The Infinite Propagation Speed and the Limit Behavior for the B-family Equation with Dispersive Term

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Abstract: In this paper, we study the infinite propagation speed and the limit behavior for the b-family equation with dispersive term. We prove that any nontrivial classical solution of the b-family equation with dispersive term will not have compact support if its initial data has this property. The issue of passing to the limit as the dispersive parameter tends to zero for the solution of the b-family equation with dispersive term is investigated, and we get the solution converges to that of b-family equation.

Key words: b-family equation; infinite propagation speed; compact supported solutions; limit behavior

1 Introduction

Recently, Degasperis and Procesi [1] studied the following family of spatially periodic third order dispersive conservation laws,

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{txx} = \left( c_1 u_x^2 + c_2 u_x^2 + c_3 uu_{xx} \right)_x,$$

where $\alpha, c_0, c_1, c_2,$ and $c_3$ are real constants and indices denote partial derivatives. In [1] the authors found that there are only three equations that satisfy the asymptotic integrability condition within this family: the KdV equation, the Camassa-Holm equation and the Degasperis-Procesi equation.

With $\alpha = c_2 = c_3 = 0$ in Eq.(1), it becomes the well-known Korteweg-de Veris equation. The KdV equation is completely integrable and its solitary waves are solitons [2,3]. The Cauchy problem of the KdV equation has been the subject of a number of studies, and a satisfactory local or global existence theory is now in hand [4].

For $c_1 = -\frac{3}{2} c_3 / \alpha^2, c_2 = c_3 / 2$, Eq.(1) becomes the Camassa-Holm equation

$$u_t - u_{txx} + 3 uu_x = 2 u_x u_{xx} + uu_{xxx}, \quad t > 0, \quad x \in R$$

(2)

It has a bi-Hamiltonian structure and is completely integrable [5]. In [6] Dangping Ding and Lixin Tian researched solution of dissipative Camassa-Holm equation on total space. Tian, Song, Yin [7,8] considered the generalized Camassa-Holm equation and derived some new exact peakon and compacton.

With $c_1 = -2 c_3 / \alpha^2, c_2 = c_3$ in Eq.(1), we find the Degasperis-Procesi equation of the form

$$u_t - u_{txx} + 4 uu_x = 3 u_x u_{xx} + uu_{xxx}, \quad t > 0, \quad x \in R$$

(3)

Degasperis, Holm and Hone [9] proved the integrability of Eq.(3) by constructing a Lax pair. They also showed that Eq.(3) has bi-Hamiltonian structure and an infinite sequence of conserved quantities, and admits exact peakon solutions which are analogous to the Camassa-Holm equation. After the Degasperis-Procesi

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equation (3) was derived, many papers were devoted to its study. For example, Yin proved local well-posedness to Eq.(3) with initial data \( u_0 \in H^s(R) \), \((s > \frac{3}{2})\) [10] and derived the precise blow-up scenario and a blow-up result. The global existence of strong solutions and global weak solutions to Eq.(3) was also investigated in [11-14]. Recently, Lenells [15] classified all weak traveling wave solutions. Matsuno [16] studied multi-soliton solutions and their peakon limit. Coclite and Karlsen [17] proved there exists a unique and a blow-up result. The global existence of strong solutions and global weak solutions to Eq.(3) was also

\[ u_t - u_{xxt} + (b + 1)u u_x = bu_x u_{xx} + u u_{xxx}, \quad t > 0, \quad x \in \mathbb{R} \]

(4)

when \( b = 2 \) Eq.(4) is Camassa-Holm and \( b = 3 \) Eq.(4) is Degasperis-Procesi equation.

