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Uniqueness of Positive Solutions for a Class of Quasilinear Problems with Multiple Parameters

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Abstract: we prove uniqueness of positive solution for the quasilinear problems

$$-\Delta_p u = \lambda f(u) + \mu\gamma(u) \qquad \text{in } \Omega, \quad u = 0 \qquad \text{on } \partial\Omega$$

where Ω is a bounded domain in R^N, with smooth boundary $\partial\Omega$, f, g are p-sublinear at ∞ for positive number p with $p > 1$, $\frac{f(u)}{u^{p-1}}$, $\frac{\gamma(u)}{u^{p-1}}$, are decreasing for large u, and λ, μ are large positive parameters.We also obtain the asymptotic behavior of the solution obtain as $\lambda, \mu \to \infty$.

Keywords: uniqueness; positive solutions; quasilinear problems; multiple parameters.
AMS Subject Classification: 35J60, 35B30, 35B40

1 Introduction

Consider the quasilinear boundary value problems

$$\begin{cases} -\Delta_p u = \lambda f(u) + \mu\gamma(u) & in\ \Omega \\ u(x) = 0 & on\ \partial\Omega \end{cases} \tag{1}$$

where the p-Laplacian operator $\Delta_p z = div(|\nabla z|^{p-2}\nabla z)$, $p > 1$, $f(u), \gamma(u) > 0$, for $u > 0$ λ,μ are positive parameters and Ω is bounded domain in R^N with smooth boundary $\partial\Omega$.

$$\begin{cases} -\Delta_p u = \lambda f(u) & in\ \Omega \\ u(x) = 0 & on\ \partial\Omega \end{cases} \tag{2}$$

Problem (2) has been investigated by many authors in recent years(see e.g.,[4,5,7-9,11,14,16,17]).When $p = 2$, uniqueness of positive solutions to (2) for λ large and $\frac{f(u)}{u}$ decreasing for large u was established in Angenent [2], Dancer [3], Hai and Smith [10], Lin [12], Schuchman [15] and Wiegner [18] and uniqueness of positive solutions for a class of quasilinear problems (2) when $p > 1$ and $\frac{f(u)}{u^{p-1}}$ is decreasing for u large and λ is a large positive parameter was obtain in Hai [1], uniqueness of positive solutions to (2) when $p > 1$ and $\frac{f(u)}{u^{p-1}}$ is decreasing on $(0, \infty)$ was obtain in Geo and Webb [8], Diaz and Saa [4], and Drabek and Hernandez [5]. In this paper, we give a positive answer to the above question. We also obtained asymptotic behavior of solutions as $\lambda, \mu \to \infty$. Our approach depends on sharp upper and lower estimates of solutions together with the maximum and comparison principles.

2 Main result

We make the following assumptions:

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(A.1) $f, \gamma : (0, \infty) \longrightarrow R$ are nondecreasing , continuous and of class C^1 on $(0, \infty)$.
(A.2) $\liminf_{u\to 0^+} \frac{f(u)}{u^{p-1}} > 0$, $\liminf_{u\to 0^+} \frac{\gamma(u)}{u^{p-1}} > 0$.
(A.3) $\limsup_{u\to 0^+} uf'(u) < \infty$, $\limsup_{u\to 0^+} u\gamma'(u) < \infty$.
(A.4) There exist $q \in (0, p-1)$ and a positive number a such that $\frac{f(u)}{u^q}$ and $\frac{\gamma(u)}{u^q}$ are decreasing on $[a, \infty)$.
(A.5) $g : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing, and there exists $q_1 \in (0, p-1)$ such that $\frac{g(u)}{u^{q_1}}$ is decreasing on $(0, \infty)$
(A.6) $\lim_{u\to\infty} \frac{g(cu)}{g(u)}$ exists and is finite for each $c \in (0, 1)$.
Our main result are

Theorem. *Let (A.1)-(A.4) hold. Then there exist* $\lambda_0 > 0, \mu_0 > 0$ *such that problem (1) has a unique positive solution for* $\lambda > \lambda_0$, $\mu > \mu_0$.

Let g satisfy (A.5). Then, for each $\lambda > 0$, there exists a unique positive solution v_λ to the problem

$$\begin{cases} -\Delta_p v_\lambda = \lambda g(v_\lambda) & in\ \Omega \\ v_\lambda = 0 & on\ \partial\Omega \end{cases} \tag{3}$$

(see e.g., [4,5,8] or Proposition A in the Appendix)

3 Preliminary lemmas

As usual, we shall denote by $|.|_{k,\alpha}$ and $||.||_k$ the norms in $C^{k,\alpha}(\overline{\Omega})$ and $L^k(\Omega)$ respectively. Let λ_1 be the first eigenvalue of $-\Delta_p$ with zero boundary conditions, and ϕ_1 a corresponding normalized eigenfunction, i.e., $||\phi_1||_\infty = 1$, and

$$\begin{cases} -\Delta_p \phi_1 = \lambda_1 \phi_1 & in\ \Omega \\ \phi_1 = 0 & on\ \partial\Omega \end{cases} \tag{4}$$

Then $\lambda_1 > 0$ and we can assume that $\phi_1 > 0$ in Ω (see [14]).

