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On the Positive and Decreasing Solutions of a Class of p–Laplacian BVP with Neumann–Robin conditions

M. Khaleghy Moghaddam [a] , G.A. Afrouzi [b] *
[a] Department of Basic Sciences, Faculty of Agriculture Engineering, Sari Agricultural Sciences and Natural Resources University, Sari, Iran
[b] Department of Mathematics, Faculty of Basic Sciences, University of Mazandaran, Babolsar, Iran.

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Abstract: In this paper, we consider the following Neumann–Robin boundary value problem

$$\begin{cases} -(\varphi_p(u'(x)))' = |u(x)|^p - \lambda, & x \in (0,1), \\ u'(0) = 0, \\ u'(1) + \alpha u(1) = 0, \end{cases}$$

where $p > 1$, $\lambda \in$ R and $\alpha > 0$ are parameters. We study the positive and decreasing solutions of this problem with respect to a parameter ρ (i.e. $u(0) = \rho$) when $\lambda < \frac{\rho^p}{p+1}$. By using a quadrature method, the results are obtained.
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1 Introduction

In this work we consider the boundary value problem

$$-(\varphi_p(u'(x)))' = |u(x)|^p - \lambda, \qquad x \in (0,1), \tag{1}$$

$$u'(0) = 0, \tag{2}$$

$$u'(1) + \alpha u(1) = 0, \tag{3}$$

where $p > 1$, $\lambda \in$ R and $\alpha > 0$ are parameters and $\varphi_p(s) := |s|^{p-2}s$ for all $s \neq 0$ and $\varphi_p(0) = 0$. Here $(\varphi_p(u'))'$ is the one dimensional p-Laplacian with $p > 1$. We study the existence of positive and decreasing solution of this problem with respect to a parameter ρ (that is the value of the solutions at zero, i.e. $u(0) = \rho$). In order to study this problem we make use of the quadrature method. In [3] problem (1) with Dirichlet boundary value conditions have been studied by Ammar Khodja for the case Laplacian and in [1] the same problem with the same boundary value conditions have been extended by Addou to the general quasilinear case p –Laplacian with $p > 1$. In [4] Anuradha, Maya and Shivaji considered a problem involving the one–dimensional Laplacian with Neumann–Robin boundary conditions by using a quadrature method. In [5] for semipositone problems, existence and multiplicity results have been established for the case $p = 2$ with Neumann boundary value conditions. There exist a number of papers devoted to the existence of solutions of the p–Laplacian operator together with Robin condotion [2**?**].

This paper is organized as follows. In Section 2, we first state some notations, remarks and our main result and finally in Section 3, we provide the proof of our main result that contains several lemmas and claims.

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[*] **Corresponding author**. *E-mail address*: afrouzi@umz.ac.ir

2 Notation and Main Result

By a solution of (1)–(3) we mean a function $u \in C^1([0,1])$ for which $\varphi_p(u') \in C^1([0,1])$ and both the equation and the boundary value conditions are satisfied. Throughout this paper we denote by ρ, the value of the solution at zero (i.e. $u(0) = \rho$).

Remark 1 *Suppose $u(x)$ is a solution of (1)–(3) at λ then $\int_0^1 \{|u(x)|\}^p dx = \lambda - \varphi_p(u'(1))$.*

In fact, by integrating of both sides of the equation (1) on $(0,1)$ and applying the conditions (2) and (3), one can conclude the last equality.

It is well-known that the initial value problem

$$\begin{cases} -(\varphi_p(u'(x)))' = |u(x)|^p - \lambda, \\ u(0) = \rho, \\ u'(0) = 0, \end{cases} \tag{4}$$

has a local solution (by applying the Schauder fixed point theorem) and since $f(u) = |u|^p - \lambda$ is locally Lipschitz, one can conclude from the classical theory for ODE the solution is locally unique. It is clear that every solution to the BVP (1)-(3) at λ with $u(0) = \rho$ is a solution to IVP (4). Now, we state the existence of decreasing and positive to the problem (1)-(3) as described below:

> **Theorem** *Let $\alpha > 0$ and $\rho > 0$, then there exists a real number $\rho^* > 0$ and for any $\rho \in (0, \rho^*)$ there exists a real number $\lambda_\rho \in (-\infty, \frac{\rho^p}{p+1})$ for which the problem (1)–(3) has a solution u at $\lambda = \lambda_\rho$ such that is positive and decreasing and $||u||_\infty = \rho$.*

3 Proof

Let u be a decreasing solution of (1)–(3) at λ with $u(0) = \rho$, thus by (3), $u \geq 0$. Now multiplying (1) throughout by u' and integrating over $(0, x)$, we obtain $|u'(x)|^p = p'\{-\frac{u|u|^p}{p+1} + \lambda u + C\}$, where C is a constant. Applying the conditions $u(0) = \rho$, $u'(0) = 0$, $u' < 0$ and $u > 0$, we have

$$\{u'\}^p = p'\{\frac{\rho^{p+1}}{p+1} - \frac{u^{p+1}}{p+1} + \lambda(u - \rho)\}, \quad x \in (0,1). \tag{5}$$

