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On the Solutions of Time-fractional Bacterial Chemotaxis in a Diffusion Gradient Chamber

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Abstract: In this paper, a new solution constructed in power series for reaction-diffusion- chemotactic model of fractional-orders for bacterial chemotaxis in a diffusion gradient chamber. The fractional derivatives are described in the Caputo sense. The obtained results are presented graphically.

Keywords: Decomposition Method; Bacteria Growth; Gradient Chamber; Fractional Calculus

1 Introduction

The study of pattern formation in bacterial colonies is of particular interest both from the biological and physical points of view [1]. Growth and development of bacteria population shows a great variety of geometrical shapes. Adler [2-3] was the first to obtain bacterial population waves in the form of concentric circle. Golding et al. [4] are interested in the opposite case, where nutrient supply is limited. For a more realistic description of colonial development on a nutrient-poor surface, they take into account the interaction of bacteria with the nutrient field$n(x,t)$. Chemotaxis means changes in the movement of the cell in response to a gradient of certain chemical fields [5-8]. The movement is biased along the gradient either in the gradient direction or in the opposite direction. Miyata and Sasaki [9] estimate the distance between spots generated by the bacteria colony model with chemotactic activity.

A diffusion gradient chamber (DGC) is a bacterial growth device that can be used to study chemotaxis and other biological properties [10]. A DGC is square in shape and has a reservoir on each side. Our mathematical model of this device describes the motion of three chemical species: the cell population, the chemoattractant and the nutrient. It also describes the interaction between these species. Details on construction of the DGC and description of the biological experiments using DGCs are given in [10] and [11].

Our mathematical model of this device describes the motion of three chemical species: the cell population, the chemoattractant and the nutrient. It also describes the interaction between these species. Let $u(x,t)$, $g(x,t)$, $s(x,t)$ and $q(x,t)$ be the concentrations of cells, nutrient, chemoattractant substrate and oxygen. Then u, g, $s$and q are in general governed by the following equations:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= D_u\,\Delta u - \nabla(u\chi_1(s)\nabla s) - \nabla(u\chi_2(q)\nabla q) + f_1(g)\,u \\
\frac{\partial g}{\partial t} &= D_g\,\Delta g - f_1(g)\,u/Y_g \\
\frac{\partial s}{\partial t} &= D_s\,\Delta s - f_2(s)\,u/Y_s \\
\frac{\partial q}{\partial t} &= D_q\,\Delta q - f_3(q)\,u/Y_q
\end{aligned} \tag{1}$$

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$D_u > 0, D_g > 0, D_s > 0$,and $D_q > 0$ are constant diffusivities, $Y_g > 0$, $Y_s > 0$and $Y_q > 0$ are bacterial yield constants. The divergence terms simulate a random walk or a diffusion process, and the term, $f_1(g)u$, describes growth of cells due to nutrient conversion. Here we assume that loss of nutrient is due solely to uptake by cells. And the terms,$f_2(s)u$ and$f_3(q)u$, describe the rates at whic chemoattractant and oxygen are taken up by cells, however growth of cells does not depend on s and q[10]. The terms, $\nabla(u\chi_1(s)\nabla s)$ and $\nabla(u\chi_2(q)\nabla q)$describe directional random walks of cells that are modulated by the presence or absence of chemoattractants.

In this paper, we implemented the Adomian's decomposition method (ADM) [12,13] to the nonlinear time-fractional reaction diffusion model in one dimensional:

$$\begin{aligned}
\frac{\partial^\alpha u}{\partial t^\alpha} &= D_u\frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}(u\chi_1(s)\frac{\partial s}{\partial x}) - \frac{\partial}{\partial x}(u\chi_2(q)\frac{\partial q}{\partial x}) + f_1(g)u \\
\frac{\partial^\alpha g}{\partial t^\alpha} &= D_g\frac{\partial^2 g}{\partial x^2} - f_1(g)u/Y_g \\
\frac{\partial^\alpha s}{\partial t^\alpha} &= D_s\frac{\partial^2 s}{\partial x^2} - f_2(s)\,u/Y_s \\
\frac{\partial^\alpha q}{\partial t^\alpha} &= D_q\frac{\partial^2 q}{\partial x^2} - f_3(q)\,u/Y_q
\end{aligned} \tag{2}$$