With a strong dispersive term in Eq.(4), we get

\[ \begin{aligned}
  u_t - u_{xxt} + (b + 1)u u_x + \gamma (u - u_{xx})_x &= bu_x u_{xx} + u u_{xxx}, \quad t > 0, \quad x \in \mathbb{R} \\
  u(x, 0) &= u_0(x), \quad x \in \mathbb{R}
\end{aligned} \]

(5)

Theorem 1 For any constant \( b \), given \( u_0 \in H^s, s > \frac{3}{2} \), there exist a maximal \( T = T(u_0) > 0 \) and a unique solution \( u \) to Eq.(5), such that

\[ u = u(\cdot, u_0) \in C((0, T); H^s) \cap C^1([0, T]; H^{s-1}) \]

Moreover, the solution depends continuously on the initial data, i.e. the mapping \( u_0 \rightarrow u(\cdot, u_0) : H^s \rightarrow C((0, T); H^s) \cap C^1([0, T]; H^{s-1}) \) is continuous.

Theorem 2 Let \( b \geq 1 \), assume \( u_0 \in H^s(R) \cap L^1(R) \), \((s > \frac{3}{2})\) and \( m_0 = u_0 - u_{0,xx} \) does not change sign on \( R \), then Eq.(5) has a global strong solution \( u = u(\cdot, u_0) \in C((0, T); H^s) \cap C^1([0, T]; H^{s-1}) \).

In section 2 of this paper, we prove that even though the classical solution \( m \) of (6) will have compact support for all times \( t \), whenever its initial data \( m_0 \) has this property, a similar result does not apply to the classical solution \( u \) of Eq.(5). This situation parallels that encountered for the CH equation and DP equation [19,20]. In section 3, we study the behavior of solutions of the Cauchy problem for Eq.(6) as the dispersive parameter \( \gamma \) tends to zero.

2 Infinite propagation speed

In this section, our main results are the two following theorems.

Theorem 3 Let \( m_0 : \mathbb{R} \rightarrow \mathbb{R} \) be a smooth function with compact support. If the solution \( m(x, t) \) of function (6) has \([0, T]\) as maximal interval of existence in the future, then, at every moment \( t \in [0, T) \), the smooth function \( m(\bullet, t) \) has compact support.
Proof. We shall employ a device from [21]. Let us introduce the initial value problem

\[
\begin{aligned}
\begin{cases}
\frac{dq}{dt} = u(t, q) + \gamma, & t \in [0, T), \\
q(0) = x.
\end{cases}
\end{aligned}
\]  

(7)

Since \(u_0 = p * m_0\), where \(u(x, 0) = u_0(x)\) for \(x \in R\), \(u(t, \bullet)\) is smooth on \([0, T)\). Therefore, via standard qualitative theory for ordinary differential equations [22], the existence in \([0, T)\), uniqueness and smooth dependence of data \(x\) for the solution \(q = q(t; x)\) of (7) is ensured. By integration in \([0, t)\), with \(t < T\), we get

\[q(t; x) = x + \int_0^t u(s, q(s; x))ds + \gamma t\]

which yields

\[q_x(t; x) = 1 + \int_0^t u_x(s, q(s; x))q_x(s; x)ds, \quad t \in [0, T),\]  

(8)

Let \(\eta\) denotes the left of (8), we get

\[
\frac{d\eta}{dt} = u_x(t, q(t; x))\eta.
\]

Finally, since \(\eta(0) = 1\), we obtain

\[q_x(t, x) = \eta(t) = \exp \left(\int_0^t u_x(s, q(s; x))ds\right), \quad t \in [0, T).\]

By introducing the function \(\varphi(x, t) = \psi(t, x)\), we have obtained an element of \(C_1(R \times [0, T), R)\). Furthermore, since, due to the Sobolev imbeddings, \(u_x(t, R)\) is bounded in \(R\) for all \(t \in [0, T)\), we deduce that

\[0 < C_1(t) \leq q_x(t; x) \leq C_2(x) < +\infty, \quad x \in R, t \in [0, T),\]  

