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Nonlocal Boundary Value Problem of a Fractional-Order Functional Differential Equation

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Abstract: The topic of fractional calculus, (integration and differentiation of fractional-order) is a one of the singular integral and integro-differential operators (see [1], [5]-[8], [10]-[11],[14]-[16] and [18] and the references therein). In this work, we prove some local and global existence theorems for a nonlocal nonlinear boundary value problem of a fractional-order functional differential equation.

Keywords: fractional calculus; nonlocal nonlinear boundary value problem of a fractional-order functional differential equation

1 Introduction

The three-point boundary value problem has been studied by many authors (see [9, 12] and [19, 20]) for instance. In [13], they study the existence of positive solutions to the three-point boundary value problem

$$\begin{cases} u'' + a(t)\, f(u) = 0, \; t \in (0,1), \\ u(0) = 0, \; \alpha\, u(\eta) = u(1), \; 0 < \eta < 1, \; 0 < \alpha < \frac{1}{\eta} \end{cases}$$

They proved the existence of at least one positive solution if f is either superlinear or sublinear by applying the fixed point theorems in cones. Also, in [17], the author concerned with determining values for λ so that the three-point nonlinear second order boundary value problem

$$\begin{cases} u''(t) + \lambda\, a(t)\, f(u(t)) = 0, \; t \in (0,1), \\ u(0) = 0, \; \alpha\, u(\eta) = u(1), \; 0 < \eta < 1, \; 0 < \alpha < \frac{1}{\eta} \end{cases}$$

has positive solutions.

Now let $\beta \in (1,2)$ and $\gamma \in (0,1]$, we deal here with the nonlocal nonlinear boundary value problem of a fractional-order functional differential equation

$$\begin{cases} D^{\beta}\, u(t) + f(t, u(\phi(t))) = 0, \; t \in (0,1), \\ I^{\gamma}\, u(t)|_{t=0} = 0, \; \alpha\, u(\eta) = u(1), \; 0 < \eta < 1, \; 0 < \alpha\, \eta^{\beta-1} < 1. \end{cases} \tag{1}$$

We investigate the behavior of solutions for problem (1) with certain nonlinearities, using the equivalence of the problem with the corresponding integral equation, we prove the existence of L_1-solution such that the function f satisfies the Caratheodory conditions and the growth condition.

2 Preliminaries

Let $L_1(I)$ be the class of Lebesgue integrable functions on the interval $I = [a, b]$, $0 \leq a < b < \infty$ and $\Gamma(.)$ be the gamma function.

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Recall that the operator T is compact if it is continuous and maps bounded sets into relatively compact ones. The set of all compact operators from the subspace $U \subset X$ into the Banach space X is denoted by $C(U, X)$. Moreover, we set $B_r = \{u \in L_1(I) : ||u|| < r, r > 0\}$.

Definition 2.1 *The fractional integral of the function $f(.) \in L_1(I)$ of order $\beta \in R^+$ is defined by (see [14]-[16] and [18])*

$$I_a^\beta f(t) = \int_a^t \frac{(t - s)^{\beta - 1}}{\Gamma(\beta)} f(s)\, ds.$$

Definition 2.2 *The Riemann-Liouville fractional-order derivative of $f(t)$ of order $\alpha \in (0, 1)$ is defined as (see [14]-[16] and [18])*

$$D_a^\alpha f(t) = \frac{d}{dt} I_a^{1 - \alpha} f(t), \quad t \in [a, b].$$

Now the following theorem (some properties of the fractional-order integration) can be easily proved.

Theorem 2.1 *Let $\beta,\ \gamma \in R^+$ and $\alpha \in (0, 1]$. Then we have:*

(i) $I_a^\beta : L_1 \to L_1$, *and if* $f(t) \in L_1$, *then* $I_a^\gamma\, I_a^\beta f(t) = I_a^{\gamma+\beta} f(t)$.

(ii) $\lim_{\beta \to n} I_a^\beta f(t) = I_a^n f(t)$, $n = 1, 2, 3, \cdots$ *uniformly.*

Now, let us recall some results which will be needed in the sequel.

