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New Double Periodic and Solitary Wave Solutions to the Modified Kawahara Equation

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Abstract: Making use of the generalized elliptic equation, the mapping method with aid of the symbolic computation system Mathematica is used for constructing new traveling wave solutions of the modified Kawahara equation. The solutions found in this paper include periodic solutions and complex periodic wave solutions. In the limit cases the multiple solitons solutions, complex solitons solutions, trigonometric solutions and rational solutions can be obtained. The properties of some solutions for the modified Kawahara equation are shown by some figures.

Keywords: periodic solutions; modified Kawahara equation; elliptic functions; solitary solutions; mapping method

1 Introduction

The world around us is inherently nonlinear. Nonlinear partial differential equations (NLPDEs) in mathematical physics are widely used to describe complex phenomena in various fields of sciences, especially in physics. So, it is greatly important to find periodic solutions of NLPDEs to provide more information for understanding many physical phenomena arising in numerous scientific and engineering fields. In the last decades, direct search for exact solutions of NLPDEs has become increasingly attractive partly due to the availability of computer symbolic software like Mathematica or Maple, which allow us to perform complicated and tedious algebraic calculations as well as help us to find exact solutions of NLPDEs. There has been a great amount of activity aiming to find powerful methods for obtaining such solutions. We can cite, the Backlund transformation [1], the Darboux transformation [2], the Jacobi elliptic function method [3], the tanh- function method [4], the sine- cosine function method [5,6], the homogenous balance method [7] and the Jacobi function expansion method [8,9]. Very recently, a unified method called the mapping method has been developed to obtain Jacobi elliptic functions, solitons and periodic solutions to some NLPDEs [10]. The remarkable observation about the mapping method is that it allows one to find Jacobi elliptic functions, triangular functions and hyperbolic functions using the same procedure. Moreover, this method permits the classification of solutions depending on four parameters.

Therefore, the motivation of this paper is to take full advantage of the elliptic equation that Jacobi elliptic functions satisfy and use its solutions to obtain new periodic, complex line periodic, solitary and complex line soliton solutions of the modified Kawahara equation. Before discussing the solutions for the modified Kawahara equation, let us simply review the technique of solution. For a given nonlinear partial differential equation

$$H(u, u_t, u_x, u_{xx}, ...) = 0, \tag{1}$$

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we seek its traveling wave solution of the form

$$u(x,t)=u(\zeta),\quad \zeta=x-\lambda t. \tag{2}$$

Substituting Eq.(2) into Eq.(1) yields an ordinary differential equation of $u(\zeta)$. Then $u(\zeta)$ is expanded into a polynomial in $f(\zeta)$

$$u(\zeta)=\sum_{i=0}^{n} a_i f^i, \tag{3}$$

where a_i are constants to be determined and n is fixed by balancing the linear term of the highest order derivative with nonlinear term. In the present work, we shall introduce the generalized auxiliary ordinary differential equation

$$\begin{array}{l} f'^2(\zeta)=pf^2(\zeta)+\frac{1}{2}qf^4(\zeta)+\frac{1}{3}sf^6(\zeta)+r \\ f''(\zeta)=pf(\zeta)+qf^3(\zeta)+sf^5(\zeta), \end{array} \tag{4}$$

here the prime means derivatives with respect to ζ and p, q, s, r are constants. After Eq.(3) with Eq.(4) is substituted into the ordinary differential equation, the coefficients $a_i, k,\ \lambda,\ p, q, s$ and r may be determined. Thus, equation (3) establishes an algebraic mapping relation between the solution of Eq.(1) and that of Eq.(4). We shall construct periodic, complex periodic, solitary and complex soliton solutions for the modified Kawahra equation by using solutions of Eq.(4), see appendix A. I would to point out that Jacobi elliptic functions sn(ζm), cn(ζm), dn(ζm), where m $(0 < m < 1)$ is the modulus of elliptic function, are doubly periodic and possess properties of triangular functions. In addition we see that other solutions are obtained from appendix A in case of degeneracy. As we know, when $m \to 1$, Jacobi elliptic functions degenerate as hyperbolic functions as indicated in Appendix B. Also, Jacobi elliptic functions degenerate into trigonometric functions as shown in appendix C when $m \to 0$.

