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Exact Solutions for First Order Quasi Linear Partial Differential Equations: A New approach

Saeed Otarod[1], M. R. Khodakarim Ardekani[2] *
[1]Departement of Physics, Yasouj University, Yasouj, Iran
[2] Department of Physics, Zanjan University, Zanjan, Iran
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Abstract: Four first order nonlinear partial differential equations are chosen as examples. The way we treat the problem is different from what has been documented in the literature

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1 Introduction

Otarod 1980 [1], in an analysis of hydrodynamical equations governing interstellar media [2] (the well known Navier Stokes equations), proved that by the application of the separation method, particular solutions of these equations are derivable, with good consistency with some restricted physical boundary conditions. The solutions found through this method, were not general enough to include complicated realistic physical conditions. In order to improve the technique, the Author started to study the characteristics of simple nonlinear partial differential equations. In the first attempt, Burger's equation was selected as the subject of the study [3]. And the general solutions found were consistent with those documented in the literature. Later on Hopf's equation was considered [4], and in a very simple way the general solution of Hopf's equation was derived. The technique was very successful, such that when applied to Fischer equation, a variety of solutions were found for that[5]. Coming back to Navier Stokes equations, we regularly encountered the equations in the form $\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=\alpha v^n$. Since the solutions of these equations is critical for solving Navier Stokes equations, in this article, we will deal with these equations for $n=0$, $n=1$, $n=2$, and $n=n$, successively and will prove the effectiveness of the method.

2 Example 1

In Navier Stokes equations, we regularly come to the term $\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}$ [2]. In many physical situations, the solution of

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=\alpha, \tag{1}$$

is ultimately needed in order to be able to solve Navier Stokes equations.

The method used for solving the equations in this article is:

(a) Take $v=X(x)T(t)$.

Substituting this in equation (1) will lead to:

$$X(x)\frac{dT(t)}{dt}+T(t)^2X(x)\frac{dX(x)}{dx}=\alpha. \tag{2}$$

(b) Now, we will look for especial cases in which the variables can be separated .These are:

1. $\frac{dT(t)}{dt}=0$.

*Corresponding author**. *E-mail address*: sotarod mail.yu.ac.ir;Khodakarimyahoo.com

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This implies:

$$T(t) = \beta \tag{3}$$

and

$$\beta^2 X(x)\frac{dX(x)}{dx} = \alpha, \tag{4}$$

where β is a constant of integration.

The solution to equation (4) is:

$$X(x) = \pm\sqrt{\frac{2\alpha}{\beta^2}x + \gamma}, \tag{5}$$

where γ is a constant of integration.

Therefore as a particular solution we may write:

$$v_1 = \pm\sqrt{2\alpha x + \gamma\beta^2}. \tag{6}$$

2. Let us choose $\frac{dX(x)}{dx} = 0$.

This implies:

$X(x) = \lambda$(where λ is a constant) and when substituted in Eq(2) gives: $\lambda\frac{dT(t)}{dt} = \alpha$. Therefore, the second particular solution will be found as:

$$v_2 = \alpha t + \lambda\delta, \tag{7}$$

where δ is a constant of integration.

(c) To find more general solutions, take, $v = v(f(v_1, v_2))$ i.e. v is a function of an unknown function f, where f itself is a function of particular solutions v_1 and v_2.

Substituting this into Eq.(1) gives:

$$\frac{dv}{df}(\frac{\partial f}{\partial v_1}\frac{\partial v_1}{\partial t} + \frac{\partial f}{\partial v_2}\frac{\partial v_2}{\partial t}) + v\frac{dv}{df}(\frac{\partial f}{\partial v_1}\frac{\partial v_1}{\partial x} + \frac{\partial f}{\partial v_2}\frac{\partial v_2}{\partial x}) = \alpha. \tag{8}$$

From Eqs.(6) and (7) $\frac{\partial v_1}{\partial t} = \frac{\partial v_2}{\partial x} = 0$. Therefore Equation (8) will reduce to:

$$\frac{dv}{df}\frac{\partial f}{\partial v_2} + v\frac{dv}{df}(\frac{1}{v_1}\frac{\partial f}{\partial v_1}) = 1. \tag{9}$$

Since f is an unknown float function of v_1 and v_2, we are allowed to choose any desired form for that. The simplest choice could be:

$$\frac{\partial f}{\partial v_2} = \xi, \tag{10}$$

and

$$\frac{1}{v_1}\frac{\partial f}{\partial v_1} = \kappa, \tag{11}$$

where ξ and κ are constants. Simultaneous solution of Equation (10) and Equation (11) will give:

$$f = \frac{1}{2}\kappa v_1^2 + \xi v_2 + k_1. \tag{12}$$

Here k_1 is a constant of integration.

