ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.7(2009) No.4,pp.443-447

Optical Solitons with Fourth Order Dispersion and Dual-power Law Nonlinearity

Anjan Biswas[1] *, Daniela Milovic[2]
[1]Center for Research and Education in Optical Sciences and Applications Department of Applied Mathematics and Theoretical Physics Delaware State University Dover, DE 19901-2277 USA
[2] Faculty of Electronic Engineering Department of Telecommunications University of Nis Aleksandra Medvedeva 14, 1800 Nis SERBIA
(Received 4 November 2008, accepted 29 March 2009)

Abstract: This letter studies optical solitons that is governed by the nonlinear Schrödinger's equation with dual-power law nonlinearity and including the fourth order dispersion term. The solitary wave hypothesis is used to obtain the closed form 1-soliton solution. The restriction on the parameters are also obtained, during the course of derivation of the solution, for the solitons to exist.

Keywords:solitons; integrability; non-Kerr law
AMS Codes: 35Q51, 35Q53, 37K10, 78A60.
PACS Codes: 02.30.Jr, 02.30.Ik

1 Introduction

The theoretical possibility of existence of optical solitons in a dielectric dispersive fiber was first predicted by Hasegawa and Tappert [4]. A couple of years later Mollenauer et al succesfully performed the famous experiment to verify this prediction [4]. Important characterestic properties of these solitons are that they posses a localized waveform which remains intact upon interaction with another soliton. Because of their remarkable robustness, they attracted enormous interest in optical and telecommunication community. At present optical solitons are regarded as the natural data bits for transmission and processing of information in future, and an important alternative for the next generation of ultra high speed optical communication systems [1-15].

The fundamental mechanism of soliton formation namely the balanced interplay of linear group velocity dispersion (GVD) and nonlinearity induced self-phase modulation (SPM) is well understood. In the pico second regime, the nonlinear evolution equation that takes into account this interplay of GVD and SPM and which describes the dynamics of soliton is the well known nonlinear Schrödinger's equation (NLSE). The NLSE, which is the ideal equation in an ideal Kerr media, is in its original form found to be completely integrable by the method of Inverse Scattering Transform (IST) and tremendous success has been acheived in the development of soliton theory in the framework of the NLSE model. In this paper the NLSE is going to be studied with the inclusion of the fourth order dispersion (4OD) term in a dual-power law media [2, 9]. The NLSE does not give correct prediction for pulse widths smaller than 1 picosecond. For example, in solid state solitary lasers, where pulses as short as 10 femtoseconds are generated, the approximation breaks down. Thus, quasi-monochromaticity is no longer valid and consequently higher order dispersion terms creep in. One needs to consider the higher order dispersion for performance enhancement along trans-oceanic and trans-continental distances. Also, for short pulse widths where the group velocity dispersion changes, within the spectral bandwidth of the signal, can no longer be neglected, one needs to take into account the presence of 4OD [7, 9].

* **Corresponding author**. *E-mail address*: biswas.anjan@gmail.com

IJNS.2009.06.30/246

2 Mathematical analysis

The NLSE with the dual-power law of nonlinearity is used to describe the saturation of the nonlinear refractive index. Also, this serves as a basic model to describe the solitons in photovoltaic-photorefractive materials such as $\mathrm{LiNbO_3}$. The propagation of ultrashort optical pulses in a nonlinear media can be charaterized by the nonliner refractive index that is given by [2, 6, 7]

$$n = n_0 + n_2|E|^{2m} + n_4|E|^{4m} \tag{1}$$

here, in (1), E represents the electric field intensity and $n_0 > n_2|E|^{2m} > n_4|E|^{4m}$. The dimensionless form of the NLSE with 4OD is given by [2, 7, 9]

$$iq_t + aq_{xx} - bq_{xxxx} + c\left(|q|^{2m} + k\,|q|^{4m}\right)q = 0 \tag{2}$$

in (2), a, b and c are real numbers. If $b = 0$, (2) reduces to NLSE with dual-power law nonlinearity. In addition, if $m = 1$, (2) reduces to parabolic law nonlinearity. The coefficient of a represents the GVD while the coefficient of c represents the SPM with dual-power law nonlinearity. The constant k binds the two nonlinear terms and the exponent m governs the power laws. The range of validity of the parameter k, for the solitons to exist, will be determined in this paper. Also, the coefficient of b is the 4OD term. The solitons are the result of a delicate balance between dispersion and nonlinearity.

