New Method for Constructing Exact Solutions to Nonlinear PDEs

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Abstract: We propose in this paper a new approach to construct exact solutions of nonlinear PDEs. The method used is called "the travelling profiles method". The travelling profiles method enables us to obtain many exact solutions to large classes of nonlinear PDEs.

Keywords: Nonlinear PDE; exact solutions; travelling profiles method

1 Introduction

In recent years, a certain number of methods had been developed for seeking exact solutions to nonlinear PDEs, a variety of powerful methods such as the Hirota’s bilinear methods [4] based on the Hirota transformation, the truncated Painlevé expansion method [1,12]; the homogeneous balance method [12,13], the special ”separation” of the variables [6] were used to investigate nonlinear problems.

In this paper, we present a new approach to find exact solutions to some nonlinear PDEs. The approach presented one will be called "the traveling profiles method" (TPM).

Consider the following equation:
\[ \frac{\partial u}{\partial t} = A_x u, \]  
(1.1)

where \( A_x u \) is a linear or nonlinear differential operator.

2 The travelling profiles method (TPM):

The principle of this method is to seek the solution of the problem (1.1) in the form
\[ u(x,t) = c(t) \psi(\xi) \quad \text{with} \quad \xi = \frac{x - b(t)}{a(t)}, \quad a, b, c \in \mathbb{R}, \]
(2.1)

where \( \psi \) is in \( L^2 \), that one will call "the based-profile". The parameters \( a(t), b(t), c(t) \) are real valued functions of \( t \).

The coefficients \( c(t), a(t), b(t) \) are determined by the solution of minimization problem:
\[ \min_{c,a,b} \int_{-\infty}^{+\infty} \left| \frac{\partial u}{\partial t} - A_x u \right|^2 dx, \]
(2.2)

therefore, we obtain three orthogonality equations which are read
\[
\left\{ \begin{array}{l}
\left( \frac{\partial u}{\partial t} - A_x u, \psi \right) = 0 \\
\left( \frac{\partial u}{\partial t} - A_x u, \xi \psi' \right) = 0 \\
\left( \frac{\partial u}{\partial t} - A_x u, \psi' \xi \right) = 0
\end{array} \right.
\]
(2.3)
where \((.,.)\) is the inner product in \(L^2\) space.

The PDE (1.1) is then transformed into a set of three coupled ODE’s:
\[
\begin{align*}
\dot{\xi} \langle \psi, \psi \rangle - \frac{b}{a} \langle \xi \psi', \psi \rangle - \frac{b}{a} \langle \psi', \psi \rangle &= \frac{1}{a} \langle A_{\xi} u, \psi \rangle \\
\dot{\xi} \langle \psi', \psi \rangle - \frac{b}{a} \langle \xi \psi', \psi \rangle - \frac{b}{a} \langle \psi', \psi \rangle &= \frac{1}{a} \langle A_{\xi} u, \psi' \rangle \\
\dot{\xi} \langle \psi, \psi' \rangle - \frac{b}{a} \langle \xi \psi', \psi' \rangle - \frac{b}{a} \langle \psi', \psi \rangle &= \frac{1}{a} \langle A_{\xi} u, \psi'^2 \rangle
\end{align*}
\] (2.4)

2.1 A priori estimates of solutions:

Let:
\[
V_t = \{ \psi, \xi \psi', \psi' \}
\]
the subspace of \(L^2\) generated by associated functions to \(\psi\) at the moment \(t\).

From relations (2.3), it is deduced that \(\frac{\partial u}{\partial t} - A_{x} u\) is orthogonal to subspace \(V_t\).

In particular we have \(\frac{\partial u}{\partial t} - A_{x} u, \frac{\partial u}{\partial t}\) = 0, thus if also \(A_{x} u\) belongs to \(V_t\) then the method provides us a weakly exact solution, which is written under the form
\[
u(x, t) = c(t) \psi \left[ \frac{x - b(t)}{a(t)} \right].
\] (2.5)

Now we want to establish conditions on the method to find exact solutions to equation (1.1).

2.2 Exact solutions to some nonlinear PDEs

Theorem 1

For \(\psi \in C^2 \cap L^2\), the equation (1.1) admits an exact solution in the form
\[
u(x, t) = c(t) \psi \left[ \frac{x - b(t)}{a(t)} \right],
\]
if

1. \(A_{x} u = \frac{c^p}{a^q} A_{\xi} \psi\), for \(p, q \in \mathbb{R}\),
2. the "based profile" \(\psi\) is a solution of the following equation:
\[
A_{\xi} \psi = \alpha \psi + \beta \xi \psi' + \gamma \psi''
\] (2.6)

in this case, the coefficients \(c(t), a(t), b(t)\) are determined by the system:
\[
\begin{align*}
\dot{c} &= \frac{c^p}{a^{q+1}} \alpha \\
\dot{a} &= -\frac{c^{p-1}}{a^q} \beta \\
\dot{b} &= -\frac{c^{p-1}}{a^q} \gamma
\end{align*}
\] (2.7)

Proof. According to the estimation principle of this method, if \(A_{x} u\) belongs to the subspace \(V_t\), then the function \(\nu(x, t) = c(t) \psi(x, t)\) is an exact solution of equation (1.1), in this case the term \(A_{\xi} \psi\) can be expressed as a linear combination of functions \(\psi, \xi \psi',\) and \(\psi'\), thus \(A_{x} \psi = \alpha \psi + \beta \xi \psi' + \gamma \psi''\), for \(\alpha, \beta, \gamma \in \mathbb{R}\).