Using Eq.(10) and Eq.(11) in Eq.(9) results in:

$$\xi\frac{dv}{df} \pm \kappa v\frac{dv}{df} = 1, \tag{13}$$

which have a simple solution

$$v = \frac{-\xi \pm \sqrt{\xi^2 \pm 2\kappa(f + c)}}{\pm\kappa}, \tag{14}$$

where c is constant of integration. Since f is given by Eq.(12) in terms of x and t, it could be written as

$$v = \frac{-\xi \pm \sqrt{\xi^2 \pm 2\kappa(\alpha\kappa x + \alpha\xi t + \xi\lambda\delta + \frac{1}{2}\kappa\gamma\beta^2 + \kappa + c)}}{\pm\kappa}. \tag{15}$$

It is probable that by choosing other forms for f we may come to other solutions.

3 Example 2

As the second example we choose the following differential equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \alpha v. \tag{16}$$

This has a trivial solution, and we directly solve it by the application of separation method. The answer to this equation is

$$v = \frac{\alpha(\beta x + \delta)}{-e^{-\alpha t} + \beta}, \tag{17}$$

where β and δ are constants of integration.

We attract the readers, attention to the point that this is not the only possible solution to the above equation. To derive more solutions we follow the orders introduced in the previous section. i.e we assume $v = T(t)X(x)$. If we substitute this into Eq.(16) and divide the result by $T(t)X(x)$, we will have:

$$\frac{1}{T(t)}\frac{dT(t)}{dt} + T(t)\frac{dX(x)}{dx} = \alpha. \tag{18}$$

As in the previous section, we first suppose $\frac{dX(x)}{dx} = 0$ and, therefore, Eq.(18) gives $X(x) = \beta$, $T(t) = \gamma e^{\alpha t}$ and ultimately

$$v_1 = \beta\gamma e^{\alpha t}, \tag{19}$$

where β and γ are constants of integration. Secondly, we suppose $\frac{dT(t)}{dt} = 0$ which results in $T(t) = \lambda$ and $X(x) = \frac{\alpha}{\lambda}x + \mu$. Consequently, as the second particular solution, we will have:

$$v_2 = \alpha x + \lambda\mu, \tag{20}$$

where λ and μ are constants of integration. Here, there is another possible choice i.e $\frac{dX(x)}{dx} = \xi$ which by application to Equation (17) results in $X(x) = \xi x + \psi$ and $T(t) = \frac{\alpha}{\xi - e^{-\alpha t}}$, in which ξ is constant and ψ is a constant of integration. Therefore,

$$v_3 = \frac{\alpha(\xi x + \psi)}{\xi - e^{-\alpha t}}. \tag{21}$$

In fact, this is the same as Eq.(17).

So far, we have found three particular solutions. To find the general solution, we consider v to be $v = v(f(v_1, v_2))$. Substituting this into Eq.(16) yield:

$$\frac{dv}{df}\frac{\partial f}{\partial v_1}\frac{\partial v_1}{\partial t} + v\frac{dv}{df}\frac{\partial f}{\partial v_2}\frac{\partial v_2}{\partial x} = \alpha v. \tag{22}$$

Since f is a float function, we choose it such that:

$$\frac{\partial f}{\partial v_1}\frac{\partial v_1}{\partial t} = A, \tag{23}$$

and

$$\frac{\partial f}{\partial v_2}\frac{\partial v_2}{\partial x} = C \tag{24}$$

where A and C have been chosen to be constant.

