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New Similarity Reductions and Exact Solutions of the Coupled KdV Equations with Variable Coefficients

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Abstract: Using the machinery of Lie group analysis the nonlinear variable coefficients coupled KdV equations is studied. Using the infinitesimal generators in the optimal system of subalgebra of the said Lie algebras, it is shown that this lead to different reductions in the form of coupled nonlinear ordinary differential equations. The search for solutions of these said equations has yielded certain exact solutions.

Keywords: coupled KdV equations; variable coefficients; symmetry method; tanh method

1 Introduction

In the last few years, the KdV and KdV-type equations are encountered in many apparently unrelated phenomena in different physical systems such as in plasma, fluid and in lattice vibrations of a crystal at low temperatures. All these applications start from a more or less general physical models and end up in the KdV equation by considering a special limit of the physical phenomena. In this sense, the KdV equation is universal due to the dispersion of linear waves is counterbalance by the non-linearity. The interaction of dispersion and non-linearity stabilities the solution. Keeping in view the above remarks and the rich treasure of nonlinear coupled KdV equations with variable coefficients we have in this work carried the application of the Lie group for obtaining exact solutions of nonlinear coupled KdV equations.

The non-linear system of partial differential equations which represents the KdV equations with variable coefficients is given by

$$u_t + \alpha(t)\, uu_x + \beta(t)\, vv_x + \gamma(t)\, u_{xxx} = 0, \tag{1.1}$$

$$v_t + \delta(t)\, uv_x + \gamma(t)\, v_{xxx} = 0. \tag{1.2}$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\delta(t)$ are all smooth functions of variable t only. Eqs. (1.1) and (1.2) are considered a simple generalization of Hirota-Satsuma coupled KdV equations [1] and if $v = v(x,t) = 0$, then the Eqs. (1.1) and (1.2) become a single KdV equation with variable coefficients which have been studied by some authors [2-5] . Recently, the coupled KdV equations with variable coefficients have been studied by using F-expansion method for a periodic wave solutions[6].

The plane of this paper is as follows: In section 2, we apply the symmetry method and derive the Lie algebra which then helps us to obtain the optimal system of generators. In section 3, we perform all the associated reductions and deduction of some exact solutions.

2 Lie symmetries

In order to determined the Lie transformations of the coupled KdV equations (1.1) and (1.2), we exploit the symmetry method due to Steinberg [7-11] as follows.

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Let the coupled equations (1.1) and (1.2) be considered as a manifold $\overline{N} = (N_1, N_2)$,

$$N_1(u,v) \equiv u_t + \alpha(t)\,uu_x + \beta(t)\,vv_x + \gamma(t)\,u_{xxx} = 0, \tag{2.1}$$

$$N_2(u,v) \equiv v_t + \delta(t)\,uv_x + \gamma(t)\,v_{xxx} = 0. \tag{2.2}$$

in the space of variables $\overline{X} = (t,x)\,, \overline{U} = (u,v)\,, \overline{U_t}, \overline{U}_x, \overline{U}_{xx}$ and $\overline{U}_{xxx}$.

The one-parameter group of point transformations of Eqs. (1.1) and (1.2) are as follows

$$\begin{aligned} t^* &= t + \varepsilon A\left(\overline{X}, \overline{\eta}\right) + O(\varepsilon^2) \\ x^* &= x + \varepsilon B\left(\overline{X}, \overline{\eta}\right) + O(\varepsilon^2), \\ \eta^* &= \overline{\eta} - \varepsilon \overline{C}\left(\overline{X}, \overline{\eta}\right) + \overline{O}(\varepsilon^2), \end{aligned} \tag{2.3}$$

where $\overline{\eta} = (u^*, v^*)$ and $\overline{C} = (C_1, C_2)$.

