Uniqueness of Positive Solutions for a Class of Quasilinear Problems with Multiple Parameters

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Abstract: we prove uniqueness of positive solution for the quasilinear problems
\[-\Delta_p u = \lambda f(u) + \mu \gamma(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega\]
where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\), with smooth boundary \(\partial \Omega\), \(f, g\) are \(p\)-sublinear at \(\infty\) for positive number \(p\) with \(p > 1\), \(\frac{f(u)}{u^{p-1}}, \frac{\gamma(u)}{u^{p-1}}\), are decreasing for large \(u\), and \(\lambda, \mu\) are large positive parameters. We also obtain the asymptotic behavior of the solution obtain as \(\lambda, \mu \to \infty\).

Keywords: uniqueness; positive solutions; quasilinear problems; multiple parameters.
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1 Introduction
Consider the quasilinear boundary value problems
\[
\begin{cases}
-\Delta_p u &= \lambda f(u) + \mu \gamma(u) \quad \text{in } \Omega \\
u(x) &= 0 \quad \text{on } \partial \Omega
\end{cases}
\tag{1}
\]
where the \(p\)-Laplacian operator \(\Delta_p z = div(|\nabla z|^{p-2}\nabla z), p > 1, f(u), \gamma(u) > 0\), for \(u > 0 \lambda, \mu\) are positive parameters and \(\Omega\) is bounded domain in \(\mathbb{R}^N\) with smooth boundary \(\partial \Omega\).

\[
\begin{cases}
-\Delta_p u &= \lambda f(u) \quad \text{in } \Omega \\
u(x) &= 0 \quad \text{on } \partial \Omega
\end{cases}
\tag{2}
\]
Problem (2) has been investigated by many authors in recent years (see e.g., [4, 5, 7-9, 11, 14, 16, 17]). When \(p = 2\), uniqueness of positive solutions to (2) for \(\lambda\) large and \(\frac{f(u)}{u}\) decreasing for large \(u\) was established in Angenent [2], Dancer [3], Hai and Smith [10], Lin [12], Schuchman [15] and Wieger [18] and uniqueness of positive solutions for a class of quasilinear problems (2) when \(p > 1\) and \(\frac{f(u)}{u^{p-1}}\) is decreasing for large \(u\) and \(\lambda\) is a large positive parameter was obtain in Hai [11], uniqueness of positive solutions to (2) when \(p > 1\) and \(\frac{f(u)}{u^{p-1}}\) is decreasing on \((0, \infty)\) was obtain in Geo and Webb [8], Diaz and Saa [4], and Drabek and Hernandez [5]. In this paper, we give a positive answer to the above question. We also obtained asymptotic behavior of solutions as \(\lambda, \mu \to \infty\). Our approach depends on sharp upper and lower estimates of solutions together with the maximum and comparison principles.

2 Main result
We make the following assumptions:

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(A.1) \( f, \gamma : (0, \infty) \rightarrow \mathbb{R} \) are nondecreasing, continuous and of class \( C^1 \) on \( (0, \infty) \).

(A.2) \( \lim \inf_{u \to 0^+} f(u) > 0, \lim \inf_{u \to 0^+} \frac{\gamma(u)}{u^p} > 0 \).

(A.3) \( \lim \sup_{u \to 0^+} uf'(u) < \infty, \lim \sup_{u \to 0^+} u\gamma'(u) < \infty. \)

(A.4) There exist \( q \in (0, p - 1) \) and a positive number \( a \) such that \( f(u) \) and \( \frac{\gamma(u)}{u^p} \) are decreasing on \( [a, \infty) \).

(A.5) \( g : [0, \infty) \rightarrow [0, \infty) \) is continuous, nondecreasing, and there exists \( q_1 \in (0, p - 1) \) such that \( \frac{g(u)}{u^p} \) is decreasing on \( (0, \infty) \).

(A.6) \( \lim_{u \to \infty} \frac{g(u)}{u^p} \) exists and is finite for each \( c \in (0, 1) \).

Our main result are

**Theorem.** Let (A.1)-(A.4) hold. Then there exist \( \lambda_0 > 0, \mu_0 > 0 \) such that problem (1) has a unique positive solution for \( \lambda > \lambda_0, \mu > \mu_0 \).