The system (2.7) is obtained as follow: when one replaces \(A_{\xi} \psi\) by the combination \(\alpha \psi + \beta \xi \psi' + \gamma \psi''\) in (2.4), we obtain the system:
\[
\begin{align*}
M X &= \frac{c^{p-1}}{a^q}MF
\end{align*}
\] (2.8)

with
\[
M = \begin{pmatrix}
\langle \psi, \psi \rangle & \langle \xi \psi', \psi \rangle & \langle \psi', \psi \rangle \\
\langle \xi \psi', \psi \rangle & \langle \xi \psi', \psi \rangle & \langle \psi', \psi \rangle \\
\langle \psi', \psi \rangle & \langle \psi', \psi \rangle & \langle \psi', \psi \rangle \\
\end{pmatrix},
X = \begin{pmatrix}
\frac{\dot{c}}{c} \\
\frac{\dot{a}}{a} \\
\frac{\dot{b}}{a}
\end{pmatrix}
\text{ and } F = \begin{pmatrix}
\alpha \\
\beta \\
\gamma
\end{pmatrix}
\]

where \((.,.)\) is the inner product in \(L^2\).

The matrix in system (2.8) is symmetric and invertible, then (2.8) can be written under the form (2.7).
2.2.1 Resolution of the differential system:

We can resolve the system (2.7) as follow: from (2.7) we have

\[
\begin{align*}
    c(t) &= K_0 a(t) \frac{\partial c}{\partial t}, \\
    b(t) &= \frac{\gamma}{\beta} a(t) + K'_0
\end{align*}
\]

with \( K_0, K'_0 \) constants, \((2.9)\)

If we replace (2.9) in (2.7), we can deduce finally:

\[
\begin{align*}
    a(t) &= \left( A(-K_0^{p-1} t + K_1) \right)^{\frac{1}{3}} \\
    c(t) &= K_0 \left( A(-K_0^{p-1} t + K_1) \right)^{\frac{2}{3}} \\
    b(t) &= \frac{\gamma}{\beta} \left( A(-K_0^{p-1} t + K_1) \right)^{\frac{1}{3}} + K'_0
\end{align*}
\]

with \( K_0, K'_0, K_1 \) constants and \( A = q + \frac{\alpha}{\beta} (p - 1) \neq 0. \)

Now, to illustrate the idea of this method we have this example.

2.2.2 Example:

Let the equation

\[ \frac{\partial u}{\partial t} = (u^2)_{xx}, \]

in this case we have \( A_\xi u = \frac{\partial^2}{\partial t} A_\xi \psi \), if we seek an exact solution like \( u(x, t) = c(t) \psi \left( \frac{x - b(t)}{a(t)} \right) \), then the "based-profile" \( \psi \) must verify the following ODE:

\[ (\psi^2)_\xi = c \psi + \beta \psi'_\xi + \gamma \psi''_\xi. \]

If we take for example \( \alpha = \beta \), and for \( \gamma \), the equation (2.12) can be written in the form

\[ \frac{d}{d\xi} \left[ (\psi^2)_\xi - (\beta \xi + \gamma) \right] = 0 \]

then we obtain

\[ (\psi^2)_\xi - (\beta \xi + \gamma) = k \]

for \( k = 0 \) we have

\[ \psi(\xi) = \frac{1}{2} \left( \beta \xi + \gamma \xi + k' \right), \quad \text{with} \ k' \ \text{constant.} \]

Then an exact solution to equation (2.11) takes the form:

\[ u(x, t) = \frac{1}{2} \left[ \beta \left( \frac{x - b(t)}{a(t)} \right)^2 + \gamma \left( \frac{x - b(t)}{a(t)} \right) + k' \right], \quad \text{with} \ k' \ \text{constant.} \]

where \( c(t), a(t), \) and \( b(t) \) are given by:

\[
\begin{align*}
    a(t) &= \left[ -\beta K_0 t + K_1 \right]^{\frac{1}{2}}, \\
    c(t) &= K_0 \left[ -\beta K_0 t + K_1 \right]^{\frac{2}{3}}, \\
    b(t) &= -\frac{\gamma}{\beta} \left[ -\beta K_0 t + K_1 \right]^{\frac{1}{3}} + K'_0
\end{align*}
\]

with \( K_0, K'_0, K_1 \) constants.

3 Conclusion

A method to construct exact solutions to some PDEs is presented in this paper. This method enables us to obtain exact solutions to large classes of nonlinear PDEs. It gives us the possibility to obtain very varied choice of classes of exact solutions. The idea of our method (TPM) is well illustrated by an example. This approach is very promising and can also bring new results for other applications in PDEs.

IJNS homepage: http://www.nonlinearscience.org.uk/
References