Theorem 2.2 ***(Rothe Fixed Point Theorem)*** *[3]*
Let U be an open and bounded subset of a Banach space E, let $T \in C(\bar{U}, E)$. Then T has a fixed point if the following condition holds

$$T(\partial U) \subseteq \bar{U}.$$

Theorem 2.3 ***(Nonlinear alternative of Laray-Schauder type)*** *[3]*
Let U be an open subset of a convex set D in a Banach space E. Assume $0 \in U$ and $T \in C(\bar{U}, E)$. Then either

(A1) *T has a fixed point in $\bar{U}$, or*

(A2) *there exists $\ell \in (0, 1)$ and $x \in \partial U$ such that $x = \ell\, Tx$.*

Theorem 2.4 ***(Kolmogorov compactness criterion)*** *[4]*
Let $\Omega \subseteq L^p\,(0, 1),\ 1 \leq p < \infty$. If

(i) *Ω is bounded in $L^p\,(0, 1)$, and*

(ii) *$x_h \to x$ as $h \to 0$ uniformly with respect to $x \in \Omega$, then Ω is relatively compact in $L^p\,(0, 1)$, where*

$$x_h(t) = \frac{1}{h} \int_t^{t+h} x(s)\, ds.$$

3 Main results

We begin this section by proving the equivalence of the problem (1) with the functional integral equation:

$$\begin{aligned} u(t) \quad = \quad & - I^\beta f(t, u(\phi(t))) - \frac{\alpha\, t^{\beta-1}}{1 - \alpha\, \eta^{\beta-1}} \int_0^\eta \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)} f(s, u(\phi(s)))\, ds \\ + \quad & \frac{t^{\beta-1}}{1 - \alpha\, \eta^{\beta-1}} \int_0^1 \frac{(1 - s)^{\beta-1}}{\Gamma(\beta)} f(s, u(\phi(s)))\, ds. \end{aligned} \tag{2}$$

Lemma 3.1 *The nonlocal nonlinear boundary value problem*

$$D^{\beta}\,u(t)\;+\;f(t,u(\phi(t)))\;=\;0,\;\beta\;\in\;(1,2),\;t\;\in\;(0,1),\tag{3}$$

$$I^{\gamma}\,u(t)|_{t=0}\;=\;0,\;\gamma\;\in\;(0,1],\;\alpha\,u(\eta)\;=\;u(1),\;0\;<\;\eta\;<1,\;0\;<\;\alpha\,\eta^{\beta-1}\;<\;1\tag{4}$$

is equivalent to the fractional-order functional integral equation (2)

Proof: Equation (3) (see [2]) can be reduced to an equivalent integral equation

$$u(t)\;=\;-\;I^{\beta}\,f(t,u(\phi(t)))\;+\;C_1\,t^{\beta-1}\;+\;C_2\,t^{\beta-2}.$$

By (4), we get $C_2 = 0$ and

$$C_1=\frac{-\alpha}{1-\alpha\eta^{\beta-1}}\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}f(s,u(\phi(s)))ds+\frac{1}{1-\alpha\eta^{\beta-1}}\int_0^{1}\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}f(s,u(\phi(s)))ds.$$

Therefore, the solution of problem (3), (4) is given by the formula (2).

Conversely, let $u(t)$ be a solution of (2) and operating on both sides of it by $I^{2-\beta}$, we get

$$I^{2-\beta}\,u(t)\;=\;-\;I^{2}\,f(t,u(\phi(t)))\;-\;t\,C_3\;+\;t\,C_4.$$

Differentiating the last relation two times we obtain (3), also it is easy to check that conditions (4) are satisfied. The proof is complete. ■

Now, we present our main result by proving some local and global existence theorems for problem (1) in L_1.

To facilitate our discussion, let us first state the following assumptions:

(i) $f:(0,1)\times R\rightarrow R^{+}$ be a function with the following properties:

(1) for each $t\in(0,1)$, $f(t,.)$ is continuous,

(2) for each $u\in R$, $f(.,u)$ is measurable,

(3) there exist two real functions $t\rightarrow a(t), t\rightarrow b(t)$ such that

$$f(t,u)\;\le\;a(t)\;+\;b(t)\mid u\mid,\;\;\text{for each}\;\;t\in(0,1),\;\;u\in R,$$

where $a(.)\in L_1(0,1)$ and $b(.)$ is measurable and bounded.

(ii) $\phi:(0,1)\rightarrow(0,1)$ is nondecreasing and there is a constant $M>0$ such that $\phi'\geq M$ a.e. on $(0,1)$.

Also, define the operator T as

$$\begin{aligned}Tu(t)\;\;&=\;\;-\;I^{\beta}\,f(t,u(\phi(t)))\;-\;\frac{\alpha\,t^{\beta-1}}{1-\alpha\,\eta^{\beta-1}}\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\,f(s,u(\phi(s)))\,ds\\&+\;\;\frac{t^{\beta-1}}{1-\alpha\,\eta^{\beta-1}}\int_0^{1}\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\,f(s,u(\phi(s)))\,ds.\end{aligned}$$

To solve equation (2) it is necessary to find a fixed point of the operator T.

