

Homotopy Perturbation Combined with Padé Approximation for Solving Two Dimensional Viscous Flow in the Extrusion Process

M. M. Rashidi¹ *, D. D. Ganji²

¹Mechanical Engineering Department, Engineering Faculty of Bu-Ali Sina University, Hamedan, Iran

²Mechanical Engineering Department, Mazandaran University, Babol, P.O. Box 484, Iran

(Received 16 December 2007, accepted 13 December 2008)

Abstract: In this letter, we will consider homotopy perturbation method (HPM) and Padé-approximant, for finding approximate solutions of two dimensional viscous flow. The HPM technique provides a sequence of functions which converges to the exact solution of the problem. The Padé-approximant increase the convergence of the solutions obtained by the HPM. It is of interest to be noted that Padé-approximants give results with no greater error bound than approximation by polynomials. The solutions is compared with the exact solution.

Keywords: homotopy perturbation method; Padé-approximants; viscous flow, extrusion

1 Introduction

Most of engineering problems, especially some heat transfer and fluid flow equations are nonlinear, therefore some of them solved using computational fluid dynamic (numerical) method and some are solved using the analytical perturbation method [1-3].

In the numerical method, stability and convergence should be considered, so as to avoid divergent or inappropriate results. In the analytical perturbation method, we should exert the small parameter in the equation [4]. Finding the small parameter and exerting it into the equation are therefore the problems with this method. Perturbation method is one of the well-known methods to solve the nonlinear equations which was studied by a large number of researchers such as Bellman [5] and Cole [6]. Actually, these scientists had paid more attention to the mathematical aspects of the subject which included a loss of physical verification. This loss in the physical verification of the subject was recovered by Nayfeh [7] and Van Dyke [8].

In recent years, there has appeared an ever increasing interest of scientist and engineers in analytical techniques for studying nonlinear problems. Such techniques have been dominated by the perturbation methods and have found many applications in science, engineering and technology. However, like other analytical techniques, perturbation methods have their own limitations. For example, all perturbation methods require the presence of a small parameter in the nonlinear equation and approximate solutions of equation containing this parameter are expressed as series expansions in the small parameter. Selection of small parameter requires a special skill and very important. Therefore, an analytical method is welcome which does not require a small parameter in the equation modeling the phenomena.

Since there are some limitations with the common perturbation method, and also because the basis of the common perturbation method was upon the existence of a small parameter, developing the method for different applications is very difficult. Therefore, many different new methods have recently introduced some ways to eliminate the small parameter such as artificial parameter method introduced by Liu [9], the homotopy analysis method by Liao [10] and the variational iteration method by He [11-13]. One of the semi-exact methods is the HPM [14-22].

* Corresponding author. E-mail address: mm_rashidi@yahoo.com

In this paper we solve the flow field due to stretching boundary with partial slip by the HPM. The flow due to a stretching boundary is important in extrusion processes. The bathing fluid is entrained by the tangential velocity of the extrusate and thus affects convective cooling [23]. Vleggaar showed experimentally the velocity of an extrusate is initially proportional to the distance from the orifice [24].

2 Basic concepts of HPM

The homotopy perturbation method is a combination of classical perturbation technique and homotopy technique. To explain the basic idea of homotopy perturbation method for solving nonlinear PDEs, we consider the following ODE:

$$A(u) - f(r) = 0, \quad r \in \Omega \quad (1)$$

Subject to boundary condition

$$B(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma \quad (2)$$

where A is general nonlinear differential operator, B a boundary operator, $f(r)$ is a known analytical function, Γ is the boundary of domain Ω and $\frac{\partial}{\partial n}$ denotes differentiation along the normal drawn outwards from Ω .

The operator A can, generally speaking, be divided into two parts: a linear part L and a nonlinear part N . Eq. (1) therefore can be rewritten as follows:

$$L(v) + N(v) - f(r) = 0 \quad (3)$$

We construct a homotopy of Eq. (1) $v(r, p) : \Omega \times [0, 1] \rightarrow \Re$ which satisfies

$$H(v, p) = (1 - p) [L(v) - L(u_0)] + p [A(v) - f(r)], \quad p \in [0, 1], \quad r \in \Omega \quad (4)$$

where $p \in [0, 1]$ is an embedding parameter and u_0 is an initial guess approximation of Eq. (3) which satisfies the boundary conditions.

3 Governing equations

Consider a two-dimensional stretching boundary (Fig. 1) where the lateral surface velocity is proportional to the distance x from the origin.

