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Numerical Approach to Obtain Positive solution for Classes of Laplacian Systems

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Abstract: Using a numerical method based on sub-super solution, we will obtain positive solution to the coupled-system of boundary value problems of the form

$$-\Delta u = \lambda_1 f(v) + \mu_1 h(u) \quad in\ \Omega$$

$$-\Delta v = \lambda_2\, g(u) + \mu_2 \gamma(v) \quad in\ \Omega$$

$$u = 0 = v \qquad on\ \partial\Omega$$

where $-\Delta$ is the Laplacian operator $\lambda_1, \lambda_2, \mu_1, \mu_2$ are nonnegative parameters, and Ω is a bounded region in R^n, with smooth boundary $\partial\Omega$. We obtain numerically a large positive solution for $\lambda_1 + \mu_1$ and $\lambda_2 + \mu_2$ large when

$$\lim_{x\to\infty} \frac{f(M[g(x)])}{x} = 0$$

for every $M > 0$, $\lim_{x\to\infty} \frac{h(x)}{x} = 0$ and $\lim_{x\to\infty} \frac{\gamma(x)}{x} = 0$. In particular, we do not assume any sign conditions on $f(0), g(0), h(0)$, or $\gamma(0)$.

Keywords: positive solutions; sub and super-solutions; Laplacian systems
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1 Introduction

Consider positive solutions to the coupled-system of boundary value problems

$$\begin{cases} -\Delta u = \lambda_1 f(v) + \mu_1 h(u) \quad in\ \Omega \\ -\Delta v = \lambda_2\, g(u) + \mu_2 \gamma(v) \quad in\ \Omega \\ u = 0 = v \qquad on\ \partial\Omega \end{cases} \tag{1}$$

where $-\Delta$ is the laplacian operator, $\lambda_1, \lambda_2, \mu_1, \mu_2$ are nonnegative parameters, and Ω is a bounded region in R^n, with smooth boundary $\partial\Omega$. Dalmasso in [5] discussed the system (1) when $\mu_1 = \mu_2 = 0, \lambda_1 = \lambda_1$ and f, g are increasing and $f, g \geq 0$. In [6], Hai and Shivaji extended the study of [5] to the case when no sign conditions on $f(0)$ or $g(0)$ were required. Our results apply to the case when $f(0), g(0), h(0)$ or $\gamma(0)$ is negative (semipositone case), which is mathematically a challenging area in the study of positive solutions (see [2] and [7]). For a review on semipositone problems, see [3].

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We make the following assumptions:

$(H1)$ $f, g, h, \gamma \in C^1(0,\infty) \cap C[0,\infty)$ be monotone functions such that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = \lim_{x\to\infty} \gamma(x) = \infty$.

$(H2)$ $\lim_{x\to\infty} \frac{f(M[g(x)])}{x} = 0$ for every $M > 0$.

$(H3)$ $\lim_{x\to\infty} \frac{h(x)}{x} = \lim_{x\to\infty} \frac{\gamma(x)}{x} = 0$.

We will investigate the following results that have been proved in [1] ;

Theorem 1 : *Let (H1)-(H3) hold. Then (1) has a large positive solution provided* $\lambda_1 + \mu_1$ *and* $\lambda_2 + \mu_2$ *are large.*

In this paper, we want to investigate numerically positive solution of (1) and we employ the method of sub-super solutions to obtain the numerical solution. A super solution to (1) is defined as an ordered pair of smooth functions $(\bar{u}, \bar{v})$ on Ω satisfying

$$\begin{cases} -\Delta \bar{u} \geq \lambda_1 f(x,\bar{v}) + \mu_1 h(x,\bar{u}) & x \in \Omega \\ -\Delta \bar{v} \geq \lambda_2 g(x,\bar{u}) + \mu_2 \gamma(x,\bar{v}) & x \in \Omega \\ \bar{u} \geq 0; \;\; \bar{v} \geq 0 & x \in \partial\Omega. \end{cases} \tag{2}$$

Sub solutions are similarly defined with inequalities reversed. Let $D = [\underline{\rho}_1, \overline{\rho}_1] \times [\underline{\rho}_2, \overline{\rho}_2]$ where $\underline{\rho}_1 = \inf\{\underline{u}(x) : x \in \partial\Omega\}$, $\overline{\rho}_1 = \sup\{\overline{u}(x) : x \in \partial\Omega\}$, $\underline{\rho}_2 = \inf\{\underline{v}(x) : x \in \partial\Omega\}$, $\overline{\rho}_2 = \sup\{\overline{v}(x) : x \in \partial\Omega\}$.