It is to be noted that (2) is not integrable by the classical method of IST since it will fail the Painleve test of integrability [1]. It is, however, still possible to obtain a closed form 1-soliton solution of (2). It is assumed that the 1-soliton solution to (2) is given by the following phase-amplitude format [3, 7].

$$q = P(x,t)e^{i\phi} \tag{3}$$

where P is the amplitude portion while ϕ is the phase portion of the soliton. It is also assumed that [7]

$$\phi(x,t) = -\kappa x + \omega t + \theta \tag{4}$$

where κ is the fequency of the soliton, ω is the wave number, while θ is the phase constant. On substituting these into (2) yields

$$iq_t = \left(i\frac{\partial P}{\partial t} - P\frac{\partial \phi}{\partial t}\right)e^{i\phi} \tag{5}$$

$$q_{xx} = \left(\frac{\partial^2 P}{\partial x^2} - 2i\kappa\frac{\partial P}{\partial x} - \kappa^2 P\right)e^{i\phi} \tag{6}$$

$$q_{xxx} = \left(\frac{\partial^3 P}{\partial x^3} - 3i\kappa\frac{\partial^2 P}{\partial x^2} - 3\kappa^2\frac{\partial P}{\partial x} + i\kappa^3 P\right)e^{i\phi} \tag{7}$$

$$q_{xxxx} = \left(\frac{\partial^4 P}{\partial x^4} - 4i\kappa\frac{\partial^3 P}{\partial x^3} - 6\kappa^2\frac{\partial^2 P}{\partial x^2} + 4i\kappa^3\frac{\partial P}{\partial x} + \kappa^4 P\right)e^{i\phi} \tag{8}$$

Substituting (5)-(8) into (2) and equating the real and imaginary parts yields

$$\frac{\partial P}{\partial t} - 2\kappa\left(a + 2b\kappa^2\right)\frac{\partial P}{\partial x} + 4b\kappa\frac{\partial^3 P}{\partial x^3} = 0 \tag{9}$$

and

$$\left(\omega + a\kappa^2 + b\kappa^4\right)P - cP^3 - \left(a + 6b\kappa^2\right)\frac{\partial^2 P}{\partial x^2} + b\frac{\partial^4 P}{\partial x^4} = 0 \tag{10}$$

For optical solitons with dual-power law nonlinearity, a proper choice for the function $P(x,t)$ will be [7]

$$P = \frac{A}{(\lambda + \cosh\tau)^p} \tag{11}$$

with

$$\tau = B(x - vt) \tag{12}$$

where A is the amplitude, B is the inverse width of the soliton and v is the soliton velocity while the exponent p is unknown at this stage. This unknown exponent will be determined in terms of m. The parameter λ depends on the coefficients a, b, c and k and the exact form of these dependency will also be determined during the course of the derivation of the soliton solution to (2). Thus, from (11), (9) reduces to

$$\begin{aligned} &\frac{pvB\sinh\tau}{(\lambda+\cosh\tau)^p} + \frac{pAB\left(2a\kappa + 4b\kappa^3\right)\sinh\tau}{(\lambda+\cosh\tau)^{p+1}} \\ &\quad -\frac{4b\kappa p^3AB^3\sinh\tau}{(\lambda+\cosh\tau)^{p+1}} + \frac{4b\kappa\lambda p(p+1)(2p+1)AB^3\sinh\tau}{(\lambda+\cosh\tau)^{p+2}} \\ &\quad -\frac{4b\kappa\left(\lambda^2-1\right)p(p+1)(p+2)AB^3\sinh\tau}{(\lambda+\cosh\tau)^{p+3}} = 0 \end{aligned} \tag{13}$$