The velocity is as follows:

$$U = a x \quad (5)$$

Let (u, v) be the fluid velocities in the (x, y) directions. Navier's condition is then [18]

$$u(x, 0) - U = k v \frac{\partial u}{\partial y}(x, 0) \quad (6)$$

where k is a proportional constant and v is kinematic viscosity of the bulk fluid. The steady 2-D Navier-Stokes equations are:

$$u_x + v_y = 0 \quad (7)$$

$$u u_x + v u_y = \nu (u_{xx} + u_{yy}) - p_x / \rho \quad (8)$$

$$u v_x + v v_y = \nu (v_{xx} + v_{yy}) - p_y / \rho \quad (9)$$

where p and ρ are pressure and density. For solving Eqs. (7)-(9) we must apply boundary conditions (Eqs. (5) and (6)). Other boundary conditions are no lateral velocity and pressure gradient far from the stretching surface. For similarity solutions we set [23]

$$u = a x f'(t) \quad (10)$$

$$v = -\sqrt{a \nu} f(t) \quad (11)$$

$$t = y \sqrt{a/v} \tag{12}$$

Continuity Eq. automatically is satisfied. Eq. (8) can be written as follows:

$$f'''(t) - f'^2(t) + f(t) f''(t) = 0 \tag{13}$$

With the boundary conditions:

$$f(0) = 0 \tag{14}$$

$$f'(\infty) = 0 \tag{15}$$

$$f'(0) = K f''(0) + 1 \tag{16}$$

Eq. (15) shows there isn't lateral velocity at infinity, on the other hand in $y = 0$ the v velocity is equal zero (Eq. (14)). Eq. (16) is from Eq. (6) and $K = k \sqrt{a v}$ is a non-dimensional parameter indicating the relative importance of partial slip. For $K = 0$, the fluid is inviscid.

In this section He's HPM is used to find approximate solutions of the Eq. (13). Suppose the solution have the form:

$$f(t) = f_0(t) + p f_1(t) + p^2 f_2(t) + p^3 f_3(t) + p^4 f_4(t) + p^5 f_5(t) + \dots \tag{17}$$

If we apply Eq. (4) to Eq. (13), then:

$$(1 - p) f'''(t) + p [f'''(t) - f'^2(t) + f(t) f''(t) = 0] = 0 \tag{18}$$

Then substituting Eq. (17) into Eq. (18) and rearranging based on powers of p - terms, we have:

$$p^0 : f_0''' = 0 \tag{19}$$

$$p^1 : f_1''' - f_0'^2 + f_0 f_0'' = 0 \tag{20}$$

$$p^2 : f_2''' - 2 f_0' f_1' + f_0 f_1'' + f_1 f_0'' = 0 \tag{21}$$

$$p^3 : f_3''' - f_1'^2 - 2 f_0' f_2' + f_0 f_2'' + f_1 f_1'' + f_2 f_0'' = 0 \tag{22}$$

$$p^4 : f_4''' - 2 f_0' f_3' - 2 f_1' f_2' + f_0 f_3'' + f_1 f_2'' + f_2 f_1'' + f_3 f_0'' = 0 \tag{23}$$

$$p^5 : f_5''' - f_2'^2 - 2 f_0' f_4' - 2 f_1' f_3' + f_0 f_4'' + f_1 f_3'' + f_2 f_2'' + f_3 f_1'' + f_4 f_0'' = 0 \tag{24}$$

$$p^6 : f_6''' - 2 f_0' f_5' - 2 f_1' f_4' - 2 f_2' f_3' + f_0 f_5'' + f_1 f_4'' + f_2 f_3'' + f_3 f_2'' + f_4 f_1'' + f_5 f_0'' = 0 \tag{25}$$

$$lp^7 : f_7''' - f_3'^2 - 2 f_0' f_6' - 2 f_1' f_5' - 2 f_2' f_4' + f_0 f_6'' + f_1 f_5'' + f_2 f_4'' + f_3 f_3'' + f_4 f_2'' + f_5 f_1'' + f_6 f_0'' = 0 \tag{26}$$