(9)

which allows us to conclude that \(q(R, t), t \in [0, T)\), are all diffeomorphisms of \(R\). Also, from (7) we get \(q_{xt} = u_xq_x\). Further, let us multiply the equation (6) by \(q^b_x\). Then, if we take \(q(t, x)\) and \(t\) as arguments of \(u, m\) instead of the usual \(x, t\), the result of multiplication reads as

\[
\begin{aligned}
0 &= (m_t + bu_xm + um_x + \gamma m_x)q^b_x \\
&= m_tq^b_x + bmq_xq^b_x + um_xq^b_x + \gamma m_xq^b_x \\
&= m_tq^b_x + bmq_xq^b_x - (q_t - \gamma)m_xq^b_x + \gamma m_xq^b_x \\
&= m_tq^b_x + bmq_xq^b_x - q_xt + q_t + m_xq^b_x \\
&= \frac{d}{dt}(mq^b_x) \\
\end{aligned}
\]

Via an integration,

\[m(t, q(t, x))q^b_x(x, t) = m(0, q(0, x))q^b_x(x, 0) = m_0(x).\]

Finally, due to (9) if the support of \(m_0\) is included in \([a, b]\) then the support of \(m(\bullet, t)\) will be included in \([q(t, a), q(t, b)]\). The proof is complete. \(\Box\)

**Theorem 4** Let \(u_0 : R \rightarrow R\) is a smooth function with compact support. If the solution \(u(x, t)\) with initial data \(u_0(x)\) of (5) exists on some time interval \([0, \varepsilon)\) for \(\varepsilon > 0\) and, at every instant \(t \in [0, \varepsilon)\), the function \(u(\bullet, t)\) has compact support, then \(u\) is identically zero.

**Proof.** According to the preceding theorem, \(u(\bullet, t)\) has compact support at every moment \(t \in [0, \varepsilon)\). We recall that, by the Paley-Wiener theorem [23], an entire (analytic) function \(g(\xi)\), where \(\xi = \eta + i\zeta\) and \(\eta, \zeta \in R\), is the Fourier transform of a smooth function \(f : R \rightarrow R\) with compact support in \([-\alpha, \alpha]\) (for \(\alpha > 0\), namely

\[g(\xi) = F_f(\xi) = \int_R f(q)e^{-i\xi q}dq.
\]

If and only if for every integer \(n \geq 0\) there exists \(c_n > 0\) such that

\[|g(\xi)| \leq \frac{c_n e^{\alpha|\xi|}}{(1 + |\xi|^\alpha)^n}, \quad \xi \in R.
\]
Since  
\[ F_{m(\bullet, t)}(\xi) = (1 + \xi^2)F_{u(\bullet, t)}(\xi), \]  
\( \xi \in R, t \in [0, \varepsilon) \),
it is obvious that the analyticity of \( F_{u(\bullet, t)} \), if assumed, will imply the analyticity of \( F_{m(\bullet, t)}(\xi) \). In such a case, the function \( F_{m(\bullet, t)} \) has value zero at \( i, -i \) for all \( t \in [0, \varepsilon) \), yielding
\[ \int_R e^\varepsilon m(x, t) dx = \int_R e^{-\varepsilon} m(x, t) dx = 0, \quad t \in [0, \varepsilon). \]

Since both \( u \) and \( m \) have compact support, we deduce that
\[ 0 = \frac{d}{dt} \int_R e^\varepsilon m(x, t) dx \]
\[ = \int_R e^\varepsilon m(x, t) dx - \int_R e^\varepsilon u m(x, t) dx - \int_R e^\varepsilon m u(x, t) dx \]
\[ = -b \int_R e^\varepsilon u m dx + \int_R e^\varepsilon m u dx - \int_R e^\varepsilon m u(x, t) dx \]
\[ = -b \int_R e^\varepsilon u m dx + \int_R e^\varepsilon m u dx - \int_R e^\varepsilon m u(x, t) dx \]
\[ = \int_R e^\varepsilon m u(x, t) dx \]
\[ = \int_R e^\varepsilon (b u^2 + \frac{3b}{2} u_x^2) dx \]
which implies that \( u(\bullet, t) \equiv 0 \) for all \( t \) when \( 0 < b \leq 3 \). The proof is complete. \( \blacksquare \)

**Remark 5** From the proof, it is clear that actually we show that either the strong solution itself or its first order derivative can not, or neither of them can decay faster than \( e^{-\varepsilon} \) at infinite uniformly in any small time interval \( [0, \varepsilon] \), although the nonzero initial datum \( u_0(x) \) belongs to \( C_0^\infty(R) \) (or \( u_0 \in C^\infty \) and decays faster than \( e^{-\varepsilon} \) at infinite).