Lemma 3.1. *Let (A.1)-(A.2) hold. Then there exist positive number* k, η *such that any positive solution of (1)satisfies*

$$u \geq \eta\phi_1 \quad in\ \Omega \quad for\ \ \lambda > \frac{\lambda_1}{k}, \mu > \frac{\mu_1}{k}$$

Proof. By (A.2), there exist $k, \eta > 0$ such that

$$f(u) > ku^{p-1}, \gamma(u) > ku^{p-1} \qquad \text{for } u \in (0, \eta].$$

Suppose that $\lambda > \frac{\lambda_1}{k}$, $\mu > \frac{\mu_1}{k}$ and let u be positive solution of (1). By the strong maximum principle [17], there exists $\epsilon > 0$ such that $u \geq \epsilon\phi_1$ in Ω. Let η_0 be the largest number such that $u \geq \eta_0\phi_1$ in Ω and suppose by contradiction that $\eta_0 < \eta$. Let

$$\Omega_0 = \{x \in \Omega : u(x) < \eta\phi_1(x)\}$$

and $m = min\{(\frac{\lambda k}{\lambda_1})^{\frac{1}{p-1}}, (\frac{\mu k}{\mu_1})^{\frac{1}{p-1}}, \frac{\eta}{\eta_0}\}$. Then $\Omega_0 \neq \emptyset$ and

$$-\Delta_p u = \lambda f(u) + \mu\gamma(u) \geq (\lambda + \mu)k(\eta_0\phi_1)^{p-1} \geq (\lambda_1 + \mu_1)(m\eta_0\phi_1)^{p-1} \quad \text{in } \Omega_0$$

$$u = \eta\phi_1 \quad \text{on } \partial\Omega_0$$

Since

$$-\Delta_p(m\eta_0\phi_1) = (\lambda_1 + \mu_1)(m\eta_0\phi_1)^{p-1} \quad \text{in } \Omega_0$$

$$m\eta_0\phi_1 \leq \eta\phi_1 \quad \text{on } \partial\Omega_0$$

It follows from the weak comparison principle (see, e.g.,[14,16]) that

$$u \geq m\eta_0\phi_1 \quad \text{in } \Omega_0$$

Clearly,

$$u \geq \eta\phi_1 \geq m\eta_0\phi_1 \quad \text{in } \Omega\backslash\Omega_0$$

Hence $u \geq m\eta_0\phi_1$ in Ω, and since $m > 1$, this contradicts the maximality of η_0. This complete the proof of lemma 3.1. ☐

Next, we define $H(u) = \frac{u}{f^{\frac{1}{p-1}}(u)}, I(u) = \frac{u}{\gamma^{\frac{1}{p-1}}(u)}$. Then H, I are increasing in $(0, \infty)$ and $\lim_{u\to\infty} H(u) = \infty, \lim_{u\to\infty} I(u) = \infty$ if (A.4) holds.

Lemma 3.2 *Let (A.1) and (A.4) hold. Then for each* $C > 0$, *there exist* M_1, M_2, *and* $\tilde{\lambda}, \tilde{\mu} > 0$ *such that*

$$M_1(H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}})) \leq H^{-1}(\lambda^{\frac{1}{p-1}}C) + I^{-1}(\mu^{\frac{1}{p-1}}C) \leq M_2(H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}})) \\ \text{for } \lambda > \tilde{\lambda}, \mu > \tilde{\mu}$$

Proof. By writing $H^{-1}(\lambda^{\frac{1}{p-1}})$ as $H^{-1}(\frac{\eta^{\frac{1}{p-1}}}{C}), I^{-1}(\mu^{\frac{1}{p-1}})$ as $I^{-1}(\frac{\eta'^{\frac{1}{p-1}}}{C})$, where $\eta^{\frac{1}{p-1}} = \lambda^{\frac{1}{p-1}}C$, $\eta'^{\frac{1}{p-1}} = \mu^{\frac{1}{p-1}}C$, we see that the left-hand inequality follows from the right hand one.