To determine $f(t)$, the above equations should be solved with appropriate boundary conditions (eqs. (14)-(16)). The solutions of above equations for $K = 0$ and $K = 20$, are as follows

$$f(t) = -\frac{31}{1307674368000}t^{15} - \frac{116383}{20922789888000}t^{16} + \frac{626881}{355687428096000}t^{17} - \frac{1718011}{11058645491712000}t^{19} + \frac{70525699}{2432902008176640000}t^{20} + \frac{419}{12934088294400}t^{18} - \frac{1323139}{4644631106519040000}t^{21} - \frac{6597271}{28820531481477120000}t^{22} + \frac{18992159}{1988616672221921280000}t^{23} + t - \frac{1}{2}t^2 + \frac{1}{6}t^3 - \frac{1}{24}t^4 + \frac{1}{120}t^5 - \frac{1}{720}t^6 + \frac{1}{5040}t^7 + \frac{1}{13440}t^8 - \frac{1}{51840}t^9 - \frac{13}{1209600}t^{10} + \frac{13}{4435200}t^{11} + \frac{29}{479001600}t^{12} - \frac{307}{1245404160}t^{13} + \frac{839}{12454041600}t^{14} \tag{27}$$

and

$$\begin{aligned}
 f(t) = & -\frac{26454129170347965799}{13359654 \times 10^{32}}t^{18} - \frac{34981592008374802708157}{290095344 \times 10^{33}}t^{19} + \\
 & \frac{9610620390380100302367991}{1099944846 \times 10^{36}}t^{20} + 0.124t - 0.0219t^2 - \frac{152109012706142059}{1135134 \times 10^{27}}t^{15} + \\
 & \frac{961}{375 \times 10^3}t^3 + \frac{15987}{1 \times 10^9}t^5 - \frac{94928946234172707119}{108972864 \times 10^{28}}t^{16} \\
 & - \frac{2263}{1 \times 10^7}t^4 - \frac{6636411496225660841695231}{1466593128 \times 10^{38}}t^{21} \\
 & - \frac{4137391052244534387257825723}{48397573224 \times 10^{40}}t^{22} + \frac{7342381}{1575 \times 10^{11}}t^7 - \frac{70153}{75 \times 10^9}t^6 + \\
 & \frac{1384631564415805048337910253}{107033094630 \times 10^{41}}t^{23} + \frac{49660951}{14 \times 10^{15}}t^8 - \\
 & \frac{679770225707}{756 \times 10^{19}}t^{10} - \frac{3799829309}{567 \times 10^{16}}t^9 + \frac{30866371688293}{31185 \times 10^{20}}t^{11} + \\
 & \frac{470250127393223949061}{4288284 \times 10^{31}}t^{17} - \frac{87749876813408687}{972972 \times 10^{24}}t^{13} \\
 & - \frac{3550152787477}{6237 \times 10^{22}}t^{12} + \frac{28171951570650389}{2837835 \times 10^{24}}t^{14}
 \end{aligned} \tag{28}$$

It is well known that Padé-approximants will converge on the entire real axis [26] if $f(t)$ is free of singularities on the real axis. The Padé-approximants of $f(t)$ for different values of K are as follows:

$$K = 0 : f(t)_{[4,4]} = \left(\frac{1}{6}t^4 + \frac{1}{42}t^3 + 10t^2 + t\right) / \left(1 + \frac{21}{2}t + \frac{143}{28}t^2 + \frac{85}{84}t^3 + \frac{47}{560}t^4\right) \tag{29}$$

$$\begin{aligned}
 lK = 0.3 : f(t)_{[4,4]} = & (0.7897t + 6.574906590t^2 - 0.0216062264t^3 + \\
 & 0.09162528278t^4) / (1 + 8.769667708t + 3.733347630t^2 + 0.6480062260t^3 + \\
 & 0.04720179853t^4)
 \end{aligned} \tag{30}$$

$$\begin{aligned}
 lK = 1 : f(t)_{[4,4]} = & (0.5699999998t + 3.756923919t^2 - 0.005201718795t^3 + \\
 & 0.03721878535t^4) / (0.9999999998 + 6.968287578t + 2.524263352t^2 \\
 & + 0.3733598838t^3 + 0.02316897904t^4)
 \end{aligned} \tag{31}$$

$$\begin{aligned}
 lK = 2 : f(t)_{[4,4]} = & (0.4320000001t + 2.354704940t^2 + 0.006419254672t^3 \\
 & + 0.01675447130t^4) / (1 + 5.779409582t + 1.842572721t^2 \\
 & + 0.2401598196t^3 + 0.01311000213t^4)
 \end{aligned} \tag{32}$$

$$\begin{aligned}
 lK = 5 : f(t)_{[4,4]} = & (0.275t + 1.01050853t^2 + 0.02348047362t^3 + \\
 & 0.003069671114t^4) / (1 + 3.938212837t + 1.077806319t^2 + \\
 & 0.1208516239t^3 + 0.005617674397t^4)
 \end{aligned} \tag{33}$$

$$\begin{aligned}
 lK = 20 : f(t)_{[4,4]} = & (0.1240000001t + 0.2901258355t^2 + 0.002483762474t^3 + \\
 & 0.0004911686858t^4) / (1 + 2.516337383t + 0.4437813266t^2 + \\
 & 0.03215924034t^3 + 0.0009716443589t^4)
 \end{aligned} \tag{34}$$