In fact, if both the strong solution itself and its first derivative decay faster than \( e^{-\varepsilon} \) in some time interval, that is, \( |u|, |u_x| \sim o(e^{-\varepsilon}) \), as \( x \to \infty \), then the above computations in the proof still hold. In particular (10) is true, then consequently we obtain a contradiction that the solution \( u(x, t) \) is zero with nonzero initial datum \( u_0 \).

How about the behavior of the solution at \( -\infty \)? Can it (itself and its first order derivative) decays faster than \( e^\varepsilon \) at negative infinite? The answer is NO.

In fact, let
\[ F(t) = \int_R e^{-\varepsilon} m(x, t) dx, \]
with \( m(x, t) = (1 - \partial^2_x) u(x, t) \). For \( u_0 \in C_0^\infty(R) \), if the solution itself and its first derivative decays faster than \( e^\varepsilon \) at negative infinity in some time interval, say \( [0, t_1] \). By the similar computation, we obtain
\[ F(t) = 0 \]
\[ \frac{dF(t)}{dt} = -\int_R e^{-\varepsilon} (\frac{b}{2} u_x^2 + \frac{3b}{2} u^2) dx \leq 0, \]
for \( 0 < b \leq 3, 0 \leq t \leq t_1 \). Hence we get a contradiction that the solution is zero with nonzero initial datum.

**Remark 6** How about the behavior as time goes on? We would like to make a conjecture here: the solution \( u(x, t) \) can not have compact \( x \)-support any longer in its lifespan even with compact supported initial datum. Although we can not prove it at present, from the above proof, it is clear that the collection of times at which the solution \( u(x, t) \) has compact \( x \)-support only could be a finite set.

## 3 Weak and Strong limit as \( \gamma \to 0 \)

We now turn to the study of the behavior of solutions of the Cauchy problem for (5) as the dispersive parameter \( \gamma \to 0 \). Consider the initial-value problem (5) and the analogous problem for the b-family equation with the same initial condition, we get the weak and strong limit of the solution as \( \gamma \to 0 \).

**Theorem 7** Under the assumption \( m_0 \geq 0, \gamma > 0 \), if \( u = u^{\gamma_1}, v = u^{\gamma_2} \) are the solutions of the problem (5) in \( C([0, T), H^s) (s \geq 3) \), with \( \gamma = \gamma_1 \) and \( \gamma = \gamma_2 \) respectively. Then \( |u - v|_2 \) converges to zero as \( \gamma_1, \gamma_2 \to 0 \).

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To prove Theorem 3.1, we need the following two lemmas.

**Lemma 8** Let \( u_0 \in H^s (R) \), \( s \geq 3 \), such that \( m_0 + \gamma \geq 0 \), where \( m_0 = (1 - \partial_x^2) u_0 \) and \( u (t, x) \) the solution of equation (5). Then there is a constant \( K (> 0) \) such that

\[
|\partial_x u|_\infty \leq K.
\]

**Remark:** If \( 0 < \gamma \leq M \), then the constant \( K \) depends only on \( M \). In the case, \( |\partial_x u|_\infty \) is uniformly bounded independent of \( \gamma \).