The infinitesimals A, B, C_1, C_2 are to be found under the following conditions:

$$F_i\left(N_i, \overline{\eta}, \overline{S}\right)\big|_{\overline{N}=\overline{0}} = \overline{0} \quad for\ \ i = 1,2 \tag{2.4}$$

In Eq. (2.4) , $F_i\left(N_i, \overline{\eta}, \overline{S}\right)$ denotes the Fréchet derivative of N_i at $\overline{\eta} = (u,v)$ in the direction of the symmetry operator $\overline{S} = (S_1, S_2)$ where the Fréchet derivative of N$(\overline{\eta})$ in the direction of $\overline{\eta_1} = (u_1, v_1)$ is given by

$$F_i\left(N_i, \overline{\eta}, \overline{\eta_1}\right) = \frac{d}{d\varepsilon}\left[N_i\left(\overline{\eta} + \varepsilon\overline{\eta_1}\right)\right]\big|_{\varepsilon=0} \tag{2.5}$$

where $i = 1,2$ and the symmetry operator $\overline{S} = (S_1, S_2)$ has the following form :

$$\begin{aligned} S_1(u) &\equiv A\left(\overline{X}, \overline{\eta}\right)\frac{\partial u}{\partial t} + B\left(\overline{X}, \overline{\eta}\right)\frac{\partial u}{\partial x} + C_1\left(\overline{X}, \overline{\eta}\right), \\ S_2(v) &\equiv A\left(\overline{X}, \overline{\eta}\right)\frac{\partial v}{\partial t} + B\left(\overline{X}, \overline{\eta}\right)\frac{\partial v}{\partial x} + C_2\left(\overline{X}, \overline{\eta}\right). \end{aligned} \tag{2.6}$$

Eq. (2.5) is used to fined the Fréchet derivative of each two nonlinear operators defined through Eqs. (2.1) and (2.2), we arrived at the following expressions

$$F_1\left(N_i, \overline{\eta}, \overline{S}\right) = [S_1]_t + \alpha(t)\,u\,[S_1]_x + \alpha(t)\,u_x\,[S_1] + \beta(t)\,v\,[S_2]_x + \beta(t)\,v_x\,[S_2] + \gamma(t)\,[S_1]_{xxx} = 0, \tag{2.7a}$$

$$F_2\left(N_i, \overline{\eta}, \overline{S}\right) = [S_2]_t + \delta(t)\,u\,[S_2]_x + \delta(t)\,v_x\,[S_1] + \gamma(t)\,[S_2]_{xxx} = 0. \tag{2.7b}$$

Substituting the values of different derivatives of S_1 and S_2 in Eqs. $(2.7.a)$ and $(2.7.b)$ and collecting the coefficients of like powers of derivatives of u and v in F_1 and F_2, we get a polynomial expression in $u_x, u_t, u_{xt}, u_{xx}, v_x, v_t, v_{xt}, v_{xx}, ...$, etc. On making use of Eqs. (1.1) and (1.2) in the polynomial expression for F_1 and F_2 rearranging terms of like powers of derivatives of u, v and equating them to zero, we arrive at the following equations.

$$\begin{gathered} A_x = A_u = A_v = 0, \\ B_u = B_v = B_{xx} = 0, \\ C_{1_{uv}} = C_{1_{ux}} = C_{1_{uu}} = 0, \\ 3\gamma B_x - \frac{d}{dt}(A\gamma) = 0, \\ \alpha C_1 + \alpha u\left(B_x - A_t\right) + \beta v C_{2_u} - uA\alpha_t + B_t = 0, \\ \beta C_2 + \beta v C_{2_v} + \beta v B_x - C_{1_u} v\beta - v\beta_t A - v\beta A_t = 0, \\ \beta v C_{2_x} + u\alpha C_{1_x} + \gamma C_{1_{xxx}} + C_{1_t} = 0, \end{gathered} \tag{2.8}$$

$$C_{2_{uv}} = C_{2_{vx}} = C_{2_{vv}} = C_{2_{ux}} = 0,$$
$$(\delta - \alpha)\, uC_{2_u} = 0,$$
$$\delta C_1 + \delta u B_x + B_t - u\delta A_t - Au\delta_t - \beta v C_{2_u} = 0,$$
$$C_{2_t} + \gamma C_{2_{xxx}} + u\delta C_{2_x} = 0.$$

Solving the system of partial differential Eqs. (2.8) for the infinitesimals A, B, C_1 and C_2 we get

$$A = \frac{1}{\Gamma'(t)}\left[3c_1\Gamma(t) + c_4\right], \gamma(t) = \Gamma'(t)$$
$$B = c_1 x + c_5, C_1 = c_2 u, C_2 = c_3 v. \tag{2.9}$$