Let \( g \) satisfy (A.5). Then, for each \( \lambda > 0 \), there exists a unique positive solution \( v_\lambda \) to the problem

\[
\begin{array}{ll}
-\Delta_p v_\lambda = \lambda g(v_\lambda) & \text{in } \Omega \\
\phi_\lambda = 0 & \text{on } \partial \Omega
\end{array}
\]

(3)

(see e.g., [4,5,8] or Proposition A in the Appendix)

### 3 Preliminary lemmas

As usual, we shall denote by \( ||.||_{k,\alpha} \) and \( ||.||_k \) the norms in \( C^{k,\alpha}(\overline{\Omega}) \) and \( L^k(\Omega) \) respectively. Let \( \lambda_1 \) be the first eigenvalue of \(-\Delta_p\) with zero boundary conditions, and \( \phi_1 \) a corresponding normalized eigenfunction, i.e., \( ||\phi_1||_\infty = 1 \), and

\[
\begin{align*}
-\Delta_p \phi_1 = \lambda_1 \phi_1 & \quad \text{in } \Omega \\
\phi_1 = 0 & \quad \text{on } \partial \Omega
\end{align*}
\]

(4)

Then \( \lambda_1 > 0 \) and we can assume that \( \phi_1 > 0 \) in \( \Omega \) (see [14]).

**Lemma 3.1.** Let (A.1)-(A.2) hold. Then there exist positive number \( k, \eta \) such that any positive solution of (1) satisfies

\[
u \geq \eta \phi_1 \quad \text{in } \Omega \quad \text{for } \lambda > \frac{\lambda_1}{k}, \mu > \frac{\mu_1}{k}
\]

**Proof.** By (A.2), there exist \( k, \eta > 0 \) such that

\[f(u) > ku^{p-1}, \gamma(u) > ku^{p-1} \quad \text{for } u \in (0, \eta].\]

Suppose that \( \lambda > \frac{\lambda_1}{k}, \mu > \frac{\mu_1}{k} \) and let \( u \) be positive solution of (1). By the strong maximum principle [17], there exists \( \epsilon > 0 \) such that \( u \geq \epsilon \phi_1 \) in \( \Omega \). Let \( \eta_0 \) be the largest number such that \( u \geq \eta_0 \phi_1 \) in \( \Omega \) and suppose by contradiction that \( \eta_0 < \eta \). Let

\[
\Omega_0 = \{ x \in \Omega : u(x) < \eta \phi_1(x) \}
\]

and \( m = \min \{ (\frac{\lambda_1}{k})^{p-1}, (\frac{\mu_1}{k})^{p-1}, \frac{\eta}{\eta_0} \} \). Then \( \Omega_0 \neq \emptyset \) and

\[
-\Delta_p u = \lambda f(u) + \mu \gamma(u) \geq (\lambda + \mu) k(\eta_0 \phi_1)^{p-1} \geq (\lambda_1 + \mu_1)(m \eta_0 \phi_1)^{p-1} \quad \text{in } \Omega_0
\]

\[u = \eta \phi_1 \quad \text{on } \partial \Omega_0
\]

Since

\[-\Delta_p (m \eta_0 \phi_1) = (\lambda_1 + \mu_1)(m \eta_0 \phi_1)^{p-1} \quad \text{in } \Omega_0\]
Let (A.1) and (A.4) hold. Then for each $\eta_0 \phi_1$ in $\Omega_0$

Clearly,

$u \geq \eta_0 \phi_1 \geq m \eta_0 \phi_1$ in $\Omega \setminus \Omega_0$

Hence $u \geq m \eta_0 \phi_1$ in $\Omega$, and since $m > 1$, this contradicts the maximality of $\eta_0$. This complete the proof of lemma 3.1. $\square$

Next, we define $H(u) = \frac{u}{f^{p-1}}(u)$, $I(u) = \frac{u}{\gamma^{p-1}}(u)$. Then $H$, $I$ are increasing in $(0, \infty)$ and $\lim_{u \to \infty} H(u) = \infty$, $\lim_{u \to \infty} I(u) = \infty$ if (A.4) holds.

**Lemma 3.3** Let (A.1) and (A.4) hold. Then for each $C > 0$, there exist $M_1$, $M_2$, and $\tilde{\lambda}$, $\tilde{\mu} > 0$ such that

$M_1(H^{-1}(\lambda^{\frac{1}{q-1}}) + I^{-1}(\mu^{\frac{1}{q-1}})) \leq H^{-1}(\lambda^{\frac{1}{q-1}} C) + I^{-1}(\mu^{\frac{1}{q-1}} C) \leq M_2(H^{-1}(\lambda^{\frac{1}{q-1}}) + I^{-1}(\mu^{\frac{1}{q-1}}))$

for $\lambda > \tilde{\lambda}$, $\mu > \tilde{\mu}$.

**Proof.** By writing $H^{-1}(\lambda^{\frac{1}{q-1}})$ as $H^{-1}(\lambda^{\frac{1}{p-1}})$, $I^{-1}(\mu^{\frac{1}{q-1}})$ as $I^{-1}(\mu^{\frac{1}{p-1}})$, where $\eta \frac{1}{q-1} = \lambda^{\frac{1}{p-1}} C$, $\eta \frac{1}{p-1} = \mu^{\frac{1}{p-1}} C$, we see that the left-hand inequality follows from the right hand one.