Now by integrating (5) on $(0, x)$ where $x \in [0,1]$, we obtain

$$\int_{u(x)}^{\rho} M(p,\rho,\lambda,s)^{-\frac{1}{p}} ds = \{p'\}^{\frac{1}{p}} x, \quad x \in [0,1]. \tag{6}$$

$$M(p,\rho,\lambda,s) := \frac{\rho^{p+1}}{p+1} - \frac{s^{p+1}}{p+1} + \lambda(s - \rho)$$

Now, we provide two necessary conditions for the existence of decreasing solution to (1)–(3) in the following lemma:

Lemma 1 *The necessary condition for the existence of positive and decreasing solution u to* (1)–(3) *at λ* with $u(0) = \rho$ *is the existence $m \in \Omega = \Omega(\rho, \lambda)$ such that satisfies the equations of the system*

$$G(m) = \{p'\}^{\frac{1}{p}} \text{ and } H(m) = \{p'\}^{\frac{1}{p}}, \tag{7}$$

where

$$G(m) = \int_{\frac{m}{\alpha}}^{\rho} \{M(p,\rho,\lambda,s)\}^{-\frac{1}{p}} ds \text{ and } H(m) = m\{M(p,\rho,\lambda,\frac{m}{\alpha})\}^{-\frac{1}{p}}, \tag{8}$$

Proof of Lemma 1.
By substituting $x = 1$ in (5) and (6), we have

$$-u'(1)\{M(p,\rho,\lambda,u(1))\}^{-\frac{1}{p}} = \{p'\}^{\frac{1}{p}} = \int_{u(1)}^{\rho} \{M(p,\rho,\lambda,s)\}^{-\frac{1}{p}} ds,$$

By setting $u'(1) = -m$, where $m > 0$, from (3), we have $u(1) = \frac{m}{\alpha} \in (0, \rho)$. Then

$$m\{M(p, \rho, \lambda, \frac{m}{\alpha})\}^{-\frac{1}{p}} = \{p'\}^{\frac{1}{p}} = \int_{\frac{m}{\alpha}}^{\rho} \{M(p, \rho, \lambda, s)\}^{-\frac{1}{p}} ds.$$

Thus for such a solution to exist, there must exist an m such that $G(m) = \{p'\}^{\frac{1}{p}} = H(m)$. $\triangle$

Now, we investigate whether such an m exists or not. For this mean, we first study the variations of the functions $G(m)$ and $H(m)$.

Claim 1 *The function $G(m)$ in (8) on $\Omega = (0, \alpha\rho)$ is decreasing and $G(\alpha\rho) = 0$. Also $G(0) < \infty$ if and only if $p > 2$.*

Proof of Claim 1. It is clear that $G'(m) = -\frac{1}{\alpha}\{M(p, \rho, \lambda, \frac{m}{\alpha})\}^{-\frac{1}{p}} < 0$ on $m \in (0, \alpha\rho)$ and $G(\alpha\rho) = 0$. For showing, $G(0) < \infty$ if and only if $p > 2$, since $\{M(p, \rho, \lambda, s)\}^{-\frac{1}{p}} \approx \{\lambda - \rho^p\}^{-\frac{1}{p}}(s-\rho)^{-\frac{1}{p}}$ near ρ^- and $\int_0^{\rho}(\rho - s)^{-\frac{2}{p}} ds < \infty$ if and only if $p > 2$. Thus one can conclude that $G(0) < \infty$ if and only if $p > 2$. $\triangle$

Now, we investigate the variations of $H(m)$.

Claim 2 *H is strictly increasing on $(0, \alpha\rho)$ and $H(0) = 0$ and $H(\alpha\rho) = \infty$.*

Proof of Claim 2. Compute $H'(m) = h(m)\{M(p, \rho, \lambda, \frac{m}{\alpha})\}^{-\frac{1}{p'}}$ where $h(m) := \frac{\rho^{p+1}}{p+1} + \frac{1}{p(p+1)}\left(\frac{m}{\alpha}\right)^{p+1} + \frac{\lambda m}{\alpha p'} - \lambda\rho$, on $(0, \alpha\rho)$. Also, $h'(m) = \frac{1}{\alpha p}\left(\left(\frac{m}{\alpha}\right)^p + \lambda(p-1)\right) > 0$ on $(0, \alpha\rho)$. Then $\min h(m) = h(0) = \rho(\frac{\rho^p}{p+1} - \lambda) > 0$, and then $H'(m) > 0$ on $(0, \alpha\rho)$. $\triangle$