while (10) reduces to

$$\begin{aligned} &-\frac{\left(\omega + a\kappa^2 + b\kappa^4\right)A}{(\lambda+\cosh\tau)^p} + \frac{cA^3}{(\lambda+\cosh\tau)^{(2m+1)p}} + \frac{ckA^5}{(\lambda+\cosh\tau)^{(4m+1)p}} \\ &\quad +\frac{p^2AB^2\left(a+6b\kappa^2\right)}{(\lambda+\cosh\tau)^p} - \frac{\lambda p(2p+1)AB^2\left(a+6b\kappa^2\right)}{(\lambda+\cosh\tau)^{p+1}} + \frac{\left(\lambda^2-1\right)p(p+1)AB^2\left(a+6b\kappa^2\right)}{(\lambda+\cosh\tau)^{p+2}} \\ &\quad -\frac{bp^4AB^4}{(\lambda+\cosh\tau)^p} + \frac{b\lambda(2p+1)\left(p^3+2p+2\right)AB^4\left(a+6b\kappa^2\right)}{(\lambda+\cosh\tau)^{p+1}} \\ &\quad -\frac{b\left\{2\lambda^2(p+1)\left(p^3+4p^2+7p+2\right) - 2p(p+1)\left(p^2+2p+2\right)\right\}AB^4}{(\lambda+\cosh\tau)^{p+2}} \\ &\quad +\frac{b\lambda\left(\lambda^2-1\right)(p+1)(p+2)\left(2p^2+7p+1\right)AB^4}{(\lambda+\cosh\tau)^{p+3}} \\ &\quad -\frac{b\left(\lambda^2-1\right)^2p(p+1)(p+2)(p+3)AB^4}{(\lambda+\cosh\tau)^{p+4}} = 0 \end{aligned} \tag{14}$$

Now, from (14) setting the exponents $(2m+1)p$ and $p+2$ equal to one another gives

$$p = \frac{1}{m} \tag{15}$$

The same value of p is recovered, when the exponents $(4m+1)p$ and $p+4$ are set equal to one another. Also noting that the functions $1/(\lambda+\cosh\tau)^{p+j}$, for j = 0, 1, 2, 3 and 4 are linearly independent, their respective coefficients in (14) must vanish. Therefore, these yield

$$A = \left[\frac{(m+1)\left(a+6b\kappa^2\right)}{bc\left(2m^3+2m^2+1\right)^2}\left\{2\lambda^2\left(2m^3+7m^2+4m+1\right) - \left(6m^3+6m^2+2m+1\right)\right\}\right]^{\frac{1}{2m}} \tag{16}$$

$$B = \left[\frac{m^2\left(a+6b\kappa^2\right)}{b\left(2m^3+2m^2+1\right)}\right]^{\frac{1}{2}} \tag{17}$$

$$\omega = \frac{AB^2}{m^4}\left\{m^2\left(a+6b\kappa^2\right)-\kappa^2\left(a+b\kappa^2\right)-bB^2\right\} \tag{18}$$

$$\lambda = \sqrt{\frac{M}{N}} \tag{19}$$

where

$$\begin{aligned} M = &\left(2m^3+2m^2+1\right)\sqrt{bc(m+1)(2m+1)(3m+1)} \\ &-\sqrt{k}(m+1)\left(6m^3+6m^2+2m+1\right)\left(a+6b\kappa^2\right) \end{aligned} \tag{20}$$

and

$$\begin{aligned} N = &\left(2m^3+2m^2+1\right)\sqrt{bc(m+1)(2m+1)(3m+1)} \\ &-2\sqrt{k}(m+1)\left(2m^3+7m^2+4m+1\right)\left(a+6b\kappa^2\right) \end{aligned} \tag{21}$$

This leads to the fact that the parameter k must lie in the interval

$$k \in \left(-\infty, \frac{bc(2m+1)(3m+1)\left(2m^3+2m^2+1\right)^2}{4(m+1)\left(2m^3+7m^2+4m+1\right)^2\left(a+6b\kappa^2\right)^2}\right) \tag{22}$$

for the solitons to exist. Finally, applying the same strategy to (13), yields

$$v = \frac{2\kappa A}{b\left(2m^3+2m^2+1\right)}\left[2\left(a+6b\kappa^2\right)-b\left(2m^3+2m^2+1\right)\left(a+2b\kappa^2\right)\right] \tag{23}$$