Let $C > 0$, $r = \frac{q}{p-1}$. Then $\frac{f}{x^{\frac{1}{q-1}}} = \frac{\gamma}{x^{\frac{1}{p-1}}}$ are decreasing on $[a, \infty)$. Let $\tilde{\lambda} = \left(\frac{H(a)}{\min(1, C)}\right)^{p-1}$, $\tilde{\mu} = \left(\frac{I(a)}{\min(1, C)}\right)^{\frac{1}{p-1}}$ and $\theta = \max(C^{\frac{1}{q-1}}, 1)$. Let $\lambda > \tilde{\lambda}$, $\mu > \tilde{\mu}$, and $x = H^{-1}(\lambda^{\frac{1}{q-1}}) + I^{-1}(\mu^{\frac{1}{q-1}})$. Then $x \geq a$ and $g(\theta x) \leq \theta^r g(x)$,

$$\frac{\theta x}{f^{\frac{1}{p-1}}(\theta x)} \geq \frac{\theta x}{\theta r f^{\frac{1}{p-1}}(x)} = \lambda^{\frac{1}{q-1}} \theta^{1-r} \geq \lambda^{\frac{1}{q-1}} C,$$

$$\frac{\theta x}{\gamma^{\frac{1}{p-1}}(\theta x)} \geq \frac{\theta x}{\theta r \gamma^{\frac{1}{p-1}}(x)} = \mu^{\frac{1}{q-1}} \theta^{1-r} \geq \mu^{\frac{1}{q-1}} C,$$

which implies $\theta x \geq H^{-1}(\lambda^{\frac{1}{q-1}} C)$, $\theta x \geq I^{-1}(\mu^{\frac{1}{q-1}} C)$, so $2 \theta x \geq H^{-1}(\lambda^{\frac{1}{q-1}} C) + I^{-1}(\mu^{\frac{1}{q-1}} C)$.

This completes the proof of lemma 3.2. $\square$

**Lemma 3.3.** (i) Let (A.1),(A.2), and (A.4) hold. There exist positive constants $C_1$, $C_2$, and $\tilde{\lambda}$, $\tilde{\mu} > 0$ such that any positive solution of (1) satisfies

$C_1[H^{-1}(\lambda^{\frac{1}{q-1}}) + I^{-1}(\mu^{\frac{1}{q-1}})]d(x, \partial \Omega) \leq u(x) \leq C_2[H^{-1}(\lambda^{\frac{1}{q-1}}) + I^{-1}(\mu^{\frac{1}{q-1}})]d(x, \partial \Omega)$

for all $x \in \Omega$ and $\lambda \geq \tilde{\lambda}$, $\mu \geq \tilde{\mu}$. Here $d(x, \partial \Omega)$ denotes the distance from $x$ to $\partial \Omega$.

(ii) Let (A.1),(A.2),(A.5), and (A.6) hold. Then there exist positive constants $C_3$, $C_4$, and $\tilde{\lambda}$, $\tilde{\mu} > 0$ such that any positive solution of (1) satisfies

$C_3[G^{-1}(\lambda^{\frac{1}{q-1}}) + G^{-1}(\mu^{\frac{1}{q-1}})]d(x, \partial \Omega) \leq u(x) \leq C_2[G^{-1}(\lambda^{\frac{1}{q-1}}) + G^{-1}(\mu^{\frac{1}{q-1}})]d(x, \partial \Omega)$

for all $x \in \Omega$ and $\lambda \geq \tilde{\lambda}$, $\mu \geq \tilde{\mu}$, $G(u) = \frac{u}{g^{\frac{1}{p-1}}(u)}$. 

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**Proof.** Suppose that (A.1),(A.2) hold. Let u be a positive solution of (1) and let λ > \( \frac{\lambda_1}{k} \), \( \mu > \frac{\mu_1}{k} \), where k is defined in Lemma 3.1. Let D be an open set such that \( \bar{D} \subset \Omega \) and let \( \phi \) be the solution of

\[
-\Delta_p \phi = \begin{cases} 
1 & \text{in } D \\
0 & \text{in } \Omega \setminus D 
\end{cases}, \quad \phi = 0 \text{ on } \partial \Omega
\]

(5)

By Lemma 3.1, there exists \( c > 0 \) such that \( u \geq c \) in \( \bar{D} \). Hence

\[
-\Delta_p u = \lambda f(u) + \mu \gamma(u) \geq \begin{cases} 
\lambda f(c) + \mu \gamma(c) & \text{in } D \\
0 & \text{in } \Omega \setminus D 
\end{cases}
\]

(6)

which implies by the weak comparison principle that

\[
u \geq (\lambda f(c) + \mu \gamma(c)) \frac{1}{p} \phi \quad \text{in } \Omega.
\]

Let \( \bar{a} \) be the largest number such that \( u \geq \lambda \frac{1}{p-1} \bar{a} \phi, \mu \frac{1}{p-1} \bar{a} \phi \) in \( \Omega \).