2 The modified Kawahara equation

We consider the modified Kawahara equation

$$u_t+u_x+u^2u_x+\alpha u_{xxx}+\beta u_{xxxxx}=0, \tag{5}$$

where α, β are nonzero real constant. This equation arises in the theory of shallow water waves [13] and its exact solutions were obtained by using the tanh-function method [14] and the sech-function method [15]. After using the transformation $u(x,t)=u(\zeta),\quad \zeta=x-\lambda t$, and integrating once, Eq.(5) becomes

$$(1-\lambda)u+\frac{1}{3}u^3+\alpha u''+\beta u^{(4)}=c_1. \tag{6}$$

The solution of Eq.(6) may be chosen as

$$u(\zeta)=\sum_{i=0}^{n} a_i f^i(\zeta), \tag{7}$$

with arbitrary constants $a_i (i=0,1,\ldots n)$ to be determined later. Balancing the highest derivative term $u^{(4)}$with the highest power nonlinear term u^3 gives the leading order $n=2$. Substituting (7) into (6) along with Eq.(4) and using Mathematica yields a system of equations with respect to $f^i(\zeta)$. Setting the coefficients of $f^i(\zeta)$ in the obtained system of equations to zero, we get the following set of algebraic equations

$$\begin{array}{l} 3a_0+a_0^3-3c_1+6a_2r\alpha+24a_2pr\beta-3a_0\lambda=0, \\ 3a_1(1+a_0^2+p\alpha+p^2\beta+6qr\beta-\lambda)=0, \\ a_0a_1^2+a_0^2a_2+a_2(1+4p\alpha+16p^2\beta+36qr\beta-\lambda)=0, \\ a_1(a_1^2+6a_0a_2+3q\alpha+30pq\beta+60rs\beta)=0 \\ 3a_2(a_1^2+a_0a_2+3q\alpha+60pq\beta+80rs\beta)=0, \\ 3a_1(a_2^2+6q^2\beta+s(\alpha+26p\beta))=0, \\ a_2(a_2^2+2(45q^2\beta+4s(\alpha+40p\beta))=0, \\ 60s\beta a_1q=0, 240s\beta a_2q=0, 35s^2\beta a_1=0, 128s^2a_2\beta=0. \end{array} \tag{8}$$

Solving these algebraic equations, by use of Mathematica, we obtain the following six sets of solutions

$$\begin{array}{l} a_0=\frac{15c_1\beta(\alpha+20p\beta)}{\alpha^3-120(2p^2-3qr)\alpha\beta^2+800p(4p^2-9qr)\beta^3}, a_1=0, a_2=\frac{450c_1q\beta^2}{\alpha^3-120(2p^2-3qr)\alpha\beta^2+800p(4p^2-9qr)\beta^3},\\ c_1=\frac{1}{15\sqrt{10}}\sqrt{-\frac{(\alpha^3-120(2p^2-3qr)\alpha\beta^2+800p(4p^2-9qr)\beta^3)^2}{\beta^3}}, \lambda=1-\frac{\alpha^2}{10\beta}-24p^2\beta+36qr\beta, q\neq 0, \beta\neq 0, \end{array} \tag{9}$$

$$\begin{array}{l} a_0=\frac{9c_1p^4-6c_1p^2qr}{(171p^6-294p^4qr-204p^2q^2r^2-40q^3r^3)\beta^3}, a_1=0, a_2=\frac{18c_1p^3q}{(171p^6-294p^4qr-204p^2q^2r^2-40q^3r^3)\beta^3},\\ c_1=\frac{1}{3}\sqrt{\frac{5}{2}}\sqrt{-\frac{(-171p^6-294p^4qr-204p^2q^2r^2-40q^3r^3)^2\beta^3}{p^6}},\\ 2\lambda=2-53p^2\beta-+52qr\beta-\frac{20q^2r^2\beta}{p^2}, \alpha=-5p-\frac{10qr}{p}, p\neq 0, q\neq 0, \beta\neq 0, \end{array} \tag{10}$$

$$\begin{array}{l} a_0=\frac{15c_1}{13936p^2\beta}, a_1=0, a_2=-\frac{225c_1q}{13936p^3\beta}, c_1=\sqrt{\frac{-388424192p^6\beta^3}{1125}}\\ r=\frac{17p^2}{10q},\ \alpha=-22p\beta,\ \lambda=1-\frac{56}{5}p^2\beta, p\neq 0, q\neq 0, \beta\neq 0, \end{array} \tag{11}$$