Let $C > 0, r = \frac{q}{p-1}$. Then $\frac{f^{\frac{1}{p-1}}}{x^r}, \frac{\gamma^{\frac{1}{p-1}}}{x^r}$ are decreasing on $[a, \infty)$. Let $\tilde{\lambda} = (\frac{H(a)}{min(1,C)})^{p-1}$, $\tilde{\mu} = (\frac{I(a)}{min(1,C)})^{p-1}$ and $\theta = max(C^{\frac{1}{1-r}}, 1)$. Let $\lambda > \tilde{\lambda}, \mu > \tilde{\mu}$, and $x = H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}})$. Then $x \geq a$ and $g(\theta x) \leq \theta^r g(x)$,

$$\frac{\theta x}{f^{\frac{1}{p-1}}(\theta x)} \geq \frac{\theta x}{\theta^r f^{\frac{1}{p-1}}(x)} = \lambda^{\frac{1}{p-1}}\theta^{1-r} \geq \lambda^{\frac{1}{p-1}}C,$$

$$\frac{\theta x}{\gamma^{\frac{1}{p-1}}(\theta x)} \geq \frac{\theta x}{\theta^r \gamma^{\frac{1}{p-1}}(x)} = \mu^{\frac{1}{p-1}}\theta^{1-r} \geq \mu^{\frac{1}{p-1}}C,$$

which implies $\theta x \geq H^{-1}(\lambda^{\frac{1}{p-1}}C), \theta x \geq I^{-1}(\mu^{\frac{1}{p-1}}C)$, so $2\theta x \geq H^{-1}(\lambda^{\frac{1}{p-1}}C) + I^{-1}(\mu^{\frac{1}{p-1}}C)$.

This completes the proof of lemma 3.2. ☐

Lemma 3.3. *(i) Let (A.1),(A.2), and (A.4) hold. There exist positive constants* C_1, C_2, *and* $\tilde{\lambda}, \tilde{\mu} > 0$ *such that any positive solution of (1) satisfies*

$$C_1[H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}})]d(x, \partial\Omega) \leq u(x) \leq C_2[H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}})]d(x, \partial\Omega)$$

for all $x \in \Omega$ *and* $\lambda \geq \tilde{\lambda}$, $\mu \geq \tilde{\mu}$. *Here* $d(x, \partial\Omega)$ *denotes the distance from x to* $\partial\Omega$.

(ii) Let (A.1),(A.2),(A.5), and (A.6) hold. Then there exist positive constants C_3, C_4, *and* $\hat{\lambda}, \hat{\mu} > 0$ *such that any positive solution of (1) satisfies*

$$C_3[G^{-1}(\lambda^{\frac{1}{p-1}}) + G^{-1}(\mu^{\frac{1}{p-1}})]d(x, \partial\Omega) \leq u(x) \leq C_2[G^{-1}(\lambda^{\frac{1}{p-1}}) + G^{-1}(\mu^{\frac{1}{p-1}})]d(x, \partial\Omega)$$

for all $x \in \Omega$ *and* $\lambda > \tilde{\lambda}, \mu > \tilde{\mu}$, $G(u) = \frac{u}{g^{\frac{1}{p-1}}(u)}$.

Proof. Suppose that (A.1),(A.2) hold. Let u be a positive solution of (1) and Let $\lambda > \frac{\lambda_1}{k}, \mu > \frac{\mu_1}{k}$, where k is defined in Lemma 3.1 Let D be an open set such that $\bar{D} \subset \Omega$ and let ϕ be the solution of

$$-\Delta_p \phi = \begin{cases} 1 & in\ D \\ 0 & in\ \Omega \backslash D \end{cases}, \qquad \phi = 0 \ \ on\ \partial\Omega \tag{5}$$

By Lemma 3.1, there exists $c > 0$ such that $u \geq c$ in $\bar{D}$. Hence

$$-\Delta_p u = \lambda f(u) + \mu\gamma(u) \geq \begin{cases} \lambda f(c) + \mu\gamma(c) & in\ D \\ 0 & in\ \Omega \backslash D \end{cases} \tag{6}$$

which implies by the weak comparison principle that

$$u \geq (\lambda f(c) + \mu\gamma(c))^{\frac{1}{p-1}} \phi \qquad in\ \Omega.$$

Let $\bar{a}$ be the largest number such that $u \geq \lambda^{\frac{1}{p-1}} \bar{a}\phi, \mu^{\frac{1}{p-1}} \bar{a}\phi$ in Ω.
Suppose that $\phi(x) \geq c_0 > 0$ for $x \in \bar{D}$.