Suppose that \( \phi(x) \geq c_0 > 0 \) for \( x \in D \).

Then we have

\[
-\Delta_p u = \lambda f(u) + \mu \gamma(u) \geq \begin{cases} 
\lambda f(\lambda \frac{1}{p-1} \bar{a} c_0) + \mu \gamma(\mu \frac{1}{p-1} \bar{a} c_0) & \text{in } D \\
0 & \text{in } \Omega \setminus D
\end{cases}
\]

(7)

which implies

\[
\lambda \frac{1}{p-1} \min(c_0, c_0(f(c)) \frac{1}{p-1}) + \mu \frac{1}{p-1} \min(c_0, c_0(\gamma(c)) \frac{1}{p-1}) = a.
\]

By the definition of \( \bar{a} \),

\[
\bar{a} \geq f(\lambda \frac{1}{p-1} \bar{a} c_0) + \gamma(\mu \frac{1}{p-1} \bar{a} c_0),
\]

or, equivalently,

\[
H(\lambda \frac{1}{p-1} \bar{a} c_0) = \lambda \frac{1}{p-1} \bar{a} c_0 \geq \lambda \frac{1}{p-1} c_0, \quad I(\mu \frac{1}{p-1} \bar{a} c_0) = \mu \frac{1}{p-1} \bar{a} c_0 \geq \mu \frac{1}{p-1} c_0
\]

\[
H(\lambda \frac{1}{p-1} \bar{a} c_0) + I(\mu \frac{1}{p-1} \bar{a} c_0) \geq \lambda \frac{1}{p-1} c_0 + \mu \frac{1}{p-1} c_0
\]

Suppose that (A.4) holds and \( \lambda \geq \lambda, \mu \geq \mu \), where

\[
\lambda \frac{1}{p-1} \min(c_0, c_0(f(c)) \frac{1}{p-1}) + \mu \frac{1}{p-1} \min(c_0, c_0(\gamma(c)) \frac{1}{p-1}) = a.
\]

Note that

\[
\lambda \frac{1}{p-1} \bar{a} c_0 + \mu \frac{1}{p-1} \bar{a} c_0, \lambda \frac{1}{p-1} c_0 + \mu \frac{1}{p-1} c_0 \geq a
\]

by the choice of \( \lambda, \mu \). Using Lemma 3.2, we deduce the existence of \( c_1 > 0 \) and \( \hat{\lambda} > \max\{ \frac{\lambda_1}{k}, \bar{\lambda} \}, \hat{\mu} > \max\{ \frac{\mu_1}{k}, \bar{\mu} \} \) such that

\[
\lambda \frac{1}{p-1} \bar{a} c_0 + \mu \frac{1}{p-1} \bar{a} c_0 \geq H^{-1}(\lambda \frac{1}{p-1} c_0) + I^{-1}(\mu \frac{1}{p-1} c_0) \geq c_1(\lambda \frac{1}{p-1} c_0) + I^{-1}(\mu \frac{1}{p-1} c_0)
\]

for \( \lambda > \hat{\lambda}, \mu > \hat{\mu} \). Hence

\[
u \geq (\lambda \frac{1}{p-1} c_0 + \mu \frac{1}{p-1} c_0) \phi \geq c_1(\lambda \frac{1}{p-1} c_0 + I^{-1}(\mu \frac{1}{p-1} c_0)) \phi \geq c_1(\lambda \frac{1}{p-1} c_0 + I^{-1}(\mu \frac{1}{p-1} c_0)) d(x, \partial \Omega).
\]

for \( \lambda > \hat{\lambda}, \mu > \hat{\mu} \), where \( c_1 \) is a positive constant independent of \( u \) and \( \lambda, \mu \). Next, we have

\[
-\Delta_p \left( \frac{u}{\lambda \frac{1}{p-1} f \frac{1}{p-1}(||u||_\infty)} \right) = \frac{f(u)}{I(||u||_\infty)} \equiv h, \quad -\Delta_p \left( \frac{u}{\mu \frac{1}{p-1} \gamma \frac{1}{p-1}(||u||_\infty)} \right) = \frac{\gamma(u)}{\gamma(||u||_\infty)} \equiv h_1.
\]
Since \( ||h||, ||h_1|| \leq 1 \), it follows from Lieberman [11] that there exist \( \alpha \in (0, 1) \) and a positive number \( C \) depending solely on \( p, N, \Omega \) such that

\[
\frac{|u|_{1,\alpha}}{\lambda^{\frac{1}{p-1}} \int_{\Omega} (|u||_{\infty})} \leq C, \quad \frac{|u|_{1,\alpha}}{\mu^{\frac{1}{p-1}} \gamma^{\frac{1}{p-1}} (|u||_{\infty})} \leq C.
\]