As the initial condition, we set $u(x,0) = B_1e^{-B_2|x|}$, $g(x,0) = G_0$, $s(x,0) = s_0(x)$ and $q(x,0) = Q_0$, where $B_1 > 0$, $B_2 > 0, G_0 > 0, S_0 > 0$ and $Q_0 > 0$ are constants. The peaked function, $B_1e^{-B_2|x|}$, approximates a delta function inoculation of bacteria at the center of the device. The initial distributions of nutrient and oxygen are assumed to be uniform, and $0 \le s_0(x) \le S_0$ give the pre-established chemoattractant gradient We present a solution of a more general model of fractional reaction-diffusion (FRD) systems (2). The system is obtained from the standard diffusion-reaction systems (1) by replacing the first time derivative term by a fractional derivative of order $\alpha > 0$.

The reason of using fractional order differential equations (FOD) is that FOD are naturally related to systems with memory which exists in most biological systems. Also they are closely related to fractals which are abundant in biological systems. The results derived of the fractional system (2) are of a more general nature. Respectively, solutions to the fractional diffusion equation spread at a faster rate than the classical diffusion equation, and may exhibit asymmetry. However, the fundamental solutions of these equations still exhibit useful scaling properties that make them attractive for applications. We would like to put your attention that time fractional derivatives change also the solutions we usually get in standard reaction-diffusion systems (1).

The concept of fractional or non-integer order derivation and integration can be traced back to the genesis of integer order calculus itself [14]. Almost most of the mathematical theory applicable to the study of non-integer order calculus was developed through the end of 19th century. However it is in the past hundred years that the most intriguing leaps in engineering and scientific application have been found. The calculation technique has in some cases had to change to meet the requirement of physical reality. The use of fractional differentiation for the mathematical modeling of real world physical problems has been widespread in recent years, e.g. the modeling of earthquake, the fluid dynamic traffic model with fractional derivatives, measurement of viscoelastic material properties, etc.

The derivatives are understood in the Caputo sense. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses.

The Adomian's decomposition method will be applied for computing solutions to the systems of fractional partial differential equations considered in this paper. This method has been used to obtain approximate solutions of a large class of linear or nonlinear differential equations. It is also quite straightforward to write computer codes in any symbolic languages. The method provides solutions in the form of power series with easily computed terms. It has many advantages over the classical techniques mainly; it provides efficient numerical solutions with high accuracy, minimal calculations. Cherruault [15] proposed a new definition of the method and he then insisted that it would become possible to prove the convergence of the decomposition method. Cherruault and Adomian [16] proposed a new convergence series. A new approach of the decomposition method was obtained in a more natural way than was given in the classical presentation [17]. Recently, the application of the method is extended for fractional differential equations [18-21].

2 Fractional calculus

There are several approaches to the generalization of the notion of differentiation to fractional orders e.g. Riemann– Liouville, Grunwald–Letnikow, Caputo and Generalized Functions approach [22]. Riemann–Liouville fractional derivative is mostly used by mathematicians but this approach is not suitable for real world physical problems since it requires the definition of fractional order initial conditions, which have no physically meaningful explanation yet. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations [23]. Unlike the Riemann–Liouville approach, which derives its definition from repeated integration, the Grunwald–Letnikow formulation approaches the problem from the derivative side. This approach is mostly used in numerical algorithms.

Here, we mention the basic definitions of the Caputo fractional-order integration and differentiation, which are used in the up coming paper and play the most important role in the theory of differential and integral equation of fractional order.
The main advantages of Caputo's approach are the initial conditions for fractional differential equations with Caputo derivatives take on the same form as for integer order differential equations.

