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Modified Variational Iteration Method for a Boundary Layer Problem in Unbounded Domain

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Abstract: In this paper, we apply the modified variational iteration method for solving the boundary layer problem in unbounded domain. The suggested modification is made by introducing He's polynomials in the correction functional. The fact that the proposed modified variational iteration method solves nonlinear problems without using Adomian's polynomials is a clear advantage of this technique over the decomposition method.

Keywords: Variational iteration method; He's polynomials; boundary layer problem; Pade-approximants

1 Introduction

With the rapid development of nonlinear science, there has appeared ever increasing interest of scientists and engineers in the analytical asymptotic techniques for nonlinear problems such as solid state physics, astrophysics, experimental and mathematical physics, nuclear charge in heavy atoms, thermal behavior of a spherical cloud of gas, thermodynamics, population models, chemical kinetics and fluid mechanics, see [1,7-13,19,20,24,25,31,32] and the references therein. In particular, a boundary layer equation in unbounded domain is of much interest. Several techniques including decomposition and perturbation have been developed for solving such problems [1,7,31,32]. He [7-13] developed the variational iteration and the homotopy perturbation methods for solving linear, nonlinear, initial and boundary value problems. Due to their compatibility with the physical problems, both these methods have been applied to a wide class of functional equations; see [1-3,5-13,16-30,32] and the references therein. In a later work Ghorbani et. al. [5,6] split the nonlinear term into a series of polynomials calling them as the He's polynomials. More recently, Noor and Mohyud-Din applied variational iteration method using He's polynomials for solving nonlinear boundary value problems, see [20-25]. Inspired and motivated by the ongoing research in this direction, we apply the modified variational method (MVIM) for finding the solution of boundary layer problem in an unbounded domain. The proposed modification is made by introducing He's polynomials in the correction functional [20-25]. It is worth mentioning that the proposed method is applied without any discretization, restrictive assumption or transformation and is free from round off errors. The Pade-approximants [4,22,31] are applied in order to make the work more concise and for the better understanding of the solution behavior. The fact that the MVIM solves nonlinear problems without using Adomian's polynomials is a clear advantage of this technique over the decomposition method. Moreover, the use of Lagrange multiplier avoids the successive application of integral operator, reduces the huge computational work and can be considered as an additional edge of this algorithm over the decomposition method.

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2 Modified Variational Iteration Method

To illustrate the basic concept of the modified variational iteration method (MVIM), we consider the following general differential equation

$$Lu + Nu = g(x), \tag{1}$$

where L is a linear operator, N a nonlinear operator and $g(x)$ is the forcing term. According to variational iteration method $[1-3, 7, 11-14, 16, 20-30]$, we can construct a correction functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left(Lu_n(\xi) + N\widetilde{u_n}(\xi) - g(\xi)\right) d\xi, \tag{2}$$

where λ is a Lagrange multiplier [7,11-14], which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\widetilde{u_n}$ is considered as a restricted variation. i.e. $\delta\widetilde{u_n} = 0$; (9) is called as a correct functional. Now, we apply the homotopy perturbation method

$$\begin{aligned}\sum_{n=0}^{\infty} p^{(n)} u_n &= u_n(x) + p\int_0^x \lambda(\xi)\left(\sum_{n=0}^{\infty} p^{(n)} L(u_n) + \sum_{n=0}^{\infty} p^{(n)} N(\widetilde{u_n})\right) d\xi \\ &\quad - \int_0^x \lambda(\xi)\, g(\xi)\, d\xi,\end{aligned}$$

which is the coupling of variational iteration method and He's polynomials and is called the modified variational iteration method (MVIM) [20-25]. The comparison of like powers of p gives solutions of various orders.

3 Numerical Applications

In this section, we apply the modified variational iteration method (MVIM) for solving boundary layer problem in an infinite domain.

Example 6.1[31,32]. Consider the following nonlinear third order boundary layer problem which appears mostly in the mathematical modeling of physical phenomena in fluid mechanics [31,32].

$$f'''(x) + (n-1)\, f(x)\, f''(x) - 2n\left(f'(x)\right)^2 = 0, \qquad n > 0,$$

with boundary conditions

$$f(0) = 0, \qquad f'(0) = 1, \qquad f'(\infty) = 0, \qquad n > 0,$$

The correction functional is given as

$$f_{n+1}(x) = f_n(x) + \int_0^x \lambda(s)\left(f'''(s) + (n-1)\,\widetilde{f}(s)\,\widetilde{f''}(s) - 2n\left(\widetilde{f'}(s)\right)^2\right) ds = 0, \quad n > 0,$$

Making the correct functional stationary, the Lagrange multipliers can be identified as $\lambda(s) = -\frac{1}{2!}(s-x)^2$, consequently

$$f_{n+1}(x) = f_n(x) - \int_0^x \frac{1}{2!}(s-x)^2\left(f'''(s) + (n-1)\, f(s)\, f''(s) - 2n\left(f'(s)\right)^2\right) ds = 0, \quad n > 0,$$