The functions $\alpha(t), \beta(t), \gamma(t)$ and $\gamma(t)$ are governed under the following restrictions:

$$\frac{d}{dt}(A(t)\alpha(t)) - (c_1 + c_2)\alpha(t) = 0,$$
$$\frac{d}{dt}(A(t)\beta(t)) - (2c_3 + c_1 - c_2)\beta(t) = 0, \tag{2.10}$$
$$\frac{d}{dt}(A(t)\delta(t)) - (c_1 + c_2)\delta(t) = 0.$$

where $c_i, i = 1, 5$ are arbitrary constants.

The symmetry Lie algebra of Eqs. (1.1) and (1.2) L^5 is generated by the operators

$$\chi_1 = \left(\frac{3\Gamma(t)}{\Gamma'(t)}\right)\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \qquad \chi_2 = u\frac{\partial}{\partial u},$$
$$\chi_3 = v\frac{\partial}{\partial v}, \quad \chi_4 = \frac{1}{\Gamma'(t)}\frac{\partial}{\partial t}, \quad \chi_5 = \frac{\partial}{\partial x}.$$

where L^5 is the direct sum of $\chi_1, ..., \chi_5$ and the commutator table of it is given by

	χ_1	χ_2	χ_3	χ_4	χ_5
χ_1	0	0	0	$-3\chi_4$	$-\chi_5$
χ_2	0	0	0	0	0
χ_3	0	0	0	0	0
χ_4	$3\chi_4$	0	0	0	0
χ_5	χ_5	0	0	0	0

2.1 Classification of Group-Invariant solutions

In general, to each $s-$parameter subgroup H of the full symmetry group G of a system of differential equations in $p > s$ independent variables, there will correspond a family of group-invariant solutions. Since there are almost always an infinite number of such subgroups, it is not usually feasible to list all possible group-invariant solutions to the system. We need an effective, systematic means of classifying these solutions, leading to an "optimal system" of group -invariant solutions from which every other such solution can be derived. Since elements $g \in G$ not in the subgroup H will transform an $H-$invariant solution to some other group-invariant solutions, only those solutions not so related need be listed in our optimal system.

Let G be a lie group. An optimal system of $s-$parameter subgroups is a list of conjugacy inequivalent $s-$parameter subgroups with the property that any other subgroup is conjugate to precisely one subgroup in the list. The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. For one dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation(Olver 1986) [12] where the adjoint action is given by the Lie series

$$Ad(\exp(\varepsilon X))\, Y = Y - \varepsilon[X, Y] + \frac{\varepsilon^2}{2}[X, [X, Y]] - ... \tag{2.11}$$

where $[X, Y]$=$XY - YX$ is the commutator for the Lie algebra, and ε is a parameter.

2.2 One-dimensional optimal system of L^5

For the problem at hand, we only need a one dimensional optimal system of L^5, consider a general element of L^5, $X = \sum_{i=1}^{5} a_i\chi_i$, and ask whether Xcan be mapped to a new element X *under the general adjoint transformation (2.11) so as to simplify it as much as possible.

The adjoint table of L^5

Ad	χ_1	χ_2	χ_3	χ_4	χ_5
χ_1	χ_1	χ_2	χ_3	$\chi_4 \exp(3\varepsilon)$	$\chi_5 \exp(\varepsilon)$
χ_2	$\chi 1$	χ_2	χ_3	χ_4	χ_5
χ_3	χ_1	χ_2	χ_3	χ_4	χ_5
χ_4	$\chi_1 - 3\varepsilon\chi_4$	χ_2	χ_3	χ_4	χ_5
χ_5	$\chi_1 - \varepsilon\chi_5$	χ_2	χ_3	χ_4	χ_5