Substituting v_1 from Eq.(19) into Eq.(22) will give

$$\alpha \frac{\partial f}{\partial v_1} v_1 = A. \tag{25}$$

By integration we come to

$$f = \frac{A}{\alpha} ln(v_1) + g(v_2), \tag{26}$$

where g is a function of v_2. Eqs.(26) and (24) along with Equation (19) will lead to

$$f = \frac{A}{\alpha} ln(v_1) + \frac{c}{\alpha} v_2. \tag{27}$$

If we use Eqs (23) and (24) in Eq.(22), we will come to

$$\frac{dv(A + Cv)}{v} = \alpha df. \tag{28}$$

Upon integration, the following will result:

$$Aln(v) + Cv = \alpha f, \tag{29}$$

or

$$v = e^{\frac{\alpha f - ALambertW(\frac{Ce^{\frac{\alpha f}{A}}}{A})}{A}}. \tag{30}$$

We note that f has already been found from Eq.(27) in terms of known function v_1 and v_2.

4 Example 3

The third choice is

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \alpha v^2. \tag{31}$$

As before, we choose $v = X(x)T(t)$. When this is inserted into the above equation, the following will result:

$$\frac{dT(t)}{dt} + T^2 \frac{dX(x)}{dx} = \alpha X(x)T(t)^2. \tag{32}$$

To find particular solutions, these choices are allowed

1. $\frac{dT(t)}{dt} = \beta T^2$.

Here β is a constant and T(t) will be found to be

$$T(t) = -\frac{1}{\beta t + \gamma}, \tag{33}$$

and

$$\beta + \frac{dX(x)}{dx} = \alpha X(x). \tag{34}$$

The answer to this equation is

$$X(x) = \frac{1}{\alpha}(e^{\alpha(x+c)} + \beta). \tag{35}$$

Therefore, v will be

$$v_1 = -\frac{e^{\alpha(x+c)} + \beta}{\alpha(\beta t + \gamma)}, \tag{36}$$

where γ and c are constants of integration.

2. $\frac{dT(t)}{dt} = 0$.

In this case, Eq.(32) reduces to $\frac{dX(x)}{dx} = \alpha X(x)$ and consequently $T(t) = \beta$, $X(x) = Ae^{\alpha x}$, ultimately

$$v_2 = \beta A e^{\alpha x}, \tag{37}$$

where β is a constant.
3. As the third choice, we take $\frac{dX(x)}{dx} = 0$

Therefore, $X(x) = \lambda$ and Eq.(32) reduces to

$$\frac{dT(t)}{dt} = \alpha\lambda T(t)^2. \tag{38}$$

Thus, $T(t) = -\frac{1}{\alpha\lambda t + c}$ and,

$$v_3 = -\frac{\lambda}{\alpha\lambda t + c}, \tag{39}$$

where λ is a constant and c is a constant of integration.
4. Now, by choosing $v = v(f(v_2, v_3))$, Eq.(32) changes to

$$\frac{dv}{df}\left(\frac{\partial f}{\partial v_2}\frac{\partial v_2}{\partial t} + \frac{\partial f}{\partial v_3}\frac{\partial v_3}{\partial t}\right) + v\frac{dv}{df}\left(\frac{\partial f}{\partial v_2}\frac{\partial v_2}{\partial x} + \frac{\partial f}{\partial v_3}\frac{\partial v_3}{\partial x}\right) = \alpha v^2. \tag{40}$$

Since $\frac{\partial v_2}{\partial t} = 0$ and $\frac{\partial v_3}{\partial x} = 0$, Eq.(40) reduces to:

$$\frac{\alpha\lambda}{\gamma}\frac{dv}{df}\frac{\partial f}{\partial v_3}v_3^2 + \alpha v\frac{dv}{df}\frac{\partial f}{\partial v_2}v_2 = \alpha v^2. \tag{41}$$

Now, suppose $\frac{\partial f}{\partial v_3}v_3^2 = \xi$, $\frac{\partial f}{\partial v_2}v_2 = \delta$. Simultaneous solution of these two equations will result in

$$f = \frac{-\xi}{v_3} + \delta ln(v_2), \tag{42}$$

where ξ and δ are constants.

Also considering the above choices in Eq.(41) will lead to

$$\frac{\xi\alpha\lambda}{\gamma}\frac{dv}{df} + \alpha\delta v\frac{dv}{df} = \alpha v^2. \tag{43}$$

This equation has the following solution

$$v = e^{\frac{\delta LambertW(\frac{\lambda\xi e^{\frac{-f}{\delta}}}{\gamma\delta}) + f}{\delta}}, \tag{44}$$

where f is given by Eq.(42).