For the local existence of a solution we have the following theorem:

Theorem 3.2 *Let the assumptions (i) and (ii) are satisfied.*

$$\textit{If}\quad\sup|b(t)|\;<\;M\,K,\tag{5}$$

where $K=(1-\alpha\,\eta^{\beta-1})\,\Gamma(1+\beta)$, *then the nonlocal boundary value problem (1) has a solution* $u\in B_r$, *where*

$$r\;\le\;\frac{\frac{1}{K}\,||\,a\,||}{1-\frac{1}{MK}\;\sup|\,b(t)\,|}.$$

Proof: Let u be an arbitrary element in B_r. Then from the assumptions (i) - (ii), we have

$$Tu(t) \ \leq \ \frac{t^{\beta-1}}{1-\alpha\, \eta^{\beta-1}} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \ f(s,u(\phi(s)))\ ds.$$

Then

$$\begin{aligned}
||Tu|| &= \int_0^1 \mid Tu(t) \mid dt \\
&\leq \int_0^1 \mid \frac{t^{\beta-1}}{1-\alpha\, \eta^{\beta-1}} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \ f(s,u(\phi(s)))\ ds \mid dt \\
&\leq \frac{1}{1-\alpha\, \eta^{\beta-1}} \int_0^1 \int_0^1 \frac{t^{\beta-1}}{\Gamma(\beta)}\ dt\ |f(s,u(\phi(s)))|\ ds \\
&= \frac{1}{1-\alpha\, \eta^{\beta-1}} \int_0^1 \left(\frac{t^{\beta}}{\Gamma(1+\beta)}\right)_0^1 |f(s,u(\phi(s)))|\ ds \\
&= \frac{1}{K} \int_0^1 |f(s,u(\phi(s)))|\ ds \\
&\leq \frac{1}{K} \int_0^1 (\ |a(s)| \ + \ |b(s)|\ |u(\phi(s))|\ )\ ds \\
&\leq \frac{1}{K} \left(||a|| \ + \ \sup|b(t)| \int_0^1 |u(\phi(s))|\ ds\right) \\
&\leq \frac{1}{K} \left(||a|| \ + \ \sup|b(t)|.\frac{1}{M} \int_0^1 \mid u(\phi(s)) \mid |\phi'|\ ds\right) \\
&= \frac{1}{K} \left(||a|| \ + \ \sup|b(t)|.\frac{1}{M} \int_{\phi(0)}^{\phi(1)} |u(x)|\ dx\right) \\
&\leq \frac{1}{K} \left(||a|| \ + \ \sup|b(t)|.\frac{1}{M}\ ||u||\right).
\end{aligned}$$

The last estimate shows that the operator T maps L_1 into itself. Now, let $u \in \partial B_r$, that is, $||u|| = r$, then the last inequality implies

$$||Tu|| \ \leq \ \frac{1}{K} \left(||\ a\ || \ + \ \sup|\ b(t)\ |.\frac{1}{M}\ r\right).$$

Then $T(\partial B_r) \subset \bar{B}_r$ (closure of B_r) if

$$||Tu|| \ \leq \ \frac{1}{K} \left(||\ a\ || \ + \ \sup|\ b(t)\ |.\frac{1}{M}\ r\right) \ \leq \ r,$$

which implies that

$$\frac{1}{K} \left(||\ a\ || \ + \ \sup|\ b(t)\ |.\frac{1}{M}\ r\right) \ \leq \ r.$$

Therefore

$$r \ \leq \ \frac{\frac{1}{K}\ ||\ a\ ||}{1-\frac{1}{MK}\ \sup|\ b(t)\ |}.$$

Using inequality (5) we deduce that $r > 0$. Moreover, we have

$$\begin{aligned}
||f|| &= \int_0^1 |f(s,u(\phi(s)))|\ ds \\
&\leq \int_0^1 (\ |a(s)| \ + \ |b(s)|\ |u(\phi(s))|\ )\ ds \\
&\leq ||a|| \ + \ \sup|b(t)|.\frac{1}{M}\ ||u||.
\end{aligned}$$

This estimation shows that f in $L_1(0,1)$.

Further, f is continuous in u (assumption 1) and I^{α} maps $L_1(0,1)$ continuously into itself, $I^{\alpha} f(t, u(\phi(t)))$ is continuous in u.Since u is an arbitrary element in B_r, T maps B_r continuously into $L_1(0,1)$.