Following (Olver 1986) we deduce the following basic fields which form an optimal system for the coupled KdV system
$(i)\ \chi_1 + a\chi_2 + b\chi_{3,}\ (ii)\ \chi_2 + c\chi_3 + k\chi_4 \pm \chi_{5,}\ (iii)\ \chi_2 + m\chi_3 + n\chi_{4,}$
$(iv)\ \chi_3 + \omega\chi_4 \pm \chi_{5,}\ (v)\ \chi_3 + \rho\chi_{4,}\ (vi)\ \chi_4 \pm \chi_{5,}\ (vii)\ \chi_{4,}\ (viii)\ \chi_{5,}$
where a, b, c, k, m, n, ω and ρ are arbitrary constants. The discrete symmetry $(t, x, u, v) \rightarrow (t, -x, u, v)$ will map the negative cases in (ii) , (iv) and (vi) to the positive cases so we will deal only with the positive cases and also neglect cases (vii) and $(viii)$ since they give trivial reductions depend on one of the variables x or t not both of them.

The symmetry variables corresponding to each case are found by solving the auxiliary equation

$$\frac{dt}{A} = \frac{dx}{B} = \frac{-du}{C_1} = \frac{-dv}{C_2}. \tag{2.12}$$

3 Reductions and exact solutions

In the following we consider, corresponding to each generator in the optimal system of sub algebras, the reductions of PDEs $(1.1)$and (1.2) into ODEs in terms of similarity variable ζ and the new dependent variables F and G obtained using the auxiliary equation (2.12) .

3.1 Generator (i)

The generator (i) in the optimal system defines the similarity variable and similarity solutions as follows:

$$\begin{aligned} \zeta &= x\Gamma(t)^{\frac{-1}{3}}, \\ u(x,t) &= F(\zeta)\,\Gamma(t)^{\frac{-a}{3}}, \\ v(x,t) &= G(\zeta)\,\Gamma(t)^{\frac{-b}{3}}. \end{aligned} \tag{3.1}$$

and the coefficients are given by the following relations:

$$\begin{aligned} \alpha(t) &= k_1\Gamma'(t)\,\Gamma(t)^{\frac{a-2}{3}}, \\ \beta(t) &= k_2\Gamma'(t)\,\Gamma(t)^{\frac{2b-a-2}{3}}, \\ \delta(t) &= k_3\Gamma'(t)\,\Gamma(t)^{\frac{a-2}{3}}. \end{aligned} \tag{3.2}$$

where $k_1, k_2$and k_3 are arbitrary constants.

Using the similarity variable, the forms of the similarity solution and the coefficient functions, the coupled PDEs $(1.1)$and (1.2) is reduced to the following system of ODEs:

$$F''' + k_1FF' + k_2GG' - \frac{1}{3}\zeta F' - \frac{a}{3}F = 0,$$

$$G''' + k_3FG' - \frac{1}{3}\zeta G' - \frac{b}{3}G = 0. \tag{3.3}$$

To solve system (3.3), we seek a special solution of the form

$$F = A_0 + A_1\zeta + A_2\zeta^2 + \frac{A_3}{\zeta} + \frac{A_4}{\zeta^2},$$
$$G = B_0 + B_1\zeta + B_2\zeta^2 + \frac{B_3}{\zeta} + \frac{B_4}{\zeta^2}. \tag{3.4}$$

where $A_0, A_1, A_2, A_3, A_4, B_0, B_1, B_2, B_3$ and B_4 are arbitrary constants. Substituting of these expressions for F and G in (3.3) and equating the coefficients of different powers of ζ to zero, we get a system of algebraic equations, then we solve it with the aid of Maple program the following relations for the constants are obtained

$$A_0 = A_2 = A_3 = B_0 = B_2 = B_3 = 0, A_1 = \frac{1}{2k_3}, A_4 = -\frac{12}{k_3},$$
$$B_1 = \mp\frac{1}{2k_3}\sqrt{\frac{k_3 - k_1}{k_2}}, B_4 = \pm\frac{12}{k_3}\sqrt{\frac{k_3 - k_1}{k_2}}, a = b = \frac{1}{2} \tag{3.5}$$

So we get the following exact solution to the coupled Eqs. (1.1) and (1.2)

$$u(x,t) = \frac{x^3 - 24\Gamma(t)}{2k_3x^2\Gamma(t)^{\frac{1}{2}}},$$
$$v(x,t) = \frac{x^3 - 24\Gamma(t)}{2k_3x^2}\sqrt{\frac{k_3 - k_1}{2\Gamma(t)}}. \tag{3.6}$$