**Proof.** Using \( p (x) = \frac{1}{2} e^{-|x|} \), \( x \in R \) one has

\[
u = p \ast m = \frac{1}{2} e^{-x} \int_{-\infty}^{x} e^y m (t, y) \, dy + \frac{1}{2} e^x \int_{x}^{\infty} e^{-y} m (t, y) \, dy
\]

Taking the derivative for \( u (t, x) \) with respect to \( x \) yields

\[
\partial_x u (t, x) = -\frac{1}{2} e^{-x} \int_{-\infty}^{x} e^y m (t, y) \, dy + \frac{1}{2} e^x \int_{x}^{\infty} e^{-y} m (t, y) \, dy
\]

On the other hand, we have

\[
\partial_x u (t, x) = -e^{-x} \int_{-\infty}^{x} e^y m (t, y) \, dy + \frac{1}{2} e^x \int_{x}^{\infty} e^{-y} m (t, y) \, dy + \frac{1}{2} e^x \int_{x}^{\infty} e^{-y} m (t, y) \, dy
\]

As \( u (t, x) \) is uniformly bounded in \( H^1 \) and independent of \( \gamma \) according to the conservation laws of Eq.(5), there exists a constant \( K > 0 \) such that

\[
|\partial_x u|_\infty \leq K
\]

which is the advertised result. "

**Lemma 9** (Kato-Ponce). Let \( \Lambda^s = (1 - \partial_x^2)^{\frac{s}{2}} \). If \( s \geq 0, 1 < p < \infty \); \( f, g \in S (R^n) \), then there exists such a constant \( C = C (s, n, p) \) that

\[
||\Lambda^s f||_p \leq C \left( |\nabla f|_{p_1} |\Lambda^{s-1} g|_{p_2} + |\Lambda^s f|_{p_3} |g|_{p_4} \right),
\]

\[
|\Lambda^s (fg)|_p \leq C \left( |f|_{p_1} |\Lambda^s g|_{p_2} + |\Lambda^s f|_{p_3} |g|_{p_4} \right),
\]

where \( 1 < p_2, p_3 < \infty \), and \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} \).

**Proof of Theorem 3.1.** Firstly, we shall prove that the solution \( u \) of (5) is bounded in \( H^s (s \geq 3) \). In fact, we can rewrite Eq.(5) as

\[
u_t + \nu_{xx} + \gamma \nu_x + \partial_x \left( 1 - \partial_x^2 \right)^{-1} \left( \frac{b}{2} \nu^2 + \frac{3 - b}{2} \nu_x^2 \right) = 0,
\]

We can rewrite the above equation as:

\[
u_t + \gamma \nu_x = -\frac{1}{2} \left( \nu_x^2 \right)_x + f (u)
\]

where \( f (u) = -\partial_x \left( 1 - \partial_x^2 \right)^{-1} \left( \frac{b}{2} \nu^2 + \frac{3 - b}{2} \nu_x^2 \right) \).

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We denote \( \langle u, v \rangle = \int_{\Omega} u(x)v(x) \, dx \) taking the scalar product of Eq.(3.1) with \( u \) we get

\[
\frac{d}{dt} \|u(t)\|_2^2 = \langle u, u_t \rangle_s = \langle u, (u^2)_x \rangle_s + 2 \langle u, f(u) \rangle_s.
\]  

From Lemma 3.2, one can see that

\[
\left| \langle u, (u^2)_x \rangle_s \right| = \left| 2 \langle u, uu_x \rangle_s - 2 \langle \Lambda^s (uu_x), \Lambda^s u \rangle ight|
\leq 2 \left| \langle u \Lambda^s u_x, \Lambda^s u \rangle \right| + \left| \langle \Lambda^s u, uu_x \rangle \right|
\leq c \| \partial_x u \|_\infty \| u \|_2^2 + c \| [\Lambda^s u, uu_x]_2 \|_2 
\leq c \| \partial_x u \|_\infty \| u \|_2^2 + c \| \partial_x u \|_\infty \| \Lambda^s u \|_2^2 
\leq c_0 \| u \|_s^2.
\]  

where in the last estimate above we used the result of the uniform boundedness of \( \partial_x u \) in Lemma 3.1 and the constant \( c_0 \) depends only on the bound of \( \gamma \), but not \( \gamma \). On the other hand, from the Cauchy-Schwarz inequality, we have

\[
\left| \langle u, f(u) \rangle_s \right| \leq \| u \|_s \| f(u) \|_s.
\]