Then we have

$$-\Delta_p u = \lambda f(u) + \mu\gamma(u) \geq \begin{cases} \lambda f(\lambda^{\frac{1}{p-1}} \bar{a} c_0) + \mu\gamma(\mu^{\frac{1}{p-1}} \bar{a} c_0) & in\ D \\ 0 & in\ \Omega \backslash D \end{cases} \tag{7}$$

which implies

$$u \geq (\lambda f(\lambda^{\frac{1}{p-1}} \bar{a} c_0) + \mu\gamma(\mu^{\frac{1}{p-1}} \bar{a} c_0))^{\frac{1}{p-1}} \phi \quad \text{in}\Omega.$$

By the definition of $\bar{a}$,

$$\bar{a} \geq (f(\lambda^{\frac{1}{p-1}} \bar{a} c_0) + \gamma(\mu^{\frac{1}{p-1}} \bar{a} c_0))^{\frac{1}{p-1}},$$

or, equivalently,

$$H(\lambda^{\frac{1}{p-1}} \bar{a} c_0) = \frac{\lambda^{\frac{1}{p-1}} \bar{a} c_0}{f^{\frac{1}{p-1}}(\lambda^{\frac{1}{p-1}} \bar{a} c_0)} \geq \lambda^{\frac{1}{p-1}} c_0 \, , I(\mu^{\frac{1}{p-1}} \bar{a} c_0) = \frac{\mu^{\frac{1}{p-1}} \bar{a} c_0}{\gamma^{\frac{1}{p-1}}(\mu^{\frac{1}{p-1}} \bar{a} c_0)} \geq \mu^{\frac{1}{p-1}} c_0$$
$$H(\lambda^{\frac{1}{p-1}} \bar{a} c_0) + I(\mu^{\frac{1}{p-1}} \bar{a} c_0) \geq \lambda^{\frac{1}{p-1}} c_0 + \mu^{\frac{1}{p-1}} c_0$$

Suppose that (A.4) holds and $\lambda \geq \check{\lambda}$, $\mu \geq \check{\mu}$, where

$\check{\lambda}^{\frac{1}{p-1}} min(c_0, c_0(f(c))^{\frac{1}{p-1}}) + \check{\mu}^{\frac{1}{p-1}} min(c_0, c_0(\gamma(c))^{\frac{1}{p-1}}) = a$. Note that

$$\lambda^{\frac{1}{p-1}} \bar{a} c_0 + \mu^{\frac{1}{p-1}} \bar{a} c_0, \lambda^{\frac{1}{p-1}} c_0 + \mu^{\frac{1}{p-1}} c_0 \geq a$$

by the choice of $\check{\lambda}, \check{\mu}$. Using Lemma 3.2, we deduce the existence of $c_1 > 0$ and $\hat{\lambda} > max\{\frac{\lambda_1}{k}, \check{\lambda}\}$, $\hat{\mu} > max\{\frac{\mu_1}{k}, \check{\mu}\}$ such that

$$\lambda^{\frac{1}{p-1}} \bar{a} c_0 + \mu^{\frac{1}{p-1}} \bar{a} c_0 \geq H^{-1}(\lambda^{\frac{1}{p-1}} c_0) + I^{-1}(\mu^{\frac{1}{p-1}} c_0) \geq c_1(H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}}))$$

for $\lambda > \hat{\lambda}$,$\mu > \hat{\mu}$. Hence

$$u \geq (\lambda^{\frac{1}{p-1}} + \mu^{\frac{1}{p-1}}) \bar{a}\phi \geq \frac{c_1(H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}}))}{c_0} \phi \geq c_1(H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}}))d(x, \partial\Omega).$$

for $\lambda > \hat{\lambda}$,$\mu > \hat{\mu}$, where c_1 is a positive constant independent of u and λ, μ. Next, we have

$$-\Delta_p\left(\frac{u}{\lambda^{\frac{1}{p-1}} f^{\frac{1}{p-1}}(||u||_\infty)}\right) = \frac{f(u)}{f(||u||_\infty)} \equiv h \, , -\Delta_p\left(\frac{u}{\mu^{\frac{1}{p-1}} \gamma^{\frac{1}{p-1}}(||u||_\infty)}\right) = \frac{\gamma(u)}{\gamma(||u||_\infty)} \equiv h_1.$$

Since $||h||, ||h_1|| \leq 1$, it follows from Lieberman [11] that there exist $\alpha \in (0,1)$ and a positive number C depending solely on p, N, Ω such that

$$\frac{|u|_{1,\alpha}}{\lambda^{\frac{1}{p-1}} f^{\frac{1}{p-1}}(||u||_\infty)} \leq C\ ,\ \frac{|u|_{1,\alpha}}{\mu^{\frac{1}{p-1}} \gamma^{\frac{1}{p-1}}(||u||_\infty)} \leq C.$$