By Claims 1 and 2, it follows that there exist a unique $m^* = m^*(\alpha, p, \rho, \lambda) \in (0, \alpha\rho)$ such that satisfies $H(m^*) = G(m^*)$. Now, it remains to find the values $\rho > 0$ and the values of $\lambda \in (-\infty, \frac{\rho^p}{p+1})$ such that one can obtain an $m^* = m^*(\alpha, p, \rho, \lambda)$ that satisfies $H(m^*) = \{p'\}^{\frac{1}{p}} = G(m^*)$. At first we consider the equation $H(m^*) = \{p'\}^{\frac{1}{p}}$ in the following Lemma 2:

Lemma 2 *Consider the equation in $m \in \Omega = (0, \alpha\rho)$*

$$H(m) = \{p'\}^{\frac{1}{p}}, \tag{9}$$

where $\alpha > 0$, $\rho > 0$, $p > 1$ and $\lambda \in (-\infty, \frac{\rho^p}{p+1})$ are real parameters. Then

***(a)** For any fixed $\alpha > 0$, $\rho > 0$, $p > 1$ and $\lambda \in (-\infty, \frac{\rho^p}{p+1})$, the equation (9) admits a unique positive zero $m^* = m^*(\alpha, p, \rho, \lambda)$.*

***(b)** The function $\lambda \mapsto m^*(\alpha, p, \rho, \lambda)$ is C^1 on $(-\infty, \frac{\rho^p}{p+1})$ and*

$$\frac{\partial m^*}{\partial \lambda} = \frac{m^*(\frac{m^*}{\alpha} - \rho)}{ph(m)} < 0,$$

for all $\alpha > 0$, $\rho > 0$, $p > 1$ and $\lambda \in (-\infty, \frac{\rho^p}{p+1})$.

Proof of Lemma 2.

(a) By similar argument in [1, Lemma 6], for any fixed $\alpha > 0$, $\rho > 0$, $p > 1$ and $\lambda \in (-\infty, \frac{\rho^p}{p+1})$, consider the function

$$m \mapsto F(\alpha, p, \rho, \lambda, m) := H(m) - \{p'\}^{\frac{1}{p}},$$

defined in Ω. By Claims 1 and 2 and the intermediate value theorem, one can conclude that the function (9) admits a unique positive zero $m^* = m^*(\alpha, p, \rho, \lambda) \in \Omega$.

(b) For any $\alpha > 0, p > 1$ and $\rho > 0$ given, consider the real-valued function

$$(\lambda, m) \mapsto F_+(\lambda, m) := H(m) - \{p'\}^{\frac{1}{p}},$$

defined on $\Omega_+ = (-\infty, \frac{\rho^p}{p+1}) \times \Omega$. It is clear that $F_+ \in C^1(\Omega_+)$ and

$$\frac{\partial F_+}{\partial m} = \frac{h(m)}{\left\{M(p,\rho,\lambda,\frac{m}{\alpha})\right\}^{-\frac{1}{p'}}}, \text{ on } \Omega_+,$$

$$\frac{\partial F_+}{\partial \lambda} = \frac{m}{p}(\rho - \frac{m}{\alpha})\left\{M(p,\rho,\lambda,\frac{m}{\alpha})\right\}^{-\frac{1}{p'}} > 0, \text{ on } \Omega_+,$$

hence by the Claim 2, $\frac{\partial F_+}{\partial m}(\lambda, m) > 0$ on Ω_+, also $m^*(\alpha,\rho,\lambda) \in \Omega$ satisfies from its definition,

$$F_+(\lambda, m^*(\alpha,p,\rho,\lambda)) = 0. \tag{10}$$

So, one can make use of the implicit function theorem to show that the function $\lambda \mapsto m^*(\alpha,p,\rho,\lambda)$ is $C^1((-\infty,\frac{\rho^p}{p+1}),\mathrm{R})$ and to obtain the expression for $\frac{\partial m^*}{\partial \lambda} = -(\frac{\partial F_+}{\partial \lambda})/(\frac{\partial F_+}{\partial m^*})$ given by (b).

Now, for finding a solution to system (7), we solve the equation $G(m^*(\alpha,p,\rho,\lambda)) = \{p'\}^{\frac{1}{p}}$ in $\rho \in (0,\infty)$ and $\lambda \in (-\infty, \frac{\rho^p}{p+1})$.

Lemma 3 *Consider the equation* $G(m^*(\alpha,p,\rho,\lambda)) = \{p'\}^{\frac{1}{p}}$ *where* $\rho \in (0,\infty)$ *and* $\lambda \in (-\infty,\frac{\rho^p}{p+1})$, *then there exists a real number* $\rho^* > 0$ *such that for any* $\rho \in (0,\rho^*)$ *there exists a unique real number* $\lambda_\rho \in (-\infty,\frac{\rho^p}{p+1})$ *for which* $G(m^*(\alpha,p,\rho,\lambda_\rho)) = \{p'\}^{\frac{1}{p}}$.