Let σ be the principal eigenvalue and $\phi_1 > 0$ with $\|\phi_1\|_\infty = 1$ the corresponding eigenfunction of $-\Delta$ with the Dirichlet boundary conditions. It is well known that $\frac{\partial \phi_1}{\partial v} < 0$ on $\partial\Omega$ where v is the unit outward normal. Hence there exist $\delta > 0, \sigma \in (0,1]$ and $m > 0$ such that

$$\begin{cases} |\nabla \phi_1|^2 - \lambda_1 \phi_1^2 \geq m & on \;\; \overline{\Omega}_\delta \\ \quad \phi_1 \geq \sigma & on \;\; \Omega - \overline{\Omega}_\delta \end{cases} \tag{3}$$

where $\Omega_\delta := \{x \in \Omega | \; d(x, \partial\Omega) \leq \delta\}$.

Let $k_0 > 0$ be such that $f(x), g(x), h(x), \gamma(x) \geq -k_0$ for all $x \geq 0$.In [1] it was proved that:

$$(\psi_1, \psi_2) = \frac{1}{2}\left(\left[\frac{(\lambda_1 + \mu_1)}{m} k_0 \right], \left[\frac{(\lambda_2 + \mu_2)}{m} k_0 \right] \right) \phi_p^2$$

is a subsolution of (1) for $\lambda_1 + \mu_1$ and $\lambda_2 + \mu_2$ large. Next, let e be the solution of $-\Delta e = 1$ in Ω, $e = 0$ on $\partial\Omega$. Let $(z_1, z_2) := (Ce, (\lambda_2 + \mu_2)\, g(C\|e\|_\infty)\, e)$ for C large enough. Similarly it was proved in [1] that (z_1, z_2) is a supersolution of (1). Further $z_i \geq \psi_i, i = 1, 2$, for C large. Thus, there exists a solution (u, v) of (1) with $\psi_1 \leq u \leq z_1, \; \psi_2 \leq v \leq z_2$. This completes the proof of Theorem 1.

Here we discuss following example:

$$\begin{cases} -\Delta u = \lambda_1 (v^{\frac{1}{2}} + v^{\frac{1}{3}} - 1) + \mu_1 (u^{\frac{1}{5}} + u^{\frac{1}{3}} - 3) & x \in \Omega, \\ -\Delta v = \lambda_2 (u^{\frac{1}{4}} + u - 2) + \mu_2 (2v^{\frac{1}{2}} + 3v^{\frac{1}{4}} - 4) & x \in \Omega, \end{cases} \tag{4}$$

2 Numerical Results

It is well known that, if there exists sub- and super-solution $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ respectively such that $(\underline{u}, \underline{v}) < (\overline{u}, \overline{v})$ then (5) has a solution (u, v) such that $(u, v) \in [(\underline{u}, \underline{v}), (\overline{u}, \overline{v})]$.

Consider the coupled-system boundary value problems

$$\begin{cases} -\Delta u(x) = f(x, u, v) & x \in \Omega \\ -\Delta v(x) = g(x, u, v) & x \in \Omega \\ u(x) = 0 = v(x) & x \in \partial\Omega. \end{cases} \tag{5}$$

Since f, g are C^1 functions, there exists positive constants K_1, K_2 such that $\frac{\partial f}{\partial u} \geq -K_1$, and $\frac{\partial g}{\partial v} \geq -K_2$ on $\overline{\Omega} \times D$. Thus we can study the equivalent system

$$\begin{cases} -\Delta u(x) + K_1 u(x) = f(x,u,v) + K_1 u(x) = \hat{f}(x,u,v) & x \in \Omega \\ -\Delta v(x) + K_2 v(x) = g(x,u,v) + K_2 v(x) = \hat{g}(x,u,v) & x \in \Omega \\ u(x) = 0 = v(x) & x \in \partial\Omega. \end{cases} \tag{6}$$