Applying the modified variational iteration method (MVIM)

$$\begin{aligned}f_0 + pf_1 + p^2 f_2 + \cdots &= x + \frac{1}{2}\alpha x^2 - p\int_0^x \frac{1}{2!}(s-x)^2\left(f_0''' + pf_1''' + \cdots\right) ds \\ &\quad - (n-1)\, p\int_0^x \frac{1}{2!}(s-x)^2 (f_0 + pf_1)\left(f_0'' + pf_1''\right) ds \\ &\quad + 2pn\int_0^x \frac{1}{2!}(s-x)^2\left(f_0' + pf_1' + p^2 f_2' + \cdots\right)^2 ds\end{aligned}$$

where $f''(0) = \alpha < 0$.Comparing the co-efficient of like powers of p

$$
\begin{aligned}
p^{(0)} &: f_0(x) = x, \\
p^{(1)} &: f_1(x) = x + \frac{1}{2}\alpha x^2, \\
p^{(2)} &: f_2(x) = x + \frac{1}{2}\alpha x^2 + \frac{1}{3}x^3 + \frac{1}{24}\alpha(3n+1)x^4 + \frac{1}{30}n(n+1)x^5, \\
p^{(3)} &: f_3(x) = x + \frac{1}{2}\alpha x^2 + \frac{1}{3}x^3 + \frac{1}{24}\alpha(3n+1)x^4 + \frac{1}{30}n(n+1)x^5 + \frac{1}{120}\alpha^2(3n+1)x^5 \\
& + \frac{1}{720}\alpha\left(19n^2 + 18n + 3\right)x^6 + \frac{1}{315}n\left(2n^2 + 2n + 1\right)x^7, \\
p^{(4)} &: f_4(x) = x + \frac{1}{2}\alpha x^2 + \frac{1}{3}x^3 + \frac{1}{24}\alpha(3n+1)x^4 + \frac{1}{30}n(n+1)x^5 + \frac{1}{120}\alpha^2(3n+1)x^5 \\
& + \frac{1}{720}\alpha\left(19n^2 + 18n + 3\right)x^6 + \frac{1}{5040}\alpha^2\left(27n^2 + 42n + 11\right)x^7 \\
& + \frac{1}{40320}\alpha\left(167n^3 + 297n^2 + 161n + 15\right)x^8 + \frac{1}{22680}n\left(13n^3 + 38n^2 + 23n + 6\right)x^9, \\
& \vdots
\end{aligned}
$$

The series solution is given as

$$
\begin{aligned}
f(x) &= x + \frac{1}{2}\alpha x^2 + \frac{1}{3}nx^3 + \left(\frac{1}{24}\alpha + \frac{1}{8}n\alpha\right)x^4 + \left(\frac{1}{30}n^2 + \frac{1}{40}n\alpha^2 + \frac{1}{120}\alpha^2 + \frac{1}{30}n\right)x^5 \\
&+ \left(\frac{19}{720}n^2\alpha + \frac{1}{240}\alpha + \frac{1}{40}n\alpha\right)x^6 \\
&+ \left(\frac{1}{120}n\alpha^2 + \frac{1}{315}n + \frac{2}{315}n^3 + \frac{11}{5040}\alpha^2 + \frac{3}{560}n^2\alpha^2 + \frac{2}{315}n^2\right)x^7 \\
&+ \left(\frac{11}{40320}\alpha^3 + \frac{33}{4480}n^2\alpha + \frac{3}{4480}\alpha^3 n^2 + \frac{23}{5760}n\alpha + \frac{1}{2688}\alpha + \frac{167}{40320}n^3\alpha + \frac{1}{960}\alpha^3 n\right)x^8 \\
&+ \left(\frac{1}{3780}n + \frac{527}{362880}n^3\alpha^2 + \frac{19}{11340}n^3 + \frac{709}{362880}n\alpha^2\right)x^9 \\
&+ \left(\frac{23}{8064}n^2\alpha^2 + \frac{23}{22680}n^2 + \frac{13}{22680}n^4 + \frac{43}{120960}\alpha^2\right)x^9 + \cdots
\end{aligned}
$$

Table 6.2 Numerical values for $\alpha = f''(0)$for $0 < n < 1$by using diagonal Pade approximants [31]

n	[2/2]	[3/3]	[4/4]	[5/5]	[6/6]
0.2	-0.3872983347	-0.3821533832	-0.3819153845	-0.3819148088	-0.3819121854
1/3	-0.5773502692	-0.5615999244	-0.5614066588	-0.5614481405	-0.561441934
0.4	-0.6451506398	-0.6397000575	-0.6389732578	-0.6389892681	-0.6389734794
0.6	-0.8407967591	-0.8393603021	-0.8396060478	-0.8395875381	-0.8396056769
0.8	-1.007983207	-1.007796981	-1.007646828	-1.007646828	-1.007792100

Table 6.3 Numerical values for for for by using diagonal Pade approximants [31]

n	α
4	-2.483954032
10	-4.026385103
100	-12.84334315
1000	-40.65538218
5000	-104.8420672

4 Conclusion

In this paper, we applied the modified variational iteration method (MVIM) for boundary layer equation in an unbounded domain. The Pade approximants were employed in order to make the work more concise and for the better understanding of the solution behavior. It may be concluded that the proposed frame work is very powerful and efficient in finding the analytical solutions for such problems. The method gives more realistic series solutions that converge very rapidly in physical problems. The fact that the proposed MVIM solves nonlinear problems without using Adomian's polynomials can be considered as a clear advantage of this technique over the decomposition method. Moreover, the use of Lagrange multiplier reduces the successive use of integral operator and it may be considered as an added advantage of this technique over the decomposition method.

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