When $K = 0$, Crane [27] found the exact solution:

$$f(t) = 1 - e^{-t} \tag{35}$$

The perturbation solution for small K is:

$$\begin{aligned}
 f(t) = & 1 - e^{-t} + \frac{K}{2} [(1-t)e^{-t} - 1] + \\
 & K^2 \{0.087459e^{-t} + 1.221835(1+te^{-t}) - 0.25[h(e^{-t}) - t + 5]\} + O(K^3)
 \end{aligned} \tag{36}$$

$$h(t) = \frac{1}{4}t^2 - \frac{1}{72}t^3 + \frac{1}{864}t^4 - \frac{1}{9600}t^5 + \frac{1}{108000}t^6 - \dots \tag{37}$$

4 Discussion

In this paper, the HPM-Padé is used to find approximate solutions of two dimensional Navier-Stokes equations. In this work, the Maple Package is used to solve the differential equations. The values of $f(\infty)$ obtained by the HPM, the HPM-Padé and the numerical method for different values of K , have been presented in the table 1. The results show that, the HPM-Padé is more accurate respect to the HPM. The

Table 1: Comparison between $f(\infty)$ by HPM,HPM-Paed and numerical method

K	0	0.3	1	2	5
HPM	0.937	0.840	0.701	0.601	0.464
HPM-Padé	1.006	0.896	0.746	0.633	0.475
Numerical method [18]	1	0.887	0.748	0.652	0.514

accuracy of the method is very good and obtained results are near to the exact solution. The approximate solution obtained in Fig. 2 in comparison with exact solution admit a remarkable accuracy.

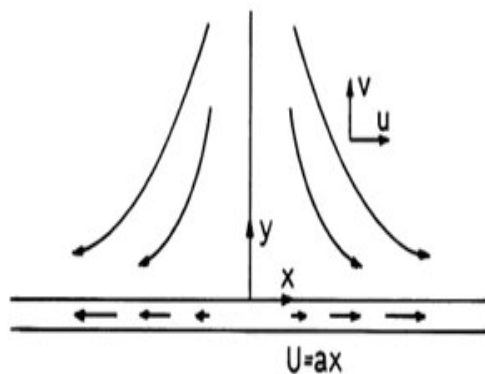


Figure 1: Two dimensional flow.

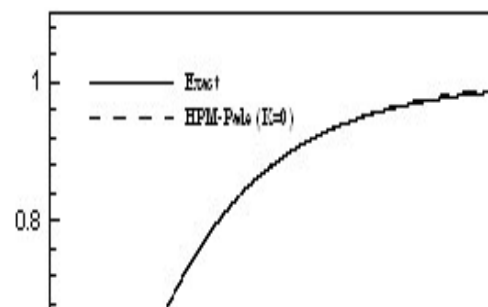


Figure 2: Comparison of the exact and approximate solution by the HPM-Padé

The approximate solutions of function $f(t)$ have been shown in the Figs. 3,4,5. The approximate solutions by the perturbation method is only valid for small values of K . For example the approximate solutions that are presented in the Figs. 4 and 5 are non physical solutions, because the value of $f(t)$ must be less (or equal) than one for different values of K . In the Figs. 6 and 7 the [4,4] Padé approximants of $f(t)$ and $f'(t)$ for different values of K are presented. Figs. 8 and 9 show the velocities in x direction for different values of time. In the Fig. 10, the distribution of velocity in y direction for different values of K are presented.

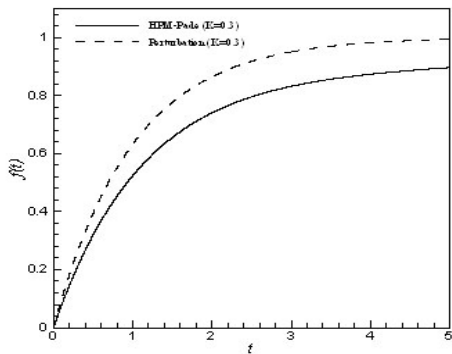


Figure 3: Comparison of the approximate solution by the HPM-Padé and the perturbation.