Definition 1 *The fractional derivative of $f(x)$ in the Caputo sense is defined as:*

$$D^{\alpha}f(x) = I^{m-\alpha}D^{m}f(x) = \frac{1}{\Gamma(m-\alpha)}\int_{0}^{x}(x-t)^{m-\alpha+1}f^{(m)}(t)dt,$$

for $m-1<\alpha\leq m$, $m\in N$, $x>0$. For the Caputo derivative we have $D^{\alpha}C=0$, C is constant

$$D^{\alpha}t^{n} = \left\{ \begin{array}{ll} 0, & (n\leq\alpha-1) \\ \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}t^{n-\alpha}, & (n>\alpha-1) \end{array} \right\}.$$

Definition 2 *For m to be the smallest integer that exceedsα, the Caputo fractional derivatives of order $\alpha>0$ is defined as*

$$D^{\alpha}u(x,t) = \frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \left\{ \begin{array}{ll} \frac{1}{\Gamma(m-\alpha)}\int_{0}^{t}(t-\tau)^{m-\alpha+1}\frac{\partial^{m}u(x,\tau)}{\partial\tau^{m}}d\tau, & for\ \mathrm{m}-1<\alpha<\mathrm{m} \\ \frac{\partial^{m}u(x,t)}{\partial t^{m}}, & for\,\alpha=m\in N \end{array} \right\}.$$

3 Adomian's decomposition method

In order to illustrate the advantages and the accuracy of the ADM for solving nonlinear fractional chemotactic model of bacteria colonies, we consider time-fractional chemotactic model of bacteria colonies (2) with

$$\chi_1(s) = ck_1\frac{D\quad k_1}{(D\,k_1+s)^2},\quad \chi_2(q) = c\,k_2\frac{D\quad k_2}{(D\,k_2+q)^2},$$

$$f_1(g) = v_1\frac{g}{k_1+g},\quad f_2(s) = v_2\frac{s}{k_2+s},\quad f_3(q) = v_3\frac{q}{k_3+q},$$

$$\begin{aligned} D^{\alpha}u &= D_u\,L_{xx}u - c\,k_1L_x(D\,k_1\frac{u}{(D\,k_1+s)^2}L_xs) - c\,k_2L_x(D\,k_2\frac{u}{(D\,k_2+q)^2}L_xq) + v_1\frac{u\,g}{k_1+g}, \\ D^{\alpha}g &= D_g\,L_{xx}g - \frac{v_1}{Y_g}\frac{u\,g}{k_1+g}, \\ D^{\alpha}s &= D_sL_{xx}s - \frac{v_2}{Y_s}\frac{u\,s}{k_2+s}, \\ D^{\alpha}q &= D_q\,L_{xx} - \frac{v_3}{Y_q},\frac{u\,q}{k_3+q} \end{aligned} \tag{3}$$

where $D^{\alpha}=\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is the fractional operator, $L_x=\frac{\partial}{\partial x}$, and $L_{xx}=\frac{\partial^2}{\partial x^2}$.

Applying the inverse operators I^{α} to the system (3)

$$\begin{aligned} u &= I^{\alpha}(D_u L_{xx}u - c\,k_1 L_x(N_1(u,s)) - ck_2 L_x(N_2(u,q)) + v_1 N_3(u,g)),\\ g &= I^{\alpha}(D_g L_{xx}g - \frac{v_1}{Y_g}N_3(u,g),\\ s &= I^{\alpha}(D_s L_{xx}s - \frac{v_2}{Y_s}N_4(u,s)),\\ q &= I^{\alpha}(D_q\, L_{xx}q - \frac{v_3}{Y_q}N_5(u,q)), \end{aligned} \tag{4}$$

subject to the initial conditions

$$\begin{aligned} u(x,0) &= u_0(x),\\ g(x,0) &= g_0(x),\\ s(x,0) &= s_0(x),\\ q(x,0) &= q_0(x). \end{aligned} \tag{5}$$

The Adomian's decomposition method suggests that the linear terms $u(x,t),\ g(x,t),\ s(x,t)$ and $q(x,t)$ are decomposed by an infinite series of components

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t),\\ g(x,t) &= \sum_{n=0}^{\infty} g_n(x,t),\\ s(x,t) &= \sum_{n=0}^{\infty} s_n(x,t),\\ q(x,t) &= \sum_{n=0}^{\infty} q_n(x,t), \end{aligned} \tag{6}$$

and the nonlinear operator is defined by the infinite series of the so-called Adomian's polynomials

$$\begin{aligned} N_1(u,s) &= \sum_{n=0}^{\infty} A_n,\\ N_2(u,q) &= \sum_{n=0}^{\infty} B_n,\\ N_3(u,g) &= \sum_{n=0}^{\infty} C_n,\\ N_4(u,s) &= \sum_{n=0}^{\infty} D_n,\\ N_5(u,q) &= \sum_{n=0}^{\infty} E_n, \end{aligned} \tag{7}$$

where $u_n(x,t),\ g_n(x,t),\ s_n(x,t)$ and $q_n(x,t)$; $n \geq 0$ are the components of $u(x,t),\ g(x,t),\ s(x,t)$ and $q(x,t)$that will be elegantly determined.