$$\begin{array}{l} a_0=\frac{15c_1}{4\beta(271\,p^2+1890\,qr)}, a_1=0, a_2=-\frac{225c_1q}{4(271p^3\beta+1890\,pqr\beta)},\\ c_1=\sqrt{\frac{p^2(-2350112p^4-32780160p^2qr-114307200q^2r^2)\beta^3}{1125}},\\ r=\frac{17p^2}{10q},\ \alpha=-22p\beta,\ \lambda=1-\frac{362p^2\beta}{5}+\frac{180qr\beta}{5},\ q\neq 0, \beta\neq 0, \end{array} \tag{12}$$

$$\begin{array}{l} a_0=0, a_1=0, a_2=\pm\frac{i\,c_1q}{8r\alpha}, c_1=24\sqrt{\frac{2}{5}}qr\alpha\sqrt{\beta},\ q=\frac{4\alpha^2-25\beta+25\beta\lambda}{900r\beta^2},\ p=-\frac{\alpha}{20\beta},\\ r\beta\neq 0, \alpha\neq 0, \quad 4\alpha^2-25\beta+25\beta\lambda\neq 0, \end{array} \tag{13}$$

and

$$\begin{array}{l} a_0=\frac{3(c_1\alpha+20c_1p\beta)}{2(-\alpha+2p\alpha^2-20p\beta+48p^2\alpha\beta-72qr\alpha\beta+160p^3\beta^2+\alpha\lambda+20p\beta\lambda)},\\ a_1=0,\ \ a_2=-\frac{3(q\alpha+20pq\beta)}{a_0},\ \ p=\pm\frac{i\sqrt{\alpha^2-10\beta-360qr\beta^2+10\beta\lambda}}{4\sqrt{15}\beta},\\ q\beta\neq 0, \alpha\neq 0, \quad \alpha-2p\alpha^2-48p^2\alpha\beta+72qr\alpha\beta-160p^3\beta^2-\alpha\lambda-20p\beta\lambda\neq 0,\\ c_1=\pm\frac{2}{9\sqrt{5}}\sqrt{A_1+A_2+A_3+A_4},\\ A_1=-8448p^4\alpha^2\beta-576p^3\alpha(\alpha\beta(\lambda-1))\\ A_2=1296pqr\alpha(\alpha^2+5\beta(\lambda-1))+8p^2(32\alpha^2(2+99qr\beta-2\lambda)\\ A_3=35\beta(\lambda-1)^2)-3(129600q^3r^3\beta^3+72q^2r^2\beta(103\alpha^2+150\beta(\lambda-1)).\\ A_4=4qr(-64\alpha^2+35\beta(\lambda-1))(\lambda-1)-15(\lambda-1)^3)) \end{array} \tag{14}$$

For the above six sets of solutions, we get six types of periodic and complex wave solutions as following

$$u=\frac{15c_1\beta(\alpha+20p\beta)}{\alpha^3-120(2p^2-3qr)\alpha\beta^2+800p(4p^2-9qr)\beta^3}+\frac{450c_1q\beta^2}{\alpha^3-120(2p^2-3qr)\alpha\beta^2+800p(4p^2-9qr)\beta^3}f^2(\zeta\,|m)\ , \tag{15}$$

$$u=\frac{9c_1p^4-6c_1p^2qr}{(171p^6-294p^4qr-204p^2q^2r^2-40q^3r^3)\beta^3}+\frac{18c_1p^3q}{(171p^6-294p^4qr-204p^2q^2r^2-40q^3r^3)\beta^3}f^2(\zeta\,|m)\ , \tag{16}$$

$$u=\frac{15c_1}{13936p^2\beta}-\frac{225c_1q}{13936p^3\beta}f^2(\zeta\,|m)\ , \tag{17}$$

$$u=\frac{15c_1}{4\beta(271\,p^2+1890\,qr)}-\frac{225c_1q}{4(271p^3\beta+1890\,pqr\beta)}f^2(\zeta\,|m)\ , \tag{18}$$

$$u=\pm\frac{i\,c_1q}{8r\alpha}f^2(\zeta\,|m)\ , \tag{19}$$

$$u=\frac{3(c_1\alpha+20c_1p\beta)}{2(-\alpha+2p\alpha^2-20p\beta+48p^2\alpha\beta-72qr\alpha\beta+160p^3\beta^2+\alpha\lambda+20p\beta\lambda)}-\frac{3(q\alpha+20pq\beta)}{a_0}f^2(\zeta\,|m) \tag{20}$$

Depending on p, q, s and r in Eq.(4), we may obtain many periodic and solitons wave solutions of Eq.(5), see the Appendix A. Due to the large number of solutions in appendix A, it is not advisable to treat every set. Therefore, we shall only deal with one set and one case from appendix A as illustrative examples.