This implies

\[
H(|u|_{1,\alpha}) + I(|u|_{1,\alpha}) = \frac{|u|_{1,\alpha}}{f^{\sqrt{\alpha}} (|u||_{1,\alpha})} + \frac{|u|_{1,\alpha}}{\gamma^{\frac{1}{p-1}} (|u||_{1,\alpha})} \leq C(\lambda^{\frac{1}{p-1}} + \mu^{\frac{1}{p-1}})
\]

and since \( |u|_{1,\alpha} \geq a \), we deduce from Lemma 3.2 that

\[
|u|_{1,\alpha} \leq H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}}) \leq C_2(H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}})) \quad (8)
\]

for \( \lambda, \mu \) large. The right-hand inequality then follows on applying the mean value theorem, which completes the proof of (i). Next, suppose that (A.5) and (A.6) hold. Then, since \( t \to \infty \) for \( \lambda > \bar{\lambda} \), \( \mu > \bar{\mu} \) large. Hence, proceeding as in part (i), we obtain the right-hand inequality in (ii).

This completes the proof of Lemma 3.3. \( \Box \)

For each \( \epsilon > 0 \), define \( \Omega_\epsilon = \{ x \in \Omega : d(x, \partial \Omega) < \epsilon \} \).

**Lemma 3.4.** Let (A.1), (A.2), and (A.4) hold. Let \( \beta_0 \leq \beta < 1 \), where \( \beta_0 = \frac{C_1}{C_2} \) and \( C_1, C_2 \) are given by Lemma 3.3. Then there exists \( \delta > 0 \) such that if \( u \) and \( u_1 \) are positive solutions of (I) then

\[
\frac{C_1 \beta_0}{2} [H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}})] \leq |t \nabla u(x) + (1 - t) \beta \nabla u_1(x)| \leq C_2[H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}})] \quad (9)
\]

for all \( t \in [0, 1] \) and \( x \in \Omega_\delta \), provided that \( \lambda > \bar{\lambda}, \mu > \bar{\mu} \).

**Proof.** Let \( t \in [0, 1] \). Using (8), we get

\[
|t u + (1 - t) \beta u_1|_{1,\alpha} \leq t|u|_{1,\alpha} + (1 - t)|u_1|_{1,\alpha} \leq C_2[H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}})]
\]

for \( \lambda > \bar{\lambda}, \mu > \bar{\mu} \), and right-hand side of (9) follows. Next, by Lemma 3.3 (i),

\[
\frac{\partial u}{\partial n}, \frac{\partial u_1}{\partial n} \leq -C_1[H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}})] \quad \text{on } \partial \Omega
\]

for \( \lambda > \bar{\lambda}, \mu > \bar{\mu} \), where \( n \) denotes the outward unit normal vector. This implies

\[
\frac{t \partial u}{\partial n} + (1 - t) \beta \frac{\partial u_1}{\partial n} \leq -C_1 \beta_0[H^{-1}(\lambda^{\frac{1}{p-1}}) + I^{-1}(\mu^{\frac{1}{p-1}})] \quad \text{on } \partial \Omega
\]

Hence

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\[
|t \nabla u(x) + (1 - t) \beta \nabla u_1(x)| \geq C_1 \beta_0 [H^{-1}(\lambda \frac{1}{\nu + 1}) + I^{-1}(\mu \frac{1}{\nu + 1})] \quad \text{on } \partial \Omega
\]  
(10)

Let \( w_t = t \nabla u(x) + (1 - t) \beta \nabla u_1(x) \). Then we have

\[
\frac{|w_t(x) - w_t(x_0)|}{|x - x_0|^\alpha} \leq C_2 [H^{-1}(\lambda \frac{1}{\nu + 1}) + I^{-1}(\mu \frac{1}{\nu + 1})] \quad \text{on } \partial \Omega
\]  
(11)

for \( x, x_0 \in \Omega, x \neq x_0 \). Let \( \delta > 0 \) satisfy \( C_2 \delta^\alpha < C_1 \frac{\beta_0}{\nu} \). Let \( x \in \Omega_\delta \) and \( x_0 \in \partial \Omega \) be such that \( d(x, \partial \Omega) = |x - x_0| \).