Proof of Lemma 3.
Claim 3

1. $\lim_{\lambda\to-\infty} G(m^*(\alpha,p,\rho,\lambda)) = 0$,
2. $\frac{\partial G(m^*(\alpha,p,\rho,\lambda))}{\partial\lambda} > 0$.
3. $\lim_{\lambda\to\frac{\rho^p}{p+1}^-} G(m^*(\alpha,p,\rho,\lambda)) = C(\rho)$ *where*

$$C(\rho) = \frac{\{p+1\}^{\frac{1}{p}}}{\{\rho\}^{\frac{1}{p}}}\int_{\frac{m^*}{\alpha\rho}}^{1}\frac{dt}{\{t-t^{p+1}\}^{\frac{1}{p}}}, \tag{11}$$

4. $\lim_{\rho\to0^+} C(\rho) = +\infty$,
5. $\lim_{\rho\to\infty} C(\rho) = 0$,

Proof of Claim 3. Easy computations show that for any $\rho > 0$ and $\lambda \in (-\infty, \frac{\rho^p}{p+1})$,

$$\{M(p,\rho,\lambda,s)\}^{-1/p} \le \{f(s)\}^{-1/p}, \text{ on } (0,\rho), \tag{12}$$

where $f(s) = (\lambda - \frac{\rho^p}{p+1})s + \frac{\rho^{p+1}}{p+1} - \lambda\rho$ and whose graph is the straight line joining the points $(0, M(p,\rho,\lambda,0))$ and $(\rho, 0)$. Now, by integration (12) on the interval $(0,\rho)$ one can conclude that

$$0 < \int_0^\rho \{M(p,\rho,\lambda,s)\}^{-1/p}ds < \int_0^\rho \{f(s)\}^{-1/p}ds = \frac{p'\{\rho\}^{\frac{1}{p'}}}{\{\frac{\rho^p}{p+1}-\lambda\}^{\frac{1}{p}}} \to 0 \text{ as } \lambda \to -\infty.$$

Also easy computations show that for any $\rho > 0$ and $\lambda \in (-\infty, \frac{\rho^p}{p+1})$,

$$\frac{\partial G(m^*)}{\partial\lambda} = \int_{\frac{m^*}{\alpha}}^{\rho}\frac{(\rho-s)ds}{p\{M(p,\rho,\lambda,s)\}^{-\frac{p+1}{p}}} - \frac{\frac{\partial m^*}{\partial\lambda}}{\{M(p,\rho,\lambda,\frac{m^*}{\alpha})\}^{\frac{1}{p}}}. \tag{13}$$

Hence, by the Lemma 2, one can conclude that $\frac{\partial G(m^*)}{\partial\lambda} > 0$. From the fact that $m^*(\alpha,\rho,\lambda) \to m_\rho$ as $\lambda \to (\frac{\rho^p}{p+1})^-$ and the monotone convergence theorem and the definition of $G(m^*(\alpha,p,\rho,\lambda))$ on $\Omega = (0,\alpha\rho)$, one can conclude that

$$\lim_{\lambda\to(\frac{\rho^p}{p+1})^-} G(m^*(\alpha,p,\rho,\lambda)) = \int_{\frac{m_\rho}{\alpha}}^{\rho} \{M(p,\rho,\frac{\rho^p}{p+1},s)\}^{\frac{1}{p}} ds, \tag{14}$$

and by setting $t = \frac{s}{\rho}$ in (14), the statement of this claim proves. By passing to the limit as ρ tends to 0^+ and ∞ in (11), $C(\rho) \to \infty$ and $C(\rho) \to 0$, respectively. $\triangle$

Now from the Claim 3 and the continuity of $C(\rho)$, one can conclude that there exists a real number $\rho^* > 0$ such that $C(\rho^*) = \{p\}^{\frac{1}{p}}$. Therefore for any $\rho \in (0,\rho^*)$ there exists a unique real number $\lambda_\rho \in (-\infty, \frac{\rho^p}{p+1})$ and then there exists a real number $m^*(\alpha,p,\rho,\lambda) \in (0,\alpha\rho)$ such that $G(m^*(\alpha,p,\rho,\lambda_\rho)) = \{p\}^{\frac{1}{p}}$. Here the proof of Lemma 3 is complete. $\triangle$.

On the other hand by the Lemma 2, $H(m^*(\alpha,p,\rho,\lambda_\rho)) = \{p\}^{\frac{1}{p}}$, therefore we can conclude that the equations of the system, (7) are simultaneously solvable in $m^* = m^*(\alpha,p,\rho,\lambda_\rho)$. $\triangle$
Hence from the Lemma 1, one can prove the theorem. $\triangle$

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