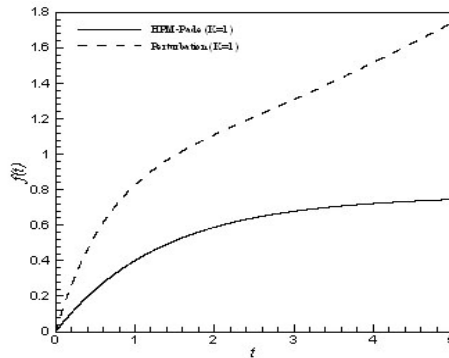


Figure 4: Comparison of the approximate solution by the HPM-Padé and the perturbation.

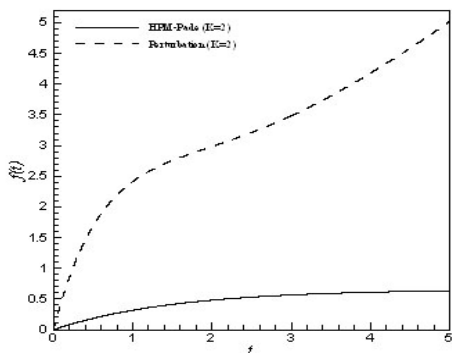


Figure 5: Comparison of the approximate solution by the HPM-Padé and the perturbation

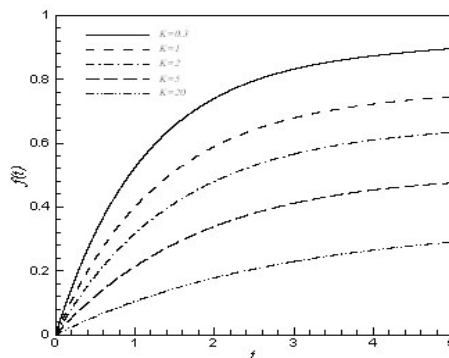


Figure 6: [4,4] Pad approximants for different values of K

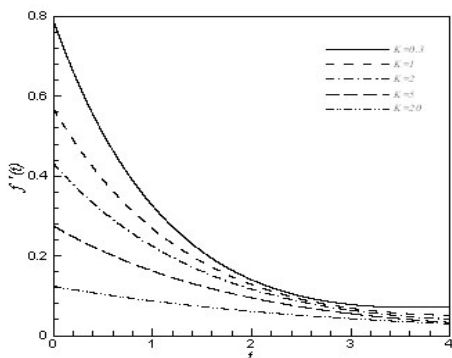


Figure 7: [4,4] Pad approximants for different values of K .

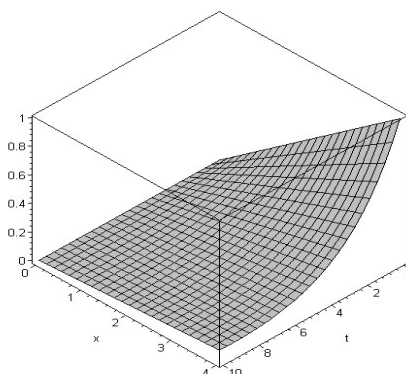


Figure 8: The distribution of velocity in x direction versus $x, t (K = 20)$.

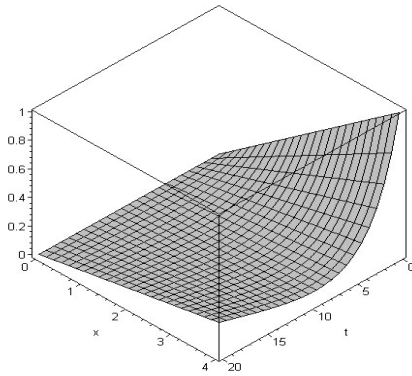


Figure 9: The distribution of velocity in x direction versus x, t ($K = 20$).

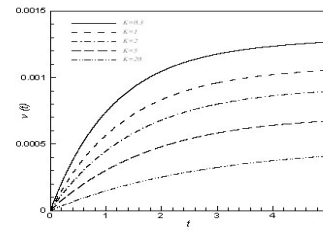


Figure 10: The distribution of velocity in y direction for different values of K ($a = 2, \nu = 10^{-6}$).