Then it follows from (10) and (11) that

\[
|w_t(x)| \geq |w_t (x_0)| - C_2 \delta^\alpha [H^{-1}(\lambda \frac{1}{\nu + 1}) + I^{-1}(\mu \frac{1}{\nu + 1})] \geq \frac{C_1 \beta_0 [H^{-1}(\lambda \frac{1}{\nu + 1}) + I^{-1}(\mu \frac{1}{\nu + 1})]}{2},
\]

which completes the proof of Lemma 3.4. \( \square \)

4 Proof of main result

In what follows, we denote by \( m_i, i = 1, 2, \ldots, \) constants depending only on \( \Omega, p, N, f, \gamma \).

**Proof of theorem.** Let \( \lambda \) be large enough so that Lemma 3.1, 3.2, 3.3(i), and 3.4 apply. Let \( u \) and \( u_1 \) be positive solution of (1). By Lemma 3.3, \( u \geq \beta_0 u_1 \) in \( \Omega \), where \( \beta_0 = \frac{C_1}{C_2} \). Let \( \beta \) be the largest number such that \( u \geq \beta u_1 \) in \( \Omega \) and suppose by contradiction that \( \beta < 1 \). Let

\[
\tilde{u} = \frac{u}{H^{-1}(\lambda \frac{1}{\nu + 1}) + I^{-1}(\mu \frac{1}{\nu + 1})}, \quad \tilde{u}_1 = \frac{u_1}{H^{-1}(\lambda \frac{1}{\nu + 1}) + I^{-1}(\mu \frac{1}{\nu + 1})}
\]

Since \( H^{-1}(\lambda \frac{1}{\nu + 1}) + I^{-1}(\mu \frac{1}{\nu + 1}) = \lambda \frac{1}{\nu} f \frac{1}{\nu} (H^{-1}(\lambda \frac{1}{\nu + 1})) + \mu \frac{1}{\nu} \gamma \frac{1}{\nu} (I^{-1}(\mu \frac{1}{\nu + 1})) \), we have

\[
\Delta_p \tilde{u} = \lambda f(u) + \mu \gamma(u) = \frac{\lambda f(u)}{f(H^{-1}(\lambda \frac{1}{\nu + 1}) + I^{-1}(\mu \frac{1}{\nu + 1}))} + \frac{\mu \gamma(u)}{\gamma(H^{-1}(\lambda \frac{1}{\nu + 1}) + I^{-1}(\mu \frac{1}{\nu + 1}))},
\]

\[
\Delta_p \tilde{u}_1 = \lambda f(u_1) + \mu \gamma(u_1) = \frac{\lambda f(u_1)}{\mu \gamma(H^{-1}(\lambda \frac{1}{\nu + 1}) + I^{-1}(\mu \frac{1}{\nu + 1}))} + \frac{\mu \gamma(u_1)}{\mu \gamma(H^{-1}(\lambda \frac{1}{\nu + 1}) + I^{-1}(\mu \frac{1}{\nu + 1}))}.
\]

Let \( \delta \) given by Lemma 3.4. Using the mean value theorem, we obtain

\[
L(\tilde{u} - \beta \tilde{u}_1) = \Delta_p \tilde{u} - (-\Delta_p \beta \tilde{u}_1) \geq \frac{\lambda f(\beta u_1) - \beta f(u)}{f(H^{-1}(\lambda \frac{1}{\nu + 1}) + I^{-1}(\mu \frac{1}{\nu + 1}))} + \frac{\mu \gamma(\beta u_1) - \beta \gamma(u)}{\gamma(H^{-1}(\lambda \frac{1}{\nu + 1}) + I^{-1}(\mu \frac{1}{\nu + 1}))}
\]

in \( \Omega_\delta \)

where

\[
Lw = - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} (a_{i,j}(x) \frac{\partial w}{\partial x_j}),
\]

where

\[
a_{i,j} = \int_0^1 \partial_{u_1} ((t \nabla \tilde{u}_1 + (1 - t) \beta \nabla \tilde{u}_1) dt, \quad a^j(\tilde{z}) = |\tilde{z}|^{p-1} \tilde{z}, i = 1, 2, \ldots, N, \quad z = (z_1, z_2, \ldots, z_N).
\]

Because of (9), see that the operator \( L \) is uniformly elliptic in \( \Omega_\delta \). In fact,

\[
\sum_{i,j=1}^{N} a_{i,j}(x) \xi_i \xi_j \geq m_0 |\xi|^2, \quad \forall x \in \Omega_\delta, \quad \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N,
\]

(12)

where \( m_0 = (\frac{C_5 \beta_0}{2})^{p-2} \) if \( p \geq 2, (p - 1)C_5^{p-2} \) if \( 1 < p < 2, \) and

\[
|a_{ij}|_{0, \alpha} \Omega_\delta \leq m_1 \quad \forall i, j = 1, \ldots, N.
\]

(13)

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Let \( D = \{ x \in \Omega_\delta : u_1(x) > \frac{a}{m_0} \} \), where \( a \) is given by (A.4). Then it follows from (A.4) and Lemma 3.3 (i) that
\[
[f(\beta u_1) - \beta^{p-1} f(u_1)] + [\gamma(\beta u_1) - \beta^{p-1} \gamma(u_1)] \geq (\beta^{p} - \beta^{p-1})[f(u_1) + \gamma(u_1)] \\
\quad \geq m_2(1 - \beta)[f(u_1) + \gamma(u_1)]
\]
for \( H^{-1}(\lambda \frac{1}{p-1}), I^{-1}(\mu \frac{1}{p-1}) \geq 1 \), where \( m_2 = \beta_0^p \min\{1, p - 1 - q\} \).