Substituting (6) and (7) into(4) gives

$$\begin{aligned}
&\sum_{n=0}^{\infty} u(x,t) = u_0(x) + I^{\alpha}(D_u\, L_{xx} \sum_{n=0}^{\infty} u(x,t) - c\, k_1 L_x(\sum_{n=0}^{\infty} A_n) - c\, k_2 L_x(\sum_{n=0}^{\infty} B_n) + v_1 \sum_{n=0}^{\infty} C_n,\\
&\sum_{n=0}^{\infty} g(x,t) = g_0(x) + I^{\alpha}(D_g\, L_{xx} \sum_{n=0}^{\infty} g(x,t) - \frac{v_1}{Y_g} \sum_{n=0}^{\infty} C_n,\\
&\sum_{n=0}^{\infty} s(x,t) = s_0(x) + I^{\alpha}(D_s L_{xx} \sum_{n=0}^{\infty} s(x,t) - \frac{v_2}{Y_s} \sum_{n=0}^{\infty} D_n,\\
&\sum_{n=0}^{\infty} q(x,t) = q_0(x) + I^{\alpha}(D_q\, L_{xx} \sum_{n=0}^{\infty} q(x,t) - \frac{v_3}{Y_q} \sum_{n=0}^{\infty} E_n,
\end{aligned} \tag{8}$$

following Adomian analysis, the nonlinear system (8) is transformed into a set of recursive relations given by

$$\begin{aligned}
&u_0(x,t) = u_0(x),\\
&u_{n+1}(x,t) = I^{\alpha}(D_u L_{xx} u_n - c k_1 L_x(A_n) - c\, k_2 L_x(B_n) + v_1 C_n) \qquad n \geq 0,
\end{aligned} \tag{9}$$

$$\begin{aligned}
&g_0(x,t) = g_0(x),\\
&g_{n+1}(x,t) = I^{\alpha}(D_g\, L_{xx} g_n - \frac{v_1}{Y_g} C_n) \qquad n \geq 0,
\end{aligned} \tag{10}$$

$$\begin{aligned}
&s_0(x,t) = s_0(x),\\
&s_{n+1}(x,t) = I^{\alpha}(D_s L_{xx} s_n - \frac{v_2}{Y_s} D_n) \qquad n \geq 0.
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&q_0(x,t) = q_0(x),\\
&q_{n+1}(x,t) = I^{\alpha}(D_q L_{xx} q_n - \frac{v_3}{Y_q} E_n) \qquad n \geq 0,
\end{aligned} \tag{12}$$

whereA_n, B_n,C_n, $D_n$and E_n are the Adomian's polynomial calculated for all forms of nonlinearity according to specific algorithms constructed by Adomian as [24]

$$A_0 = k_1 D \frac{u_0\,(s_0)_x}{(D\,k_1 + s_0)^2} = u_0 \frac{\partial}{\partial x} \frac{s_0}{D\,k_1 + s_0},$$

$$A_1 = u_1 \frac{\partial}{\partial x} \frac{s_0}{D\,k_1 + s_0} + u_0 s_1 \frac{\partial}{\partial x} \frac{D\,k_1}{(D\,k_1 + s_0)^2}, \tag{13}$$

$$B_0 = k_2 D \frac{u_0\,(q_0)_x}{(D\,k_2 + q_0)^2} = u_0 \frac{\partial}{\partial x} \frac{q_0}{D\,k_2 + q_0},$$

$$B_1 = u_1 \frac{\partial}{\partial x} \frac{q_0}{D\,k_2 + q_0} + u_0 q_1 \frac{\partial}{\partial x} \frac{D\,k_2}{(D\,k_2 + q_0)^2}, \tag{14}$$