3.2 Generator (ii)

Corresponding this generator the associated similarity variable and similarity solution are given as follows:

$$\zeta = \frac{\Gamma(t)}{k} - x,$$
$$u(t,x) = F(\zeta)\exp\left(-\frac{\Gamma(t)}{k}\right), \tag{3.7}$$
$$v(t,x) = G(\zeta)\exp\left(-\frac{c\Gamma(t)}{k}\right).$$

and the coefficient functions are given by

$$\alpha(t) = k_4\Gamma'(t)\exp\left(\frac{\Gamma(t)}{k}\right),$$
$$\beta(t) = k_5\Gamma'(t)\exp\left(\frac{(2c-1)\Gamma(t)}{k}\right), \tag{3.8}$$
$$\delta(t) = k_6\Gamma'(t)\exp\left(\frac{\Gamma(t)}{k}\right),$$

where k_4, k_5 and k_6 are arbitrary constants.

The reduced system of ODEs. is

$$F''' + k_4FF' + k_5GG' - \frac{1}{k}F' + \frac{1}{k}F = 0,$$
$$G''' + k_6FG' - \frac{1}{k}G' + \frac{c}{k}G = 0. \quad k \neq 0 \tag{3.9}$$

By using the same behavior utilized in case 1 to solve the above system, it has the following exact solution

$$F(\zeta) = -\frac{k_5}{k}(1 + B_0 k) - \frac{k_5}{k(1 + k_4 k_5)}\zeta,$$

$$G(\zeta) = B_0 + \frac{1}{k(1 + k_4 k_5)}\zeta, \tag{3.10}$$

where B_0 is an arbitrary constant and $k_6 = -\frac{1}{k_5}, c = -\frac{1}{1+k_4 k_5}$. By substituting from (3.10)into (3.7), we have the following exact solution for system (1.1-1.2) corresponding to the conditions (3.8)

$$u(x,t) = \left(-\frac{k_5}{k}(1 + B_0 k) - \frac{k_5}{k(1 + k_4 k_5)}\left(\frac{\Gamma(t)}{k} - x\right)\right)\exp\left(-\frac{\Gamma(t)}{k}\right),$$

$$v(x,t) = \left(B_0 + \frac{1}{k(1 + k_4 k_5)}\left(\frac{\Gamma(t)}{k} - x\right)\right)\exp\left(\frac{\Gamma(t)}{k(1 + k_4 k_5)}\right). \tag{3.11}$$

3.3 Generator (iii)

For this generator under consideration the associated similarity variable and similarity solution are obtained as follows

$$\zeta = x,$$

$$u(x,t) = F(\zeta)\exp\left(-\frac{\Gamma(t)}{n}\right), \tag{3.12}$$

$$v(x,t) = G(\zeta)\exp\left(-\frac{m\Gamma(t)}{n}\right).$$

and the coefficient functions are given by

$$\alpha(t) = k_7 \Gamma'(t)\exp\left(\frac{\Gamma(t)}{n}\right),$$

$$\beta(t) = k_8 \Gamma'(t)\exp\left(\frac{(2m-1)\Gamma(t)}{n}\right), \tag{3.13}$$

$$\delta(t) = k_9 \Gamma'(t)\exp\left(\frac{\Gamma(t)}{n}\right).$$

where k_7, k_8 and k_9 are arbitrary constants.

The reduced system of ODEs. is

$$F''' + k_7 F F' + k_8 G G' - \frac{1}{n}F = 0,$$

$$G''' + k_9 F G' - \frac{m}{n}G = 0, \quad n \neq 0 \tag{3.14}$$

To solve the above system use the same method used in the above cases, then by substituting with this solution in (3.12) we get the following new exact solution for the variable coefficient coupled KdV system

$$u(x,t) = \exp\left(-\frac{\Gamma(t)}{n}\right)\left(A_0 + \frac{mx}{k_9 n}\right),$$

$$v(x,t) = -\frac{\sqrt{\frac{m(k_9 - k_7 m)}{k_8}}}{k_9 m n}\exp\left(-\frac{m\Gamma(t)}{n}\right)(A_0 k_9 + mx). \tag{3.15}$$

where A_0 is an arbitrary constant.