This implies

$$H(|u|_{1,\alpha}) + I(|u|_{1,\alpha}) = \frac{|u|_{1,\alpha}}{f^{\frac{1}{p-1}}(|u|_{1,\alpha})} + \frac{|u|_{1,\alpha}}{\gamma^{\frac{1}{p-1}}(|u|_{1,\alpha})} \leq C(\lambda^{\frac{1}{p-1}} + \mu^{\frac{1}{p-1}})$$

and since $|u|_{1,\alpha} \geq a$, we deduce from Lemma 3.2 that

$$|u|_{1,\alpha} \leq H^{-1}(\lambda^{\frac{1}{p-1}} C) + I^{-1}(\mu^{\frac{1}{p-1}} C) \leq C_2(H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}})) \tag{8}$$

for λ, μ large.The right-hand inequality then follows on applying the mean value theorem, which completes the proof of (i). Next, suppose that (A.5) and (A.6) hold. Then, since

$$\lim_{u\to\infty} \frac{G(u)}{H(u)} = 1\ ,\ \lim_{u\to\infty} \frac{G(u)}{I(u)} = 1$$

it follows from (3.1) that there exist $\tilde{c_0}$ such that

$$G(\lambda^{\frac{1}{p-1}} \bar{a} c_0) + G(\mu^{\frac{1}{p-1}} \bar{a} c_0) \geq \lambda^{\frac{1}{p-1}} \tilde{c_0} + \mu^{\frac{1}{p-1}} \tilde{c_0}$$

for λ, μ large.Proceeding as in part (i) with H, I replaced by G, we obtain the left-hand inequality in (ii). $||u||_\infty \to \infty$ as $\lambda, \mu \to \infty$, and

$$-\Delta_p\left(\frac{u}{\lambda^{\frac{1}{p-1}} g^{\frac{1}{p-1}}(||u||_\infty)}\right) = \frac{f(u)}{g(||u||_\infty)} \equiv \tilde{h}\ ,-\Delta_p\left(\frac{u}{\mu^{\frac{1}{p-1}} g^{\frac{1}{p-1}}(||u||_\infty)}\right) = \frac{\gamma(u)}{g(||u||_\infty)} \equiv \tilde{h_1}.$$

it follows that $||\tilde{h}||_\infty, ||\tilde{h_1}||_\infty$ are uniformly bounded for λ, μ large. Hence, proceeding as in part (i), we obtain the right-hand inequality in (ii).

This completes the proof of Lemma 3.3. $\square$

For each $\epsilon > 0$, define $\Omega_\epsilon = \{x \in \Omega \ :\ d(x, \partial\Omega) < \epsilon\}$.

Lemma 3.4. *Let (A.1),(A.2), and (A.4) hold. Let $\beta_0 \leq \beta < 1$, where $\beta_0 = \frac{C_1}{C_2}$ and C_1, C_2 are given by Lemma 3.3. Then there exists $\delta > 0$ such that if u and u_1 are positive solutions of (1) then*

$$\frac{C_1\beta_0}{2}[H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}})] \leq |t\nabla u(x) + (1-t)\beta\nabla u_1(x)| \leq C_2[H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}})] \tag{9}$$

for all $t \in [0,1]$ and $x \in \Omega_\delta$, provided that $\lambda > \bar{\lambda}$, $\mu > \bar{\mu}$

Proof. Let $t \in [0,1]$. Using (8), we get

$$|tu + (1-t)\beta u_1|_{1,\alpha} \leq t|u|_{1,\alpha} + (1-t)|u_1|_{1,\alpha} \leq C_2[H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}})]$$

for $\lambda > \bar{\lambda}$, $\mu > \bar{\mu}$, and right-hand side of (9) follows. Next, by Lemma 3.3 (i),

$$\frac{\partial u}{\partial n}, \frac{\partial u_1}{\partial n} \leq -C_1[H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}})] \qquad \text{on } \partial\Omega$$

$\lambda > \bar{\lambda}$, $\mu > \bar{\mu}$, where n denotes the outward unit normal vector. This implies

$$t\frac{\partial u}{\partial n} + (1-t)\beta\frac{\partial u_1}{\partial n} \leq -C_1\beta_0[H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}})] \qquad \text{on } \partial\Omega$$

Hence

$$|t\nabla u(x)+(1-t)\beta\nabla u_1(x)|\geq C_1\beta_0[H^{-1}(\lambda^{\frac{1}{p-1}})+I^{-1}(\mu^{\frac{1}{p-1}})] \qquad on\ \partial\Omega \tag{10}$$

Let $w_t = t\nabla u(x)+(1-t)\beta\nabla u_1(x)$. Then we have

$$\frac{|w_t(x)-w_t(x_0)|}{|x-x_0|^{\alpha}}\leq C_2[H^{-1}(\lambda^{\frac{1}{p-1}})+I^{-1}(\mu^{\frac{1}{p-1}})] \qquad on\ \partial\Omega \tag{11}$$

for x,$x_0\in\Omega$, $x\neq x_0$. Let $\delta>0$ satisfy $C_2\delta^{\alpha}<C_1\frac{\beta_0}{2}$. Let $x\in\Omega_\delta$ and $x_0\in\partial\Omega$ be such that $d(x,\partial\Omega)=|x-x_0|$.