5 Example 4

In the most general case, we may write the following equation:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \alpha v^n. \tag{45}$$

As before, we choose $v = X(x)T(t)$. Substituting this into the above equation will give:

$$\frac{dT(t)}{dt} + T(t)^2\frac{dX(x)}{dx} = \alpha X(x)^{n-1}T(t)^n. \tag{46}$$

Again as in the previous subsections, the following options is allowed.
1. $\frac{dT(t)}{dt} = 0$.
which leads to, $T(t) = \beta$ and

$$X(x) = [(2-n)(\alpha\beta^{n-2}x + c)]^{\frac{1}{2-n}}, \tag{47}$$

where β is a constant and c is a constant of integration. As a result,

$$v_1 = \beta[(2-n)(\alpha\beta^{n-2}x + c)]^{\frac{1}{2-n}}. \tag{48}$$

2. $\frac{dX(x)}{dx} = 0$.

In this case, $X(x) = \delta$ and Eq.(46) will give $\frac{dT(t)}{dt} = \alpha\delta^{n-1}t + d$.Where δ is a constant. Therefore $T(t) = [(1-n)(\alpha\delta^{n-1}t + d)]^{\frac{1}{1-n}}$, where d is a constant of integration. Consequently, the second special solution will be

$$v_2 = \delta[(1-n)(\alpha\delta^{n-1}t + d)]^{\frac{1}{1-n}}. \tag{49}$$

3. For more general solution, we proceed as before, assuming that $v = v(f(v_1, v_2))$. Consequently, Eq.(45) will become

$$\frac{dv}{df}(\frac{\partial f}{\partial v_1}\frac{\partial v_1}{\partial t} + \frac{\partial f}{\partial v_2}\frac{\partial v_2}{\partial t}) + v\frac{dv}{df}(\frac{\partial f}{\partial v_1}\frac{\partial v_1}{\partial x} + \frac{\partial f}{\partial v_2}\frac{\partial v_2}{\partial x}) = \alpha v^n, \tag{50}$$

and since $\frac{\partial v_1}{\partial t} = 0$ and $\frac{\partial v_2}{\partial x} = 0$, Eq.(50) changes to

$$\frac{dv}{df}(\frac{\partial f}{\partial v_2}\frac{\partial v_2}{\partial t}) + v\frac{dv}{df}(\frac{\partial f}{\partial v_1}\frac{\partial v_1}{\partial x}) = \alpha v^n. \tag{51}$$

But from Eq.(48) and Eq.(49), we have $\frac{\partial v_2}{\partial t} = \alpha v_2^n$ and $\frac{\partial v_1}{\partial x} = \alpha v_1^{n-1}$. Substituting them in Eq.(51) will lead to

$$v_2^n \frac{dv}{df}\frac{\partial f}{\partial v_2} + v v_1^{n-1}\frac{dv}{df}\frac{\partial f}{\partial v_1} = v^n. \tag{52}$$

If we choose $v_2^n \frac{\partial f}{\partial v_2} = \xi$ and $v_1^{n-1}\frac{\partial f}{\partial v_1} = \kappa$ (κ and ξ are constants), simultaneous solution of these two equations will result in

$$f = \frac{\xi}{1-n}v_2^{1-n} + \frac{\kappa}{2-n}v_1^{2-n}. \tag{53}$$

Inserting the above choices in Eq.(52) will lead to

$$\frac{\xi}{1-n}v^{1-n} + \frac{\kappa}{2-n}v^{2-n} = f + h, \tag{54}$$

where h is a constant of integration.
f is given by Eq.(53) and, therefore, for $n \neq 1$ and $n \neq 2$, v could be evaluated from Eq.(54). For example, in case $n = 3$, v has two values

$v = \frac{-\kappa + \sqrt{\kappa^2 - 2\xi f}}{2f}$ and $v = \frac{\kappa + \sqrt{\kappa^2 - 2\xi f}}{-2f}$

with

$$f = -\frac{\xi}{2}v_2^{-2} + \kappa v_1^{-1}. \tag{55}$$

6 Discussion

The above examples prove the effectiveness of the method introduced in section 2. As was noted earlier, based on simple logics we may solve complicated first order quasi linear partial differential equations in a very simple manner. This method can be applied to higher order nonlinear partial differential equations, as is done by the author [3,4,5]. What deserves special attention in the present study is that the general solution is found in a very straightforward manner. We guess that the method is applicable to a vast variety of more complex partial differential equations, except those we have solved so far. This is the subject of future research.

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