The mapping $T : (u_1, v_1) \to (u_2, v_2), \;\; (u_2, v_2) = T(u_1, v_1)$

$$(u_1, v_1) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}] \quad \forall x \in \overline{\Omega}$$

where (u_2, v_2) is the unique solution of the coupled-system

$$\begin{cases} -\Delta u_2(x) + K_1 u_2(x) = f(x, u_1, v_1) + K_1 u_1(x) & x \in \Omega \\ -\Delta v_2(x) + K_2 v_2(x) = g(x, u_1, v_1) + K_2 v_1(x) & x \in \Omega \\ u_2(x) = 0 = v_2(x) & x \in \partial\Omega \end{cases} \tag{7}$$

satisfied the hypotheses of Schauder fixed point theorem, and then we can conclude that

$$\exists\, (u,v) \in D \quad T(u,v) = (u,v)$$

so (u, v) is a solution of (5) (see [4]).

By letting $\hat{f}(x,u,v) = f(x,u,v) + K_1 u$ and $\hat{g}(x,u,v) = g(x,u,v) + K_2 v$, we use the following iteration to obtain solution:

$$\begin{cases} u_0(x) = \underline{u}, \; v_0(x) = \overline{v} & \\ (\Delta - K_1) u_{n+1} = -\hat{f}(x, u_n, v_n) & x \in \Omega \\ & n = 0, 1, 2, ... \\ (\Delta - K_2) v_{n+1} = -\hat{g}(x, u_{n+1}, v_n) & x \in \Omega \\ u_{n+1} = 0 = v_{n+1} & x \in \partial\Omega \end{cases} \tag{8}$$

We can also use $u_0(x) = \overline{u}$ or $v_0(x) = \underline{v}$ as initial guesses. we use following algorithm.

sub- and super-solution algorithm

1. Find $u_0 = \underline{u}$ and $v_0 = \overline{v}$. Choose numbers $K_1, K_2 > 0$;

2. Solve the boundary value system (9);

3. If $\|u_{n+1} - u_n\| < \epsilon$ and $\|v_{n+1} - v_n\| < \epsilon$, output and stop. Else go to step 2.

We point out that in our experimental example, the eigenvalues and eigenfunctions are explicitly known. Specifically, in the ODE case when $\Omega = [0, 1]$ then $\lambda_i = (i\pi)^2$ and $\psi_i(x) = \sqrt{2}\, sin(i\pi x)$, and in the PDE case when $\Omega = [0, 1] \times [0, 1]$ then $\lambda_{ij} = (i^2 + j^2)\pi^2$ and $\phi_{ij}(x, y) = 2\, sin(i\pi x) sin(i\pi y)$. Let $\Omega = [0, 1] \times [0, 1]$ so we get $\phi_1(x, y) = sin(i\pi x) sin(i\pi y)$ which $\|\phi_1\|_\infty = 1$ in Eq.(3). According to the following table, we obtain $m = 1.3$, $\delta = 0.2$ and $\sigma = 0.34$.

Table 1: approximate value of $|\nabla \phi_1|^2 - \lambda_1 \phi_1^2$

$x \backslash\ y$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	1.3042	2.5010	4.0453	5.3470	5.9091	5.5168	4.3200	2.7757	1.4739
0.2	2.5010	1.0258	-0.5631	-1.6588	-1.8426	-1.0445	0.4308	2.0197	3.1153
0.3	4.0453	-0.5631	-5.8149	-9.7041	-10.7452	-8.5404	-3.9320	1.3198	5.2090
0.4	5.3470	-1.6588	-9.7041	-15.7160	-17.3981	-14.1078	-7.1020	0.9434	6.9552
0.5	5.9091	-1.8426	-10.7452	-17.3981	-19.2601	-15.6201	-7.8683	1.0342	7.6871
0.6	5.5168	-1.0445	-8.5404	-14.1078	-15.6201	-12.4996	-5.9383	1.5576	7.1250
0.7	4.3200	0.4308	-3.9320	-7.1020	-7.8683	-5.9383	-2.0491	2.3137	5.4837
0.8	2.7757	2.0197	1.3198	0.9434	1.0342	1.5576	2.3137	3.0136	3.3900
0.9	1.4739	3.1153	5.2090	6.9552	7.6871	7.1250	5.4837	3.3900	1.6437