Then it follows from (10) and (11) that

$$|w_t(x)|\geq|w_t(x_0)|-C_2\delta^{\alpha}[H^{-1}(\lambda^{\frac{1}{p-1}})+I^{-1}(\mu^{\frac{1}{p-1}})]\geq\frac{C_1\beta_0[H^{-1}(\lambda^{\frac{1}{p-1}})+I^{-1}(\mu^{\frac{1}{p-1}})]}{2},$$

which completes the proof of Lemma 3.4. □

4 Proof of main result

In what follows, we denote by m_i,$i=1,2,......$, constants depending only on Ω, p, N, f, γ.

Proof of theorem. Let λ be large enough so that Lemma 3.1, 3.2, 3.3(i), and 3.4 apply. Let u and u_1 be positive solution of (1). By Lemma 3.3, $u\geq\beta_0u_1$ in Ω, where $\beta_0=\frac{C_1}{C2}$. Let β be the largest number such that $u\geq\beta u_1$ in Ω and suppose by contradiction that $\beta<1$. Let

$$\tilde{u}=\frac{u}{H^{-1}(\lambda^{\frac{1}{p-1}})+I^{-1}(\mu^{\frac{1}{p-1}})} \quad , \quad \tilde{u_1}=\frac{u_1}{H^{-1}(\lambda^{\frac{1}{p-1}})+I^{-1}(\mu^{\frac{1}{p-1}})}$$

Since $H^{-1}(\lambda^{\frac{1}{p-1}})+I^{-1}(\mu^{\frac{1}{p-1}})=\lambda^{\frac{1}{p-1}}f^{\frac{1}{p-1}}(H^{-1}(\lambda^{\frac{1}{p-1}}))$,$\mu^{\frac{1}{p-1}}\gamma^{\frac{1}{p-1}}(I^{-1}(\mu^{\frac{1}{p-1}}))$, we have

$$\Delta_p\tilde{u}=\lambda f(\tilde{u})+\mu\gamma(\tilde{u})=\frac{\lambda f(u)}{f(H^{-1}(\lambda^{\frac{1}{p-1}})+I^{-1}(\mu^{\frac{1}{p-1}}))}+\frac{\mu\gamma(u)}{\gamma(H^{-1}(\lambda^{\frac{1}{p-1}})+I^{-1}(\mu^{\frac{1}{p-1}}))},$$

$$\Delta_p\tilde{u_1}=\lambda f(\tilde{u_1})+\mu\gamma(\tilde{u_1})=\frac{\lambda f(u_1)}{\mu\gamma(H^{-1}(\lambda^{\frac{1}{p-1}})+I^{-1}(\mu^{\frac{1}{p-1}}))}+\frac{\mu\gamma(u_1)}{\gamma(H^{-1}(\lambda^{\frac{1}{p-1}})+I^{-1}(\mu^{\frac{1}{p-1}}))}.$$

Let δ given by Lemma 3.4. Using the mean value theorem, we obtain

$$L(\tilde{u}-\beta\tilde{u_1})=\Delta_p\tilde{u}-(-\Delta_p\beta\tilde{u_1})\geq\frac{\lambda[f(\beta u_1)-\beta^{p-1}f(u_1)]}{f(H^{-1}(\lambda^{\frac{1}{p-1}})+I^{-1}(\mu^{\frac{1}{p-1}}))}+\frac{\mu[\gamma(\beta u_1)-\beta^{p-1}\gamma(u_1)]}{\gamma(H^{-1}(\lambda^{\frac{1}{p-1}})+I^{-1}(\mu^{\frac{1}{p-1}}))} \quad in\ \Omega_\delta$$

where

$$Lw=-\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}(a_{i,j}(x)\frac{\partial w}{\partial x_j}),$$

where $a_{i,j}=\int_0^1\frac{\partial a^i}{\partial z_j}(t\nabla\tilde{u_1}+(1-t)\beta\nabla\tilde{u_1})dt$, and $a^j(z)=|z|^{p-1}z_i$, $i=1,2,....,N$, $z=(z_1,z_2,.....z_N)$.