3 Solutions of the first set

By choosing $p = 2m^2 - 1, q = -2m^2, s = 0, r = 1 - m^2$, the solution of Eq. (4) reads $f = cn(\zeta\,|m)$. We obtain the periodic solutions

$$\begin{aligned}&u = \tfrac{15c_1\beta(\alpha+20(-1+2m^2)\beta)}{b_1} - \tfrac{900c_1m^2\beta^2}{b_1}cn^2(\zeta\,|m),\\ &\lambda = 1 - \tfrac{\alpha^2}{10\beta} - 72m^2(1-m^2)\beta - 24(-1+2m^2)^2\beta,\ c_1 = \tfrac{1}{15\sqrt{10}}\sqrt{-\tfrac{b_1^2}{\beta^3}},\\ &b_1 = \alpha^3 - 120(6m^2(1-m^2) + 2(-1+2m^2)^2)\alpha\beta^2 + 800(-1+2m^2)(18m^2(1-m^2) + 4(-1+2m^2)^2)\beta^3.\end{aligned} \tag{21}$$

In case of degeneracy, we can obtain the following solitary solution when m goes to one.

$$\begin{aligned}u &= \tfrac{15c_1\beta(\alpha+20\beta)}{\alpha^3-240\alpha\beta^2+3200\beta^3} - \tfrac{900c_1\beta^2}{\alpha^3-240\alpha\beta^2+3200\beta^3}\sec h^2(\zeta),\\ \lambda &= 1 - \tfrac{\alpha^2}{10\beta} - 24\beta, c_1 = \tfrac{1}{15\sqrt{10}}\sqrt{-\tfrac{(\alpha^3-240\alpha\beta^2+3200\beta^3)^2}{\beta^3}}.\end{aligned} \tag{22}$$

The structure graphs of Eqs.(21) and (22) are plotted in Figure.1 and Figure.2, the parameters are α=10, β=-5, m=0.3. They are valid throughout this section unless otherwise stated.

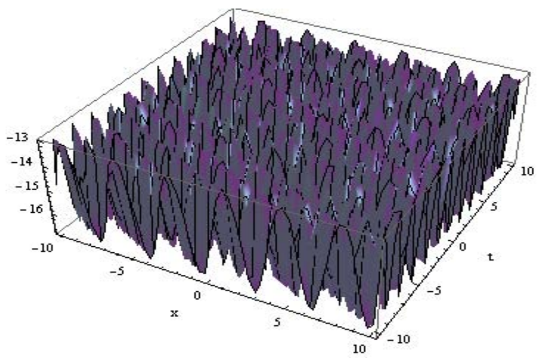

Figure 1: The periodic wave solution of Eq.(21)

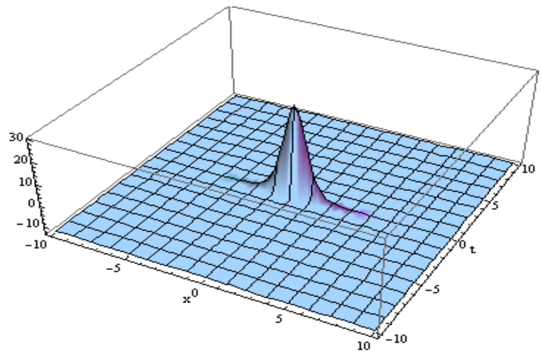

Figure 2: The solitary wave solution of Eq.(22)