According to Theorem 1, problem (4) has a large positive solution when $\lambda_1 + \mu_1$ and $\lambda_2 + \mu_2 > 0$ are large enough. For brevity we express just some of those numerical results :

Table 2: approximation of u for $\lambda_1 + \mu_1 = 20$

x / y	0.1	0.3	0.5	0.7	0.9
0.1	0.3125 + 1.2157i	0.9367 + 2.9021i	1.1797 + 3.4744i	0.9368 + 2.9021i	0.3125 + 1.2157i
0.3	0.9367 + 2.9021i	2.5053 + 7.0943i	3.0965 + 8.5458i	2.5053 + 7.0943i	0.9367 + 2.9021i
0.5	1.1797 + 3.4744i	3.0965 + 8.5458i	3.8107 +10.3141i	3.0963 + 8.5458i	1.1797 + 3.4744i
0.7	0.9368 + 2.9021i	2.5053 + 7.0943i	3.0963 + 8.5458i	2.5052 + 7.0943i	0.9367 + 2.9021i
0.9	0.3125 + 1.2157i	0.9367 + 2.9021i	1.1797 + 3.4744i	0.9367 + 2.9021i	0.3125 + 1.2157i

Table 3: approximation of v for $\lambda_1 + \mu_1 = 20$

x / y	0.1	0.3	0.5	0.7	0.9
0.1	-0.3573 + 1.8265i	-0.1043 + 3.8326i	0.0669 + 4.4077i	-0.1043 + 3.8326i	-0.3572 + 1.8265i
0.3	-0.1043 + 3.8326i	0.5900 + 8.5223i	0.9540 + 9.9429i	0.5900 + 8.5223i	-0.1043 + 3.8326i
0.5	0.0669 + 4.4077i	0.9540 + 9.9429i	1.3797 +11.6503i	0.9540 + 9.9429i	0.0669 + 4.4077i
0.7	-0.1043 + 3.8326i	0.5900 + 8.5223i	0.9540 + 9.9429i	0.5899 + 8.5223i	-0.1043 + 3.8326i
0.9	-0.3573 + 1.8265i	-0.1043 + 3.8326i	0.0669 + 4.4077i	-0.1043 + 3.8326i	-0.3572 + 1.8265i

Table 4: approximation of u for $\lambda_1 + \mu_1 = 200$

x / y	0.1	0.3	0.5	0.7	0.9
0.1	52.4070	121.9873	144.8306	121.9877	52.4070
0.3	121.9873	289.3530	345.2774	289.3537	121.9876
0.5	144.8306	345.2774	412.6501	345.2777	144.8308
0.7	121.9877	289.3537	345.2777	289.3538	121.9877
0.9	52.4070	121.9876	144.8308	121.9877	52.4070

Table 5: approximation of $v \times 10^{-3}$ for $\lambda_1 + \mu_1 = 200$

x / y	0.1	0.3	0.5	0.7	0.9
0.1	0.2327	0.5945	0.7270	0.5945	0.2328
0.3	0.5945	1.5235	1.8651	1.5235	0.5945
0.5	0.7270	1.8651	2.2842	1.8651	0.7270
0.7	0.5945	1.5235	1.8651	1.5235	0.5945
0.9	0.2328	0.5945	0.7270	0.5945	0.2328

Table 6: approximation of $u \times 10^{-5}$ for $\lambda_1 + \mu_1 = 20000$

x / y	0.1	0.3	0.5	0.7	0.9
0.1	0.2688	0.6289	0.7481	0.6289	0.2688
0.3	0.6289	1.4984	1.7909	1.4984	0.6289
0.5	0.7481	1.7909	2.1435	1.7909	0.7481
0.7	0.6289	1.4984	1.7909	1.4984	0.6289
0.9	0.2688	0.6289	0.7481	0.6289	0.2688

Table 7: approximation of $v \times 10^{-7}$ for $\lambda_1 + \mu_1 = 1000$

x / y	0.1	0.3	0.5	0.7	0.9
0.1	0.1148	0.2947	0.3610	0.2947	0.1148
0.3	0.2947	0.7582	0.9294	0.7582	0.2947
0.5	0.3610	0.9294	1.1398	0.9294	0.3610
0.7	0.2947	0.7582	0.9294	0.7582	0.2947
0.9	0.1148	0.2947	0.3610	0.2947	0.1148

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