From (16) and (17), one can conclude that the amplitude and the width of the soliton are related as

$$A^{2m} = \frac{B^2(m+1)}{cm^2\left(2m^3+2m^2+1\right)}\left\{2\lambda^2\left(2m^3+7m^2+4m+1\right)-\left(6m^3+6m^2+2m+1\right)\right\} \tag{24}$$

Hence the 1-soliton solution of (2) is given by

$$q(x,t) = \frac{A}{\left[\lambda+\cosh B(x-vt)\right]^{\frac{1}{m}}}e^{i(-\kappa x+\omega t+\theta)} \tag{25}$$

where the amplitude, width, wave number, velocity of the soliton and the parameter λ are given by (16)-(19) and (23).

3 Conclusions

In this paper, an exact optical 1-soliton solution to the NLSE, with dual-power law nonlinearity, and 4OD is obtained. The governing equation is thus integrable although the Painleve test of integrability will fail. In future, this NLSE will be studied along with its perturbation terms. This will also include the stochastic perturbation terms. The quasi-stationary soliton will be obtained in presence of such perturbation terms.

Acknowledgements

The research work of the first author (AB) was fully supported by NSF-CREST Grant No: HRD-0630388 and Army Research Office (ARO) along with the Air Force Office of Scientific Research (AFOSR) under the award number: W54428-RT-ISP and these supports are genuinely and sincerely appreciated.

References

[1] M. J. Ablowitz , H. Segur :Solitons and Inverse Scattering Transform. *SIAM. Philadelphia, PA, USA*. (1981)

[2] N. Asif, S. Shwetanshumala ,S. Konar:Photovoltaic spatial soliton pairs in two-photon photorefractive materials. *Physics Letters A*.372(5):735-740 (2008)

[3] A. Biswas ,S. Konar: Introduction to non-Kerr law optical solitons. *CRC Press, Boca Raton, FL, USA*. (2006)

[4] A. Hasegawa, Y. Kodama: Solitons in Optical Communications. *Oxford University Press, Oxford, UK.* (1995)

[5] S. Jana, S. Konar:A new family of Thirring type optical spatial solitons via electromagnetically induced transparency. *Physics Letters A*.362(5-6):435-438(2007)

[6] Z. Jovanoski , D. R. Rowland:Variational analysis of solitary waves in a homogenous cubic-quintic nonlinear medium *Journal of Modern Optics*. 48(7):1179-1193(2001)

[7] R. Kohl, A. Biswas, D. Milovic , E. Zerrad:Optical soliton perturbation in a non-Kerr law media. *Optics and Laser Technology*.40(4):647-662(2008)

[8] S. Konar, M. Mishra , S. Jana:Nonlinear evolution of cosh-Gaussian laser beams and generation of flat top spatial solitons in cubic quintic nonlinear media. *Physics Letters A*. 362(5-6):505-510(2007)

[9] A. M. Wazwaz:Exact solutions for the fourth order nonlinear Schrodinger equations with cubic and power law nonlinearities. *Mathematical and Computer Modelling.* 43(7-8):802-808(2006)

[10] P. E. Zhidkov:Korteweg-de Vries and Nonlinear Schrödinger's Equations:Qualitative Theory. *Springer Verlag. New York, NY.*(2001)

[11] K. R. Raslan:Exact solitary wave solution of equal width wave and related equations using a direct Algebraic method. *International Journal of Nonlinear Science*. 6(3):246-254(2008)

[12] Y. Wang, A. Gao, X. Chen:Global conservative solutions of the generalized shallow water wave equations. *International Journal of Nonlinear Science*. 6(3):273-280(2008)

[13] M. A. Abdou:Exact periodic wave solutions to some nonlinear evolution equations. *International Journal of Nonlinear Science*. 6(2):145-153(2008)

[14] Y. Xie:New explicit and exact solutions to the mKdV equation. *International Journal of Nonlinear Science*. 6(2):124-128(2008)

[15] H. Demiray:Modified reductive perturbation method as applied to long water waves: The Korteweg-de Vries hierarchy. *International Journal of Nonlinear Science*.6(1):11-20(2008)