According to Lemma 3.1, we estimate \( \| f(u) \|_s \) as follows:

\[
\| f(u) \|_s \leq \left( \frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 \right)^{s-1}
\leq c_1 \left( \| u \|_s \| u_x \|_s \right)^{s-1}
\leq c_1 \left( \| u \|_s + \| \partial_x u \|_s \right)^{s-1}
\leq c_0 \| u \|_s.
\]

where \( c_0, c_1 \) are constants which depend only on \( M \) with \( |\gamma| \leq M \). So combining (13) with (14), one can deduce from (12) that

\[
\frac{d}{dt} \|u(t)\|_2^2 \leq c_0 \| u \|_s^2.
\]

And by using Gronwall’s inequality, we get that the solution \( u \) of (5) is bounded in \( H^s (s \geq 3) \), for any \( t \in [0, T) \), where \( c_0 \) is independent of \( \gamma \), but \( M \) with all \( |\gamma| \leq M \).

Next, we show that the sequence of solutions of (5) is a Cauchy sequence in \( L^2 (R) \). Let \( u = u^1 \) and \( v = u^2 \) be the solutions of problem (5) with \( \gamma = \gamma_1 \) and \( \gamma = \gamma_2 \) respectively. Then the function \( w = u - v \) satisfies

\[
w_t + wu_x + w_x v = -\gamma_1 w_x + (\gamma_2 - \gamma_1) v_x + f(u) - f(v)
\]

(15)

Considering

\[
f(u) - f(v) = -\partial_x \left( 1 - \partial_x^2 \right)^{-1} \left( \frac{b}{2} w + \frac{3-b}{2} w_x (u_x + v_x) \right),
\]

one can see that

\[
\| f(u) - f(v) \|_s \leq \left( \frac{b}{2} \| w \|_s + \frac{3-b}{2} \| w_x \|_s \right)^{s-1}
\leq \left( \frac{|b|}{2} \| w \|_s + \frac{3-b}{2} \| w_x \|_s \right)^{s-1}
\]

(16)

Multiplying (15) with \( w \) and integrating on \( R \) with respect to \( x \), and integration by parts, we obtain

\[
\frac{d}{dt} \|w\|_{L^2}^2 = -\int_R w^2 (2u_x + v_x) \, dx + (\gamma_2 - \gamma_1) \int_R w (2u_x v_x + f(u) - f(v)) \, dx
\leq (2 \|u_x\|_\infty + |v|_\infty) \|w\|_{L^2}^2 + (\gamma_2 - \gamma_1) \|w\|_{L^2} (\|w\|_{L^2} + \|w_x\|_{L^2} + \|f(u) - f(v)\|_{L^2})
\leq (2 \|u\|_s + \|v\|_s) \|w\|_{L^2}^2 + (\gamma_2 - \gamma_1) \|w\|_s (\|w\|_s + \|w_x\|_s) \|w\|_{L^2}^2
\leq \left( \frac{|b| + 3-b}{2} \| w \|_s + \frac{|b| + 3-b}{2} \| w_x \|_s \right) \|w\|_{L^2}^2 + (\gamma_2 - \gamma_1) \|w\|_s (\|w\|_s + \|w_x\|_s) \|w\|_{L^2}^2,
\]

by using (16). Integrating this inequality over \([0, T]\) for any \( T > 0 \) and applying Gronwall’s inequality yield

\[
\|w\|_{L^2}^2 \leq CT \|\gamma_1 - \gamma_2\| K (\|u(t)\|_s, \|v(t)\|_s), \quad \forall t \in [0, T].
\]
Since $K (\|u (t)\|_s, \|v (t)\|_s)$ is uniformly bounded and independent of $\gamma$, $|w|_{L^2}$ converges to zero as $\gamma_1, \gamma_2 \to 0$ and $u_\gamma$ is a Cauchy sequence in $L^2$ as $\gamma \to 0$, uniformly with respect to $t \in [0, T]$. This completes the proof of Theorem 3.1.  