3.4 Generator (iv)

For this generator the associated similarity variable and similarity solution are expressed as follows

$$\zeta = \frac{\Gamma(t)}{w} - x, \quad w \neq 0$$
$$u(x,t) = F(\zeta), \tag{3.16}$$
$$v(x,t) = G(\zeta)\exp\left(-\frac{\Gamma(t)}{w}\right).$$

and the coefficient functions are given by

$$\alpha(t) = k_{10}\Gamma'(t),$$
$$\beta(t) = k_{11}\Gamma'(t)\exp\left(\frac{2\Gamma(t)}{w}\right), \tag{3.17}$$
$$\delta(t) = k_{12}\Gamma'(t).$$

where k_{10}, k_{11} and k_{12} are arbitrary constants.

The reduced system of ODEs. is

$$\begin{aligned} F''' + k_{11}GG' + k_{10}FF' - \frac{1}{w}F' &= 0, \\ G''' + k_{12}FG' - \frac{1}{w}\left(G' - G\right) &= 0. \end{aligned} \tag{3.18}$$

By solving the above system, we get the following exact solution for system $(1.1), (1.2)$ with $k_{12} = 0$, $k_{10} = -k_{11}$

$$\begin{aligned} u(x,t) &= 2\exp\left(-\frac{\Gamma(t)+4x}{8}\right) + \frac{1}{2k_{11}}, \\ v(x,t) &= 2\exp\left(\frac{\Gamma(t)-4x}{8}\right). \end{aligned} \tag{3.19}$$

3.5 Generator (v)

The associated similarity variable and similarity solution are expressed as follows

$$\begin{aligned} \zeta &= x, \\ u(x,t) &= F(\zeta), \\ v(x,t) &= G(\zeta)\exp\left(-\frac{\Gamma(t)}{\rho}\right). \end{aligned} \tag{3.20}$$

and the coefficient functions are given by

$$\begin{aligned} \alpha(t) &= k_{13}\Gamma'(t), \\ \beta(t) &= k_{14}\Gamma'(t)\exp\left(\frac{2\Gamma(t)}{\rho}\right), \\ \delta(t) &= k_{15}\Gamma'(t). \end{aligned} \tag{3.21}$$

where k_{13}, k_{14} and k_{15} are arbitrary constants.

The reduced system of ODEs. is

$$\begin{aligned} F''' + k_{14}GG' + k_{13}FF' &= 0, \\ G''' + k_{15}FG' - \frac{1}{\rho}G &= 0, \qquad \rho \neq 0 \end{aligned} \tag{3.22}$$

By solving the above system, we get the following exact solution for system (1.1) , (1.2)

$$\begin{aligned} u(x,t) &= a_0 + \frac{1}{k_{15}\rho}x, \\ v(x,t) &= \pm\left(a_0 + \frac{1}{k_{15}\rho}x\right)\exp\left(-\frac{\Gamma(t)}{\rho}\right). \end{aligned} \tag{3.23}$$

where a_0 is an arbitrary constant and $k_{14} = -k_{13}$.

3.6 Generator (vi)

For this generator the associated similarity variable and similarity solution are expressed as follows

$$\begin{gathered} \zeta = \Gamma(t) - x, \\ u(x,t) = F(\zeta), \\ v(x,t) = G(\zeta). \end{gathered} \tag{3.24}$$

and the coefficient functions are given by

$$\begin{aligned} \alpha(t) &= k_{16}\Gamma'(t), \\ \beta(t) &= k_{17}\Gamma'(t), \\ \delta(t) &= k_{18}\Gamma'(t). \end{aligned} \tag{3.25}$$

where k_{16}, k_{17} and k_{18} are arbitrary constants.