Since \( u_1(x) \leq \frac{a}{m_0} \) in \( \Omega \setminus D \), the mean value theorem gives
\[
|[f(\beta u_1) - \beta^{p-1} f(u_1)] + [\gamma(\beta u_1) - \beta^{p-1} \gamma(u_1)]| = (1 - \beta)|\frac{c}{u_1(x)}|f'(\frac{c}{u_1(x)}) + \gamma'(\frac{c}{u_1(x)})| - (p - 1)c^{p-2}[f(u_1(x)) + \gamma(u_1(x))],
\]
where \( c \in [\beta_0, 1] \). Since \( \lim_{z \to 0^+} z f'(z), \lim_{z \to 0^+} z \gamma'(z) < \infty \), this implies
\[
|[f(\beta u_1) - \beta^{p-1} f(u_1)] + [\gamma(\beta u_1) - \beta^{p-1} \gamma(u_1)]| \leq m_3(1 - \beta)
\]
where \( m_3 = \frac{1}{\beta_0} \sup_{0 < z \leq \frac{a}{m_0}} |z f'(z) + z \gamma'(z)| + (p - 1) \max\{1, \beta_0^{p-2}\} f(\frac{a}{m_0}) \).

Combining (11), (14) and (15), we obtain
\[
\begin{align*}
L(\frac{f(H^{-1}(\lambda \frac{1}{p-1}) + I^{-1}(\mu \frac{1}{p-1}))) \gamma(H^{-1}(\lambda \frac{1}{p-1}) + I^{-1}(\mu \frac{1}{p-1})))}{\beta^{p-1} + \beta \lambda \frac{1}{p-1}}) \geq & m_2[f(C_d(x, \partial \Omega)) + \gamma(C_d(x, \partial \Omega))] \quad & \text{in } D \\
- m_3 & \quad \text{in } \Omega_\delta \setminus D.
\end{align*}
\]

Let \( z \) be solution of
\[
Lz = \left\{ \begin{array}{ll}
m_2[f(C_d(x, \partial \Omega)) + \gamma(C_d(x, \partial \Omega))] & \text{in } D, \\
- m_3 & \text{in } \Omega_\delta \setminus D.
\end{array} \right.
\]
\quad \text{on } \partial \Omega_\delta
\]

Since \( \bar{u} - \beta \bar{u} \geq 0 \) on \( \partial \Omega_\delta \), it follows from the weak maximum principle that
\[
[f(H^{-1}(\lambda \frac{1}{p-1}) + (I^{-1}(\mu \frac{1}{p-1}) \gamma(H^{-1}(\lambda \frac{1}{p-1}) + I^{-1}(\mu \frac{1}{p-1})))(\bar{u} + \beta \bar{u})] \geq z \quad \text{in } \Omega_\delta
\]

Let \( \bar{z} \) satisfy
\[
L \bar{z} = m_2[f(C_d(x, \partial \Omega)) + \gamma(C_d(x, \partial \Omega))] \quad \text{in } \Omega_\delta, \quad \bar{z} = 0 \quad \text{on } \partial \Omega_\delta.
\]

Because of (13) and (14), and the maximum principle, there exists a positive number \( \nu \) depending on \( m_0, \nu_1, \Omega_\delta, f, \gamma \), such that
\[
\bar{z} \geq \theta d(x, \partial \Omega_\delta)
\]
(see Proposition in the Appendix).

Next, we have
\[
L(\bar{z} - z) = \bar{g} \equiv \left\{ \begin{array}{ll}
0 & \text{in } D, \\
m_2[f(C_d(x, \partial \Omega)) + \gamma(C_d(x, \partial \Omega))] & \text{in } \Omega_\delta \setminus D.
\end{array} \right. \\
\quad \text{on } \partial \Omega_\delta
\]
\quad \text{and note that}
\[
m_2[f(C_d(x, \partial \Omega)) + \gamma(C_d(x, \partial \Omega))] + m_3 \leq m_2[f(C_d) + \gamma(C_d)] + m_3 \equiv m_4 \quad \text{in } \Omega_\delta \setminus D.
\]
It then follows from Theorems 8.16 and 8.33 of [6] (see also the remark on page 212 of [6]) that
\[
|\bar{z} - z|_{1, \alpha, \Omega_\delta} \leq C ||g||_{r, \Omega_\delta} \leq C m_4 |\Omega_\delta \setminus D|^{\frac{1}{2}}
\]
where \( r = \frac{N}{1-\alpha}, \) C is a constant depending only on \( N, p, m_0, m_1, \Omega_\delta, \) and \( \Omega_\delta \setminus D \) denotes the Lebesgue measure of \( \Omega_\delta \setminus D. \)