4 When c_1 equals zero

In a similar way, in case of taking the constant of integration equals zero, we can obtain three types of complex periodic wave solutions

$$\begin{aligned}&u = \pm\tfrac{i(\sqrt{10}\alpha+20\sqrt{10}p\beta)}{10\sqrt{\beta}} \pm 3i\sqrt{10}q\sqrt{\beta})f^2(\zeta\,|m)\ ,\\ &\alpha^3 + \alpha\beta^2(360qr - 240p^2) - p\beta^2(7200qr - 3200p^2) = 0, \lambda = 1 - \tfrac{\alpha^2}{10\beta} - 24p^2\beta + 36qr\beta,\end{aligned} \tag{23}$$

$$\begin{aligned}&u = \pm\tfrac{i(3\sqrt{10}p^2\sqrt{\beta}-2\sqrt{10}qr\sqrt{\beta})}{2p} \pm 3i\sqrt{10}q\sqrt{\beta})f^2(\zeta\,|m)\ ,\\ &40q^3 + \tfrac{294p^4q}{r^2} + \tfrac{204p^2q^2}{r^2} - \tfrac{171p^6q}{r^3} = 0, \alpha = \beta(-5p - \tfrac{10qr}{p}), \lambda = 1 + \tfrac{188p^2\beta}{4} - \tfrac{171p^4\beta}{4qr} + \tfrac{308qr\beta}{4},\end{aligned} \tag{24}$$

and

$$u = \pm i\sqrt{\tfrac{2}{5}}p\sqrt{\beta} \pm 3i\sqrt{10}q\sqrt{\beta})f^2(\zeta\,|m)\ , r = -\tfrac{271p^2}{1890q}, \alpha = -22p\beta, lambda = 1 - \tfrac{8144p^2\beta}{105}. \tag{25}$$

By choosing $p = 2 - m^2, q = -2, s = 0, r = m^2 - 1$, the solution of Eq. (4) reads $f = dn(\zeta\,|m)$. The periodic solutions of Eq.(23) is

$$\begin{aligned}u &= \pm\tfrac{i(\sqrt{10}\alpha+20\sqrt{10}(2-m^2)\beta)}{10\sqrt{\beta}} \pm i6\sqrt{10}\sqrt{\beta}dn^2(\zeta\,|m)\ ,\\ \lambda &= 1 - \tfrac{\alpha^2}{10\beta} - 24(2-m^2)^2\beta - 72(m^2-1)\beta\end{aligned} \tag{26}$$

When $m \to 1$, this solution becomes

$$u = \pm\frac{i(\sqrt{10}\alpha+20\sqrt{10}\beta)}{10\sqrt{\beta}} \pm i6\sqrt{10}\sqrt{\beta}\sec h^2\zeta, lambda = 1 - \frac{\alpha^2}{10\beta} - 24\,\beta \tag{27}$$

The modulus of solutions (26) and (27) are shown graphically in Fig.3 and Fig.4.

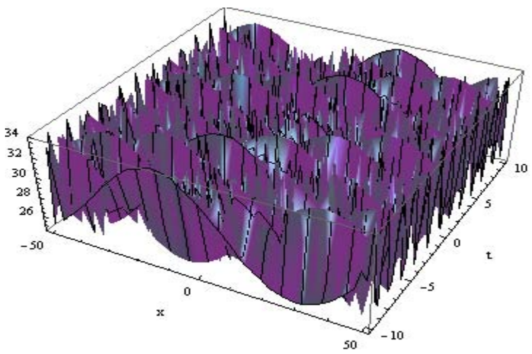

Figure 3: The modulus of the complex periodic solution of Eq.(26)

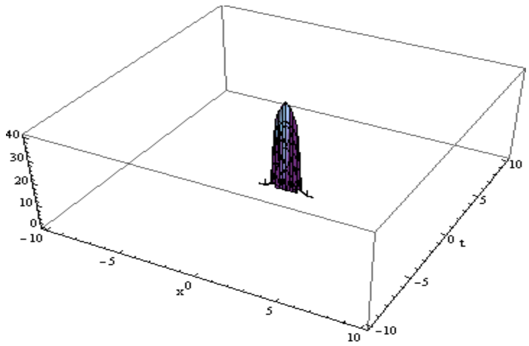

Figure 4: The modulus of the complex solitary solution of Eq.(27)

5 Conclusion and discussion

In this work, new periodic and solitary solutions of the modified Kawahara equation are obtained by using the mapping method. The idea of our method is to use the generalized elliptic equation involving four parameters instead of specific functions as in previous methods [5-13]. The remarkable observation about the mapping method is that it allows one to find Jacobi elliptic functions, triangular functions and hyperbolic functions using the same procedure. Moreover, this method permits the classification of solutions depending on four parameters.

In addition, we obtained some complex periodic and complex soliton solutions for the modified Kawahara equation. We suppose that these solutions are not just only the extension from mathematical meaning, but in the hope that they will lead to a deeper and more comprehensive understanding of the complex structure resulted from the nonlinearity of traveling wave equations.