The reduced system of ODEs. is

$$\begin{aligned} F''' + k_{17}GG' + k_{16}FF' - F' &= 0, \\ G''' + k_{18}FG' - G' &= 0. \end{aligned} \tag{3.26}$$

Herein, we apply the tanh function method used in [13] to obtain an exact travelling wave solutions for system (3.26) . Let us assume that system (3.26) admits a solution in the form

$$\begin{aligned} F &= \sum_{i=0}^{m}\phi^i(\zeta)A_i, \\ G &= \sum_{j=0}^{n}\phi^j(\zeta)B_i \end{aligned} \tag{3.27}$$

where $\phi(\zeta)$ is a solution of the following Riccati equation

$$\phi'(\zeta) = r + \phi^2(\zeta). \tag{3.28}$$

This Riccati equation has the following solutions [13]

$$\begin{aligned} \phi(\zeta) &= -\sqrt{-r}\tanh(\sqrt{-r}\zeta), & r<0, \\ \phi(\zeta) &= -\sqrt{-r}\coth(\sqrt{-r}\zeta), & r<0, \\ \phi(\zeta) &= \sqrt{r}\tan(\sqrt{r}\zeta), & r>0, \\ \phi(\zeta) &= -\sqrt{r}\cot(\sqrt{r}\zeta), & r>0, \\ \phi(\zeta) &= -\frac{1}{\zeta}, & r=0. \end{aligned} \tag{3.29}$$

Balancing the highest linear term with the nonlinear term in system (3.26),we obtain $m, n = 2$. This suggests the following form for F and G :

$$\begin{aligned} F &= A_0 + A_1\phi + A_2\phi^2, \\ G &= B_0 + B_1\phi + B_2\phi^2. \end{aligned} \tag{3.30}$$

On combining the Eqs. (3.30) and (3.26), collecting the coefficients of $\phi^i, i = 0, 1, 2, ...$ and equating them to zero, we arrive at a set of algebraic equations which can be solved by the aid of Maple program, we arrive at the following two cases of solutions for the constants $A_0, A_1, A_2, B0, B_1$,and B_2.

$$r = \frac{1}{4},\ k_{16} = -k_{18},\ A_0 = -\frac{1}{k_{18}},\ A_2 = -\frac{12}{k_{18}},$$
$$B_0 = -\frac{1}{3}\sqrt{\frac{18}{k_{17}k_{18}}},\ B_2 = \pm\sqrt{\frac{288}{k_{17}k_{18}}},\quad A_1 = B_1 = 0,$$
$$r = -\frac{1}{4},\ k_{16} = -k_{18},\ A_0 = \frac{3}{k_{18}},\ A_2 = -\frac{12}{k_{18}}, \tag{3.31}$$
$$B_0 = \sqrt{\frac{18}{k_{17}k_{18}}}\ B_2 = \pm\sqrt{\frac{288}{k_{17}k_{18}}},\quad A_1 = B_1 = 0.$$

On combining Eqs.(3.30-3.31) with (3.29) and substituting in (3.24), we get the following exact solutions for the coupled KdV system

$$\begin{aligned} u(x,t) &= \frac{3}{k_{18}} sech^2(\frac{1}{2}(\Gamma(t) - x)), \\ v(x,t) &= -\sqrt{\frac{18}{k_{17}k_{18}}} sech^2(\frac{1}{2}(\Gamma(t) - x)) \end{aligned} \tag{3.32}$$

$$\begin{aligned} u(x,t) &= -\frac{3}{k_{18}} csch^2(\frac{1}{2}(\Gamma(t) - x)), \\ v(x,t) &= \sqrt{\frac{18}{k_{17}k_{18}}} csch^2(\frac{1}{2}(\Gamma(t) - x)) \end{aligned} \tag{3.33}$$

$$\begin{aligned} u(x,t) &= -\frac{1}{k_{18}}(1 + 3\tan^2(\frac{1}{2}(\Gamma(t) - x)), \\ v(x,t) &= \sqrt{\frac{18}{k_{17}k_{18}}}(\frac{1}{3} + \tan^2(\frac{1}{2}(\Gamma(t) - x))) \end{aligned} \tag{3.34}$$

$$\begin{aligned} u(x,t) &= -\frac{1}{k_{18}}(1 + 3\cot^2(\frac{1}{2}(\Gamma(t) - x)), \\ v(x,t) &= \sqrt{\frac{18}{k_{17}k_{18}}}(\frac{1}{3} + \cot^2(\frac{1}{2}(\Gamma(t) - x))) \end{aligned} \tag{3.35}$$

4 Remark

It's worth mentioning here that all our solutions are checked up by using the Mathematica program.

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