References

- [1] A. Rajabi, D. D. Ganji, H. Taherian: Application of homotopy perturbation method in nonlinear heat conduction and convection equations. *Physics Letters A*. 360:570-573(2007)
- [2] D. D. Ganji, A. Rajabi: *International Communications in Heat and Mass Transfer*. 33(3):391 (2006)
- [3] A. M. Siddiqui, M. Ahmed, Q. K. Ghori: Thin film flow of non-Newtonian fluids on a moving belt. *Chaos, Solitons and Fractals*. 33:1006-1016(2007)
- [4] D. D. Ganji: The application of He's homotopy perturbation method to nonlinear equations arising in heat transfer. *Physics Letters A*. 355:337-341(2006)
- [5] R. Bellman: Perturbation techniques in mathematics. *Physics and Engineering, Holt, Rinehart & Winston*(New York).(1964)
- [6] J.D. Cole: Perturbation methods in applied mathematics. *Blaisedell, Waltham, MA.*(1968)
- [7] A. H. Nayfeh: Perturbation methods. *Wiley, NewYork.*(1973)
- [8] M. VanDyke: Perturbation methods in fluid mechanics. *Annotated edition, Parabolic press, Stanford, CA.*(1975)
- [9] G.L. Liu: New research directions in singular perturbation theory artificial parameter approach and inverse-perturbation technique. *Conference of 7th modern mathematics and mechanics, Shanghai.* (1997)
- [10] S.J. Liao: An approximate solution technique not depending on small parameters: a special example. *Int. J. Non-Linear Mech.* 303: 371-380(1995)
- [11] J.H. He: Approximate analytical solution for seepage flow with fractional derivatives in porous media. *J. Comput. Math. Appl. Mech. Eng.* 167:57-68(1998)
- [12] J.H. He: Approximate solution for nonlinear differential equations with convolution product nonlinearities. *Comput. Math. Appl. Mech. Eng.* 167:69-73(1998)
- [13] J.H. He: Variational iteration method: a kind of nonlinear analytical technique: some examples. *Int. J. Non-Linear Mech.* 344:699 (1999)
- [14] J. H. He: Newton-like method for solving algebraic equations. *Commun Nonlinear Sci Numer Simulat.* 3(2): 106-109(1998)

- [15] J. H. He: Comparison of homotopy perturbation and homotopy analysis method. *Appl Math Comput.* 156:359-527(2004)
- [16] J. H. He: Homotopy perturbation method: a new nonlinear analytical technique. *Appl Math Comput.* 135:73-79(2003)
- [17] J. H. He: Asymptotology by homotopy perturbation method. *Appl Math Comput.* 156:591-596(2004)
- [18] Hassan Khaleghi, D.D. Ganji, A. Sadighi: Application of Variational iteration and Homotopy-perturbation methods to nonlinear heat transfer equations with variable coefficients. *Numerical Heat Transfer, Part A.* 52(1): 25-42(2007)
- [19] D.D. Ganji, A. Sadighi: He's Homotopy-perturbation Method to Nonlinear Coupled Systems of Reaction-diffusion Equations. *International Journal of nonlinear Science and numerical Simulation.* 7(4): 413-420(2006)
- [20] A. Sadighi, D.D. Ganji: Exact solutions of Laplace equation by homotopy-perturbation and Adomian decomposition methods. *Physics Letters A.* 367:83-87(2007)
- [21] A. Sadighi, D.D. Ganji: Solution of the generalized nonlinear Boussinesq equation using homotopy perturbation and variational iteration methods. *International Journal of Nonlinear Science and Numerical Simulation.* 8(3): 435-444(2007)
- [22] D.D. Ganji, A. Sadighi: Application of homotopy perturbation and variational iteration methods to nonlinear heat transfer and porous media equations. *Journal of Computational and Applied mathematics.* 207: 24-34(2007)
- [23] C. Y. Wang: Flow due to a stretching boundary with partial slip-an exact solution of the Navier-Stokes equations. *Chemical Engineering Science.* 57:3745-3747(2002)
- [24] J. Vlegaar: Laminar boundary-layer behavior on continuous accelerated surfaces. *Chemical Engineering Science.* 32: 1517-1525.
- [25] C. Y. Wang: The three dimensional flow due to a stretching flat surface. *Physics of Fluids.* 27:1915-1917(2002)
- [26] J. P. Boyd: Padé Approximants algorithm for solving nonlinear ordinary differential equation boundary value problems on an unbounded domain. *Comput. Phys.* 11:299-303(1997)
- [27] L. J. Crane: Flow past a stretching plate. *Zeitschrift fuer Angewandte Mathematik und Physik.* 21: 645-647(1970)