, \ldots$ are known coefficients and $\alpha_j$ are unknown ones, which must be computed. We would explain this method by considering two examples.

## 3 Illustrate examples

**Example 1:** Consider the following integro-differential equation taken from [17]

$$y'(x) + y(x) = 1 + 2x + \int_0^x x\,(1 + 2x)\, \mathrm{e}^{t(x-t)}\, y(t)\, dt, \;\; y(0) = 1. \tag{11}$$

The exact solution is $y(x) = e^{x^2}$ .

For solving this equation by the NHPM we consider the following homotopy

$$(1-p)(Y'(x)-y_0(x))+p\left(Y'(x)+Y(x)-1-2x-\int_0^x x\,(1+2x)\,e^{t(x-t)}\,Y(t)\,dt\right)=0. \quad (12)$$

By integration of equation (12) we have

$$Y(x)=Y(0)+\int_0^x y_0(\xi)\,d\xi-p\int_0^x\left(y_0(\xi)+Y(\xi)-1-2\xi-\int_0^x \xi\,(1+2\xi)\,e^{t(\xi-t)}\,Y(t)\,dt\right)d\xi=0. \quad (13)$$

Now suppose that the solution of equation (13) is in the following form

$$Y(x)=Y_0(x)+pY_1(x)+p^2Y_2(x)+p^3Y_3(x)+\cdots, \quad (14)$$

Substituting (14) into (13) and equating the coefficients of $p$ with the same power lead to

$$\begin{array}{l} p^0: Y_0(x)=Y(0)+\int_0^x y_0(\xi)\,d\xi,\\ p^1: Y_1(x)=-\int_0^x(y_0(\xi)+Y_0(\xi)-1-2\xi-\int_0^x \xi\,(1+2\xi)\,e^{t(\xi-t)}\,Y_0(t)\,dt)\,d\xi,\\ p^j: Y_j(x)=-\int_0^x(Y_{j-1}(\xi)-\int_0^x \xi\,(1+2\xi)\,e^{t(\xi-t)}\,Y_{j-1}(t)\,dt)\,d\xi,\ \ j=2,3,\ldots. \end{array}$$

Now assume that $y_0(x)=\sum_{n=0}^{\infty}a_nP_n(x),\ P_k(x)=x^k,\ Y(0)=1$ and the Taylor series of $Y_1(x)$ equal to zero. Then we have

$$\begin{array}{l} -a_0x+\left(1-\frac{a_1}{2}-\frac{a_0}{2}\right)x^2+\left(\frac{1}{3}-\frac{a_1}{6}-\frac{a_2}{3}\right)x^3+\left(\frac{1}{2}+\frac{a_0}{4}-\frac{a_2}{12}-\frac{a_4}{3}\right)x^4\\ +\left(\frac{1}{30}+\frac{2a_0}{5}+\frac{a_1}{10}-\frac{a_3}{20}-\frac{a_4}{5}\right)x^5+\left(\frac{1}{18}+\frac{a_0}{36}+\frac{a_1}{6}+\frac{a_2}{18}-\frac{a_4}{30}-\frac{a_5}{6}\right)x^6\\ +\left(\frac{1}{420}+\frac{a_0}{21}+\frac{a_1}{84}+\frac{2a_1}{21}+\frac{a_3}{28}-\frac{a_5}{42}-\frac{a_6}{7}\right)x^7+\cdots=0 \end{array}$$

This implies that

$$a_0=0,a_1=2,a_2=0,a_3=2,a_4=0,a_5=1,a_6=0,a_7=\frac{1}{3},\ldots$$

Therefore, the exact solution of the integro-differential equation (11) can be expressed as

$$\begin{array}{l} y(x)=Y_0(x)=1+a_0x+\frac{a_1}{2}x^2+\frac{a_2}{3}x^3+\frac{a_3}{4}x^4+\frac{a_4}{5}x^5+\cdots\\ \quad=1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\frac{x^8}{4!}+\cdots=\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}=e^{x^2}. \end{array}$$

**Example 2:** Consider the following equation taken from [18, 19]

$$y''(x)-xy(x)=g(x)+\int_0^x x^2e^ty(t)\,dt,\ y(0)=1,y'(0)=0, \quad (15)$$

where $g(x)=-(1+x)\,\cos x-\frac{x^2}{2}(e^x(\cos x+\sin x)-1)$

The exact solution is $y(x)=\cos x$.