$$C_0 = \frac{u_0\, g_0}{k_1 + g_0},$$

$$C_1 = \frac{u_1\, g_0}{k_1 + g_0} + \frac{k_1\, u_0\, g_1}{(k_1 + g_0)^2}, \tag{15}$$

$$C_2 = \frac{u_2\, g_0}{k_1 + g_0} + \frac{k_1(u_0\, g_2 + u_1\, g_1)}{(k_1 + g_0)^2} - \frac{k_1^2 u_0\, g_1^2}{(k_1 + g_0)^3},$$

$$D_0 = \frac{u_0\, s_0}{k_2 + s_0},$$

$$D_1 = \frac{u_1\, s_0}{k_2 + s_0} + \frac{k_2\, u_0\, s_1}{(k_2 + s_0)^2}, \tag{16}$$

$$D_2 = \frac{u_2\, s_0}{k_2 + s_0} + \frac{k_2(u_0\, s_2 + u_1\, s_1)}{(k_2 + s_0)^2} - \frac{k_2^2 u_0 s_1^2}{(k_2 + s_0)^3},$$

$$E_0 = \frac{u_0\, q_0}{k_3 + q_0},$$

$$E_1 = \frac{u_1\, q_0}{k_3 + q_0} + \frac{k_3\, u_0\, q_1}{(k_3 + q_0)^2}, \tag{17}$$

$$E_2 = \frac{u_2\, q_0}{k_3 + q_0} + \frac{k_3(u_0\, q_2 + u_1\, q_1)}{(k_3 + q_0)^2} - \frac{k_3^2 u_0\, q_1^2}{(k_3 + q_0)^3}.$$

It is an essential feature of the decomposition method that the zeroth components $u_0(x,t)$,$g_0(x,t)$,$s_0(x,t)$ and $q_0(x,t)$ are defined always by all terms that arise from initial data and from integrating the inhomogeneous terms. The remaining pairs$(u_{n+1}, g_{n+1}, s_{n+1}, q_{n+1}, n \geq 0$, can be easily determined in a parallel manner. Additional pairs for the decomposition series normally account for higher accuracy.

Having determined the components of $u_n(x,t), g_n(x,t), s_n(x,t), q_n(x,t)$, the solutions of the system follow immediately in the form of a power series expansion upon using (6). The series obtained can be summed up in many cases to give a closed form solution for concrete problems, the n term approximants can be used for numerical purposes. Comparing the scheme presented above with existing techniques such as characteristics method and Riemann invariants, it is clear that the decomposition method introduces a fundamental qualitative difference in approach, because no assumptions are made. The approach is straightforward and the rapid convergence is guaranteed.

by using Eqs. (9-17), we could be able to calculate some of the terms of the decomposition series (6)as:

$$u_0(x,t) = B_1 e^{-B_2 x}, \quad u_1(x,t) = (D_u B_1 B_2^2 + \frac{v_1\, G_0 B_1}{k1 + G_0})\, e^{-B_2 x} \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2(x,t) = \frac{1}{Y_g(k1+G_0)^3}(B_1 e^{-2B_2\, x}(-B_1 G_0 k_1 v_1^2 + Y_g(k1+G_0)\, e^{B_2 x}(B_2^2 D_u(k_1+G_0) + v_1 G_0)^2)))\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$g_0(x,t) = G_0, \quad g_1(x,t) = \frac{-v_1 G_0 B_1}{Y_g(k_1+G_0)} e^{-B_2 x} \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$g_2(x,t) = -\frac{B_1 e^{-2\, B_2 x} v_1 G_0}{Y_g^2(k_1+G_0)^3}(-k_1 v_1 B_1 + Y_g(k_1+G_0)(B_2^2(D_g+D_u) \times (k_1+G_0) + v_1 G_0) e^{B_2 x}) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$s_0(x,t) = S_0, \quad s_1(x,t) = \frac{-v_2 S_0 B_1}{Y_s(k_2+S_0)} e^{-B_2 x} \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$s_2(x,t) = -\frac{B_1 e^{-2\, B_2 x} v_2 S_0}{Y_s^2(k_1+G_0)(k_2+S_0)^3}(B_1 k_2 v_2(k_1+G_0) - Y_s(k_2+S_0)^2(B_2^2(D_s+D_u)$$