Because of (9), we see that the operator L is uniformly elliptic in Ω_δ. In fact,

$$\sum_{i,j=1}^{N}a_{ij}(x)\xi_i\xi_j\geq m_0|\xi|^2, \quad \forall x\in\Omega_\delta, \quad \xi=(\xi_1,.....,\xi_N)\in R^N, \tag{12}$$

where $m_0=(\frac{C_1\beta_0}{2})^{p-2}$ if $p\geq2$, $(p-1)C_2^{p-2}$ if $1<p<2$, and

$$|a_{ij}|_{0,\alpha;\Omega_\delta}\leq m_1 \qquad \forall i,j=1,...,N. \tag{13}$$

Let $D = \{x \in \Omega_\delta \;\; : u_1(x) > \frac{a}{\beta_0}\}$, where a is given by (A.4). Then it follows from (A.4) and Lemma 3.3 (i) that

$$\begin{aligned}[][f(\beta u_1) - \beta^{p-1}f(u_1)] + [\gamma(\beta u_1) - \beta^{p-1}\gamma(u_1)] &\geq (\beta^q - \beta^{p-1})[f(u_1) + \gamma(u_1)] \\ &\geq m_2(1-\beta)[f(u_1) + \gamma(u_1)]\end{aligned}$$

$$m_2(1-\beta)[f(u_1) + \gamma(u_1)] \geq m_2(1-\beta)[f(C_1 d(x, \partial\Omega)) + \gamma(C_1 d(x, \partial\Omega))] \qquad in \Omega \tag{14}$$

for $H^{-1}(\lambda^{\frac{1}{p-1}}), I^{-1}(\mu^{\frac{1}{p-1}}) \geq 1$, where $m_2 = \beta_0^q min\{1, p-1-q\}$.

Since $u_1(x) \leq \frac{a}{\beta_0}$ in $\Omega \backslash D$, the mean value theorem gives

$$|[f(\beta u_1) - \beta^{p-1}f(u_1)] + [\gamma(\beta u_1) - \beta^{p-1}\gamma(u_1)]| =$$

$$(1-\beta)|\tfrac{cu_1(x)[f'(cu_1(x)) + \gamma'(cu_1(x))]}{c} - (p-1)c^{p-2}[f(u_1(x)) + \gamma(u_1(x))]|,$$

where $c \in [\beta_0, 1]$. Since $\limsup_{z\to 0^+} zf'(z), \limsup_{z\to 0^+} z\gamma'(z) < \infty$, this implies

$$|[f(\beta u_1) - \beta^{p-1}f(u_1)] + [\gamma(\beta u_1) - \beta^{p-1}\gamma(u_1)]| \leq m_3(1-\beta) \tag{15}$$

where $m_3 = \frac{1}{\beta_0} sup_{0<z\leq\frac{a}{\beta_0}} |[zf'(z) + z\gamma'(z)]| + (p-1)max\{1, \beta_0^{p-2}\}f(\frac{a}{\beta_0})$.

Combining (11), (14) and (15), we obtain

$$L(\tfrac{[f(H^{-1}(\lambda^{\frac{1}{p-1}}) + (I^{-1}(\mu^{\frac{1}{p-1}}))\gamma(H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}}))](\tilde{u} + \beta\tilde{u_1})}{1-\beta}) \geq$$
$$\begin{cases} m_2[f(C_1 d(x, \partial\Omega)) + \gamma(C_1 d(x, \partial\Omega))] & in\ D \\ -m_3 \quad in\ \Omega_\delta \backslash D. \end{cases}$$

Let z be solution of

$$Lz = \begin{cases} m_2[f(C_1 d(x, \partial\Omega)) + \gamma(C_1 d(x, \partial\Omega))] & in\ D \\ -m_3 \quad in\ \Omega_\delta \backslash D. \end{cases} , z = 0 \quad on\ \partial\Omega_\delta \tag{16}$$

Since $\tilde{u} - \beta\tilde{u} \geq 0$ on $\partial\Omega_\delta$, it follows from the weak maximum principle that

$$\frac{[f(H^{-1}(\lambda^{\frac{1}{p-1}}) + (I^{-1}(\mu^{\frac{1}{p-1}}))\gamma(H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}}))](\tilde{u} + \beta\tilde{u_1})}{1-\beta} \geq z \quad in\ \Omega_\delta \tag{17}$$

Let $\bar{z}$ satisfy

$$L\bar{z} = m_2[f(C_1 d(x, \partial\Omega)) + \gamma(C_1 d(x, \partial\Omega))] \quad in\ \Omega_\delta\ , \quad \bar{z} = 0\ \ on\ \partial\Omega_\delta.$$

Because of (13) and (14), and the maximum principle, there exists a positive number ν depending on $m_0, m_1, \Omega_\delta, f, \gamma$, such that

$$\bar{z} \geq \theta d(x, \partial\Omega_\delta) \tag{18}$$

(see Proposition in the Appendix).