For solving this system by the NHPM we consider the following homotopy

$$(1-p)(Y''(x)-y_0(x))+p(Y''(x)-xY(x)-g(x)-\int_0^x x^2e^tY(t)\,dt)=0. \quad (16)$$

By integration of equation (16) we have

$$\begin{array}{l} Y(x)=Y(0)+xY'(0)+\int_0^x\int_0^\tau y_0(\xi)\,d\xi\,d\tau\\ \quad -p\int_0^x\int_0^\tau(y_0(\xi)-\xi Y(\xi)-g(\xi)-\int_0^\xi \xi^2\,e^t\,Y(t)\,dt)\,d\xi\,d\tau, \end{array} \quad (17)$$

Now suppose that the solution of equation (17) is in the following form

$$Y(x)=Y_0(x)+pY_1(x)+p^2Y_2(x)+p^3Y_3(x)+\cdots, \quad (18)$$

Substituting (18) into (17) and equating the coefficients of $p$ with the same power lead to

$$p^0 : Y_0(x) = 1 + \int_0^x \int_0^\tau y_0(\xi)\, d\xi\, d\tau,$$
$$p^1 : Y_1(x) = -\int_0^x \int_0^\tau (y_0(\xi) - \xi Y_0(\xi) - g(\xi) - \int_0^\xi \xi^2\, \mathrm{e}^t\, Y_0(t)\, dt)\, d\xi\, d\tau,$$
$$p^j : Y_j(x) = -\int_0^x \int_0^\tau (-\xi Y_{j-1}(\xi) + \int_0^\xi \xi^2\, \mathrm{e}^t\, Y_{j-1}(t)\, dt)\, d\xi\, d\tau,\;\; j = 2, 3, \ldots$$

If we set the Taylor series of $Y_1(x)$ equal to zero, we will have

$$-\left(\tfrac{1}{2} + \tfrac{a_0}{2}\right) x^2 - \tfrac{a_1}{6} x^3 + \left(\tfrac{1}{24} - \tfrac{a_2}{12}\right) x^4 + \left(\tfrac{1}{40} + \tfrac{a_0}{40} - \tfrac{a_3}{20}\right) x^5 + \left(-\tfrac{1}{720} - \tfrac{a_4}{30} + \tfrac{a_1}{180}\right) x^6$$
$$+ \left(\tfrac{1}{336} + \tfrac{a_0}{252} - \tfrac{a_5}{42} + \tfrac{a_2}{504}\right) x^7 + \left(\tfrac{13}{5760} + \tfrac{a_0}{448} + \tfrac{a_3}{1120} - \tfrac{a_6}{56} + \tfrac{a_1}{1344}\right) x^8 + \cdots = 0.$$

It follows easily that

$$a_0 = -1, a_1 = 0, a_2 = \frac{1}{2}, a_3 = 0, a_4 = -\frac{1}{24}, a_5 = 0, a_6 = \frac{1}{720}, a_7 = 0, \ldots$$

Therefore, the exact solution of the integro-differential equation (15) can be expressed as

$$y(x) = Y_0(x) = 1 + \tfrac{a_0}{2} x^2 + \tfrac{a_1}{6} x^3 + \tfrac{a_2}{12} x^4 + \tfrac{a_3}{20} x^5 + \tfrac{a_4}{30} x^6 + \cdots$$
$$= 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \tfrac{x^6}{6!} + \tfrac{x^8}{8!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \tfrac{x^{2n}}{(2n)!} = \cos x.$$

## 4 Conclusion

A new homotopy perturbation method (NHPM) was successfully deduced for solving high-order integro-differential equations. In comparison with Radial basis function (RBF) [17], homotopy perturbation method (HPM) [18] Taylor polynomial method (TPM) [19] in the present method we achieve exact solutions while RBF, HPM and TPM can not give the exact solutions. Computations are performed using the Maple 10 package.

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