From (18) and the mean value theorem,

\[
|z(x) - z(\bar{x})| \leq Cm_4|\Omega_\delta \setminus D|^\frac{1}{2}d(x, \partial \Omega_\delta) \quad \text{for } x \in \Omega_\delta
\]

which, together with (17), implies

\[
z(x) \leq |\theta - Cm_4|\Omega_\delta \setminus D|^\frac{1}{2}d(x, \partial \Omega_\delta) \quad \text{for } x \in \Omega_\delta
\]

By Lemma 3.3 (i),

\[
\Omega_\delta \setminus D \subseteq \{ x \in \omega : d(x, \partial \Omega) \leq \frac{a}{\beta_0C_1[H^{-1}(\lambda \frac{1}{p-1}) + I^{-1}(\mu \frac{1}{p-1})]} \}
\]

and hence \(|\Omega_\delta \setminus D| \to 0 \) as \( \lambda, \mu \to \infty. \) Thus, for \( \lambda, \mu \) large enough

\[
z \geq \frac{\bar{d}}{2}d(x, \partial \Omega_\delta)
\]

For \( x \in \Omega_\frac{\delta}{2}, \) we have \( d(x, \partial \Omega_\delta) = d(x, \partial \Omega) \) and therefore using (17), we obtain

\[
\bar{u}(x) - \beta \bar{u}_1(x) \geq \left[ \frac{1}{f(H^{-1}(\lambda \frac{1}{p-1}) + I^{-1}(\mu \frac{1}{p-1}))} \right] (1 - \beta)z(x)
\]

\[
\geq \left[ \frac{1}{f(H^{-1}(\lambda \frac{1}{p-1}) + I^{-1}(\mu \frac{1}{p-1}))} \right] (1 - \beta)\theta d(x, \partial \Omega)
\]

\[
\geq \frac{\bar{d}}{\beta_0C_2} \left[ \frac{1}{f(H^{-1}(\lambda \frac{1}{p-1}) + I^{-1}(\mu \frac{1}{p-1}))} \right] (1 - \beta)\theta \bar{u}_1
\]

for \( x \in \Omega_\frac{\delta}{2} \) and \( \lambda, \mu \) large. In order words, there exists \( \bar{\epsilon} > 0 \) such that

\[
u > (\beta + \bar{\epsilon})u_1 \quad \text{in } \Omega_\frac{\delta}{2}
\]

for \( \lambda, \mu \) large. In particular,

\[
u > \beta u_1(x) + \bar{\epsilon} \quad \text{when } \nu = \frac{\epsilon}{2},
\]

where \( \bar{\epsilon} = \epsilon C_1 \frac{\delta}{2} [H^{-1}(\lambda \frac{1}{p-1}) + I^{-1}(\mu \frac{1}{p-1})]. \)

Let \( \Omega_1 = \Omega \setminus \Omega_\frac{\delta}{2}. \) Then, for \( \lambda, \mu \) large enough,

\[
\bar{u}_1(x) \geq C_1 \frac{\epsilon}{2} [H^{-1}(\lambda \frac{1}{p-1}) + I^{-1}(\mu \frac{1}{p-1})] \geq \frac{\bar{d}}{\beta_0} \text{ which implies}
\]

\[
-\Delta_p u = \lambda f(u) + \mu \gamma(u) \geq \lambda f(\beta u_1) + \mu \gamma(\beta u_1) > \beta^{p-1}(\lambda f(u_1) + \mu \gamma(u_1)) \quad \text{in } \Omega_1.
\]

Since

\[
-\Delta_p(\beta u_1 + \bar{\epsilon}) = \beta^{p-1}(\lambda f(u_1) + \mu \gamma(u_1)) \quad \text{in } \Omega_1,
\]

and \( u \geq \beta u_1 + \bar{\epsilon} \) on \( \partial \Omega_1, \) it follows that

\[
u \geq \beta u_1 + \bar{\epsilon} \quad \text{in } \Omega_1
\]

(21)

Combining (20) and (21), we deduce the existence of \( \beta_1 > \beta \) such that \( u \geq \beta_1 u_1 \) in \( \omega. \) This contradicts the maximality of \( \beta. \) Hence \( \beta \geq 1 \) and Theorem is proved. \( \square \)

References


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