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Appendix A: Different solutions of the elliptic equation (4)

Case	Arbitrary constants	Solutions of Eq. (4)
1	$p=-(1+m^2), q=2m^2, s=0, r=1.$	$f(\zeta)=sn\zeta,\ f(\zeta)=cd\zeta.$
2	$p=2m^2-1, q=-2m^2, s=0, r=1-m^2.$	$f(\zeta)=cn\zeta.$
3	$p=2-m^2, q=-2, s=0, r=m^2-1.$	$f(\zeta)=dn\zeta.$
4	$p=2m^2-1, q=2, s=0, r=-m^2(1-m^2).$	$f(\zeta)=ds\zeta.$
5	$p=2-m^2-1, q=2, s=0, r=1-m^2.$	$f(\zeta)=cs\zeta.$
6	$p=\frac{m^2-2}{2}, q=\frac{m^2}{2}, s=0, r=\frac{1}{4}.$	$f(\zeta)=\frac{sn\zeta}{1\pm dn\zeta}.$
7	$p=\frac{m^2-2}{2}, q=\frac{m^2}{2}, s=0, r=\frac{m^2}{4}.$	$f(\zeta)=sn\zeta\pm icn\zeta,$ $f(\zeta)=\frac{dn\zeta}{i\sqrt{1-m^2}\,sn\zeta\pm dn\zeta}.$
8	$p=\frac{1-2m^2}{2}, q=\frac{1}{2}, s=0, r=\frac{1}{4}.$	$f(\zeta)=\frac{dn\zeta}{mcn\zeta\pm i\sqrt{1-m^2}}, f(\zeta)=\frac{cn\zeta}{\sqrt{1-m^2}sn\zeta\pm dn\zeta},$ $f(\zeta)=\frac{sn\zeta}{1\pm cn\zeta}, f(\zeta)=msn\zeta\pm idn\zeta.$
9	$p=\frac{m^2+1}{2}, q=\frac{m^2-1}{2}, s=0, r=\frac{m^2-1}{4}.$	$f(\zeta)=\frac{dn\zeta}{1\pm msn\zeta}.$
10	$p=\frac{1+m^2}{2}, q=\frac{1-m^2}{2}, s=0, r=\frac{1-m^2}{4}.$	$f(\zeta)=\frac{cn\zeta}{1\pm sn\zeta}.$
11	$p=\frac{1+m^2}{2}, q=-\frac{1}{2}, s=0, r=-\frac{(1-m^2)^2}{4}.$	$f(\zeta)=mcn\zeta\pm dn\zeta.$
12	$p=\frac{1+m^2}{2}, q=\frac{(1-m^2)^2}{2}, s=0, r=\frac{1}{4}.$	$f(\zeta)=\frac{sn\zeta}{dn\zeta\pm cn\zeta}.$
13	$p=\frac{m^2-2}{2}, q=\frac{m^2}{2}, s=0, r=\frac{1}{4}.$	$f(\zeta)=\frac{cn\zeta}{\sqrt{1-m^2}\pm dn\zeta}.$
14	$p=0, q=2, s=0, r=0.$	$f(\zeta)=\frac{C}{\zeta}.$

Here C is a constant.

Appendix B: Jacobi elliptic functions degenerate into hyperbolic functions when the modulus is approaching 1

sn$\zeta\to$tanhζ	cn$\zeta\to$sechζ	dn$\zeta\to$sechζ	sc$\zeta\to$sinhζ
Sd$\zeta\to$sinhζ	cd$\zeta\to$1	dc$\zeta\to$1	ns$\zeta\to$cothζ
Nd$\zeta\to$coshζ	cs$\zeta\to$cschζ	ds$\zeta\to$cschζ	nc$\zeta\to$coshζ

Appendix C: Jacobi elliptic functions degenerate into trigonometric functions when the modulus is approaching 0

sn$\zeta\to$sinζ	cn$\zeta\to$cosζ	dn$\zeta\to$1	sc$\zeta\to$tanζ
Sd$\zeta\to$sinζ	cd$\zeta\to$cosζ	ns$\zeta\to$cscζ	nc$\zeta\to$secζ
Nd$\zeta\to$1	cs$\zeta\to$cotζ	ds$\zeta\to$cscζ	dc$\zeta\to$secζ