Now, we will show that T is compact, to achieve this goal we will apply Theorem 2.4. So, let Ω be a bounded subset of B_r. Then $T(\Omega)$ is bounded in $L_1(0,1)$, i.e. condition (i) of Theorem 2.4 is satisfied. It remains to show that $(Tu)_h \to Tu$ in $L_1(0,1)$ as $h \to 0$, uniformly with respect to $Tu \in T\,\Omega$. We have the following estimation:

$$\begin{aligned}
||(Tu)_h - Tu|| &= \int_0^1 |\,(Tu)_h(t) - (Tu)(t)\,|\,dt \\
&= \int_0^1 |\,\frac{1}{h}\int_t^{t+h} (Tu)(s)\,ds - (Tu)(t)\,|\,dt \\
&\leq \int_0^1 \left(\frac{1}{h}\int_t^{t+h} |\,(Tu)(s) - (Tu)(t)\,|\,ds\right) dt \\
&\leq \int_0^1 \frac{1}{h}\int_t^{t+h} |-I^{\beta} f(s,u(\phi(s))) + I^{\beta} f(t,u(\phi(t)))\,|\,ds\,dt \\
&+ \int_0^1 \frac{1}{h}\int_t^{t+h} |-K_1\, s^{\beta-1} + K_1\, t^{\beta-1}\,|\,ds\,dt \\
&+ \int_0^1 \frac{1}{h}\int_t^{t+h} |\,K_2\, s^{\beta-1} - K_2\, t^{\beta-1}\,|\,ds\,dt,
\end{aligned}$$

where

$$K_1 = \frac{\alpha}{1 - \alpha\,\eta^{\beta-1}} \int_0^{\eta} \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)} f(s,u(\phi(s)))\,ds$$

and

$$K_2 = \frac{1}{1 - \alpha\,\eta^{\beta-1}} \int_0^{1} \frac{(1 - s)^{\beta-1}}{\Gamma(\beta)} f(s,u(\phi(s)))\,ds.$$

Since $f \in L_1(0,1)$ we get that $I^{\beta} f(.) \in L_1(0,1)$. Moreover $t^{\beta-1} \in L_1(0,1)$. So, we have (see [21])

$$\frac{1}{h}\int_t^{t+h} |-I^{\beta} f(s,u(\phi(s))) + I^{\beta} f(t,u(\phi(t)))\,|\,ds \to 0,$$

$$\frac{1}{h}\int_t^{t+h} |-K_1\, s^{\beta-1} + K_1\, t^{\beta-1}\,|\,ds \to 0$$

and

$$\frac{1}{h}\int_t^{t+h} |\,K_2\, s^{\beta-1} - K_2\, t^{\beta-1}\,|\,ds \to 0$$

for a.e. $t \in (0,1)$. Therefore, by Theorem 2.4, we have that $T(\Omega)$ is relatively compact, that is, T is a compact operator.

Therefore, Theorem 2.2 with $U = B_r$ and $E = L_1(0,1)$ implies that T has a fixed point. This complete the proof. ■

Now for more global solution of the nonlocal boundary value problem (1), consider the following assumption:

(iii) Assume that every solution $u(.) \in L_1(0,1)$ to the equation

$$\begin{aligned}
u(t) &= \ell\,(-I^{\beta} f(t,u(\phi(t))) - \frac{\alpha\, t^{\beta-1}}{1-\alpha\,\eta^{\beta-1}} \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(\phi(s)))\,ds \\
&+ \frac{t^{\beta-1}}{1-\alpha\,\eta^{\beta-1}} \int_0^{1} \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(\phi(s)))\,ds) \ \text{ a.e. on } (0,1),\ 1 < \beta < 2.
\end{aligned}$$

satisfies $||u|| \neq r$ (r is arbitrary but fixed).

Theorem 3.3 *Let the conditions (i) - (iii) be satisfied , then the nonlocal boundary value problem (1) has at least one solution* $u \in L_1(0,1)$.

Proof: Let u be an arbitrary element in the open set $B_r = \{u : ||u|| < r, r > 0\}$. Then from the assumptions (i) - (ii), we have

$$||Tu|| \;\leq\; \frac{1}{K}\left(||a|| \;+\; \sup|b(t)|.\frac{1}{M}\,||u||\right).$$

The above inequality means that the operator T maps B_r into L_1. Moreover, we have

$$||f|| \;\leq\; ||a|| \;+\; \sup|b(t)|.\frac{1}{M}\,||u||.$$

This estimation shows that f in $L_1(0,1)$.

As a consequence of Theorem 3.2 we get that T maps B_r continuously into $L_1(0,1)$ and T is compact. Set $U = B_r$ and $D = E = L_1(0,1)$, then in the view of assumption (iii) the condition $A2$ of Theorem 2.3 does not hold. Therefore, Theorem 2.3 implies that T has a fixed point. This complete the proof. ■

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