$$\times(k_1+G_0) + v_1\, G_0) e^{B_2 x} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

and

$$q_0(x,t) = Q_0, \quad q_1(x,t) = \frac{-v_3 Q_0 B_1}{Y_q(k_3+Q_0)} e^{-B_2 x} \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$q_2(x,t) = -\frac{B_1 e^{-2\, B_2 x} v_3 Q_0}{Y_q^2(k_1+G_0)(k_3+Q_0)^3}(k_3 v_3 B_1(k_1+G_0) - Y_q(k_3+Q_0)^2(B_2^2(D_q+D_u),$$

$$\times(k_1+G_0) + v_1 G_0) e^{B_2 x}) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

and so on, the solution in a series form is:

$$u(x,t) = u_0 + u_1 + u_2 + \cdots\cdots, \quad g(x,t) = g_0 + g_1 + g_2 + \cdots\cdots$$

$$s(x,t) = s_0 + s_1 + s_2 + \cdots\cdots, \quad q(x,t) = q_0 + q_1 + q_2 + \cdots\cdots$$

See figures (1-4).

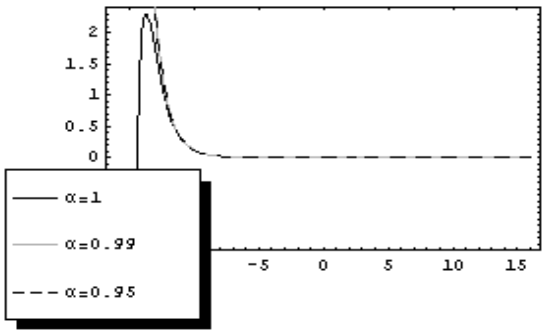

Figure 1: The Numerical results of $u(x,t)$ at $t = 0.01$

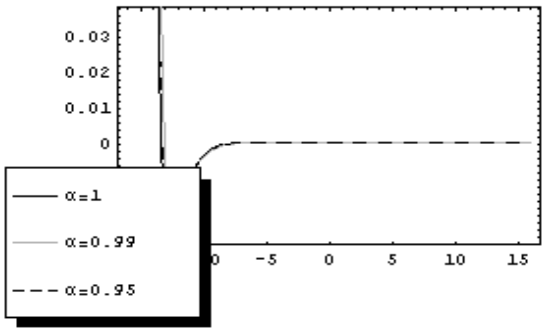

Figure 2: The Numerical results of $g(x,t)$

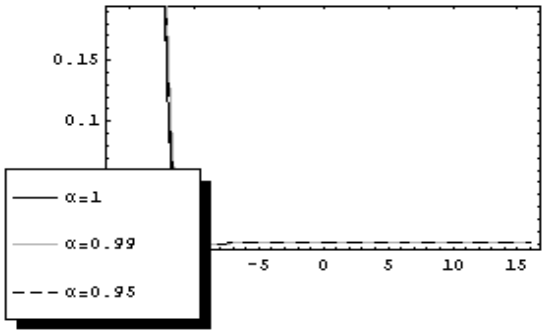

Figure 3: The Numerical results of $s(x,t)$

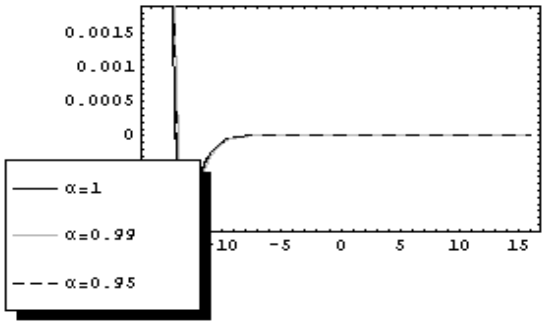

Figure 4: The Numerical results of $q(x,t)$

4 Conclusion

In this paper, Adomian's decomposition method has been successfully applied to time-fractional bacterial chemotaxis model in a diffusion gradient chamber. The main advantage of the methods is the fact that they do not require discretization of the variables, i.e. time and space, and thus it is not affected by computation round off errors and one is not faced with necessity of large computer memory and time.

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