Next, we have

$$L(\bar{z} - z) = \bar{g} \equiv \begin{cases} 0 \quad in\ D \\ m_2[f(C_1 d(x, \partial\Omega)) + \gamma(C_1 d(x, \partial\Omega))] & in\ \Omega_\delta \backslash D. \end{cases} , z = 0 \quad on\ \partial\Omega_\delta \tag{19}$$

and note that

$$m_2[f(C_1 d(x, \partial\Omega)) + \gamma(C_1 d(x, \partial\Omega))] + m_3 \leq m_2[f(C_1\delta) + \gamma(C_1\delta)] + m_3 \equiv m_4 \quad in\ \Omega_\delta \backslash D.$$

It then follows from Theorems 8.16 and 8.33 0f [6] (see also the remark on page 212 of [6]) that

$$|\bar{z} - z|_{1,\alpha;\Omega_\delta} \leq C||\bar{g}||_{r;\Omega_\delta} \leq Cm_4|\Omega_\delta \backslash D|^{\frac{1}{r}} \tag{20}$$

where $r = \frac{N}{1-\alpha}$, C is a constant depending only on $N, p, m_0, m_1, \Omega_\delta$, and $\Omega_\delta\backslash D$ denotes the Lebesgue measure of $\Omega_\delta\backslash D$.

From (18) and the mean value theorem,

$$|\bar{z}(x) - z(x)| \leq Cm_4|\Omega_\delta\backslash D|^{\frac{1}{r}} d(x, \partial\Omega_\delta) \qquad for \;\; x \in \Omega_\delta$$

which, together with (17), implies

$$z(x) \leq [\theta - Cm_4|\Omega_\delta\backslash D|^{\frac{1}{r}}] d(x, \partial\Omega_\delta) \qquad for \;\; x \in \Omega_\delta$$

By Lemma 3.3 (i),

$$\Omega_\delta\backslash D \subseteq \{x \in \omega \quad : d(x, \partial\Omega) \leq \frac{a}{\beta_0 C_1[H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}})]}\}$$

and hence $|\Omega_\delta\backslash D| \to 0$ as $\lambda, \mu \to \infty$. Thus, for λ, μ large enough

$$z \geq \frac{\theta}{2} d(x, \partial\Omega_\delta)$$

For $x \in \Omega_{\frac{\delta}{2}}$, we have $d(x, \partial\Omega_\delta) = d(x, \partial\Omega)$ and therefore using (17), we obtain

$$\begin{aligned}\tilde{u}(x) - \beta\tilde{u_1}(x) &\geq [\frac{1}{f(H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}}))} + \frac{1}{\gamma(H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}}))}](1-\beta)z(x)\\ &\geq [\frac{1}{f(H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}}))} + \frac{1}{\gamma(H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}}))}](1-\beta)\theta d(x, \partial\Omega)\\ &\geq \frac{1}{2C_2}[\frac{1}{f(H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}}))} + \frac{1}{\gamma(H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}}))}](1-\beta)\theta\tilde{u_1}\end{aligned}$$

for $x \in \Omega_{\frac{\delta}{2}}$ and λ, μ large. In order words, there exists $\bar{\epsilon} > 0$ such that

$$u > (\beta + \bar{\epsilon})u_1 \qquad in \; \Omega_{\frac{\delta}{2}}$$

for λ, μ large. In particular,

$$u > \beta u_1(x) + \bar{c} \qquad when \;\; d(x, \partial\Omega) = \frac{\delta}{2},$$

where $\bar{c} = \bar{\epsilon} C_1 \frac{\delta}{2}[H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}})]$

Let $\Omega_1 = \Omega\backslash\bar{\Omega}_{\frac{\delta}{2}}$. Then, for λ, μ large enough,

$u_1(x) \geq C_1\frac{\delta}{2}[H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}})] \geq \frac{a}{\beta_0}$ which implies

$$-\Delta_p u = \lambda f(u) + \mu\gamma(u) \geq \lambda f(\beta u_1) + \mu\gamma(\beta u_1) > \beta^{p-1}(\lambda f(u_1) + \mu\gamma(u_1)) \quad \text{in } \Omega_1.$$

since

$$-\Delta_p(\beta u_1 + \bar{c}) = \beta^{p-1}(\lambda f(u_1) + \mu\gamma(u_1)) \quad \text{in } \Omega_1,$$

and $u \geq \beta u_1 + \bar{c}$ on $\partial\Omega_1$, it follows that

$$u \geq \beta u_1 + \bar{c} \qquad in \; \Omega_1 \tag{21}$$

Combining (20) and (21), we deduce the existence of $\beta_1 > \beta$ such that $u \geq \beta_1 u_1$ in ω. This contradicts the maximality of β. Hence $\beta \geq 1$ and Theorem is proved. ☐

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