By Theorem 3.1 and the a priori estimates established for solutions of equation (3.1) independent of $\gamma$, we can obtain that the Cauchy sequence of solutions of the b-family equation with dispersive term locally strongly converges to the solution of the b-family equation as $\gamma$ tends to zero.

**Corollary 10** Under the assumption $m_0 \geq 0$, and $\gamma > 0$, let $u_\gamma$ be the solution of (5) in $H^s (s \geq 3)$. Then $u_\gamma$ converges to the solution of (4) in $H^s (s \geq 3)$ as $\gamma \to 0$.

**Proof.** First of all, according to Theorem 3.1, and letting $u = \lim_{\gamma \to 0} u_\gamma$ in $L^2$, where $u_\gamma$ is a Cauchy sequence in $L^2$ as $\gamma \to 0$ uniformly with respect to $t \in [0, T]$, we will show that $u$ is the solution of (5). Indeed, since $u_\gamma$ is the solution of (5) in $H^s$, $s \geq 3$, for $t \in [0, T]$, we have

$$u_\gamma (t) = S_\gamma (t) u_0 + \int_0^t S_\gamma (t - \tau) \left( f (u) - \frac{1}{2} (u^2) \right) d\tau, \quad (17)$$

where $S_\gamma (t) v = \frac{1}{2\pi} \int_R e^{i(\xi x - \gamma \xi t)} \hat{\nu} (\xi) d\xi$ and $S_\gamma (t)$ satisfies the relation $\|S_\gamma (t) v\|_s = \|v\|_s$, $s \geq 0, v \in H^s$.

Since $\|S_\gamma (t) u_0\| \leq \frac{1}{2\pi} \int_R |\hat{u} (\xi)| |d\xi| \leq \frac{1}{\pi} \|u_0\|_s, s \geq \frac{3}{2}$, and from Lebesgue’s dominated convergence theorem we can get that

$S_\gamma (t) u_0 \to S (t) u_0, \quad 0 < t < T$ as $\gamma \to 0$, where $S (t) u_0 = \frac{1}{2\pi} \int_R e^{i\xi x} \hat{u_0} (\xi) d\xi$.

We can also get $S_\gamma (t) (u_0 - u_{0xx}) \to S (t) (u_0 - u_{0xx})$.

On the other hand, for $s \geq 3$, $t \in [0, T]$, and $\tau \in [0, t]$, it follows from (14) that

$$|S (t - \tau) f (u)| \leq \|S (t - \tau) f (u)\|_s \leq \frac{1}{\pi} \|f (u)\|_s \leq 0 \|u\|_s .$$

So the right-hand side of (17) is bounded uniformly independent of $\gamma$. Therefore, by using Lemma 3.1, and in view of the Lebesgue dominated convergence theorem, passing to the limit as $\gamma \to 0$ in (17), we get that

$$u (t) = S (t) u_0 + \int_0^t S (t - \tau) \left( f (u) - \frac{1}{2} (u^2) \right) d\tau, \quad (18)$$

where $f (u) = \overline{f_0} (u)$. Hence $u \in L^\infty (0, T; L^2)$ satisfies Eq.(5), the local existence for the b-family equation implies that equation (5) has a unique solution in $C ([0, T); H^s)$, $s \geq 3$. This proves that $u$ is the strong solution of (4). This completes the proof of Corollary 3.1.  

**References**


**IINS homepage:** http://www.nonlinearscience.org.uk/


[18] Guilong Gui, Yue Liu and Lixin Tian: Global Existence and Blow-Up Phenomena for the Peakon b-Family of Equations. *Indiana.edu Preprints*


