Optimal Control of the Viscous KdV-Burgers Equation Using an Equivalent Index Method

Anna Gao *, Chunyu Shen, Xinghua Fan
Nonlinear Scientific Research Center, Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu 212013, China

(Received 15 October 2008, accepted 7 January 2009)

Abstract: This paper studies the problem of optimal control of the viscous KdV-Burgers’ equation. We develop a technique to utilize the Cole-Hopf transformation to solve an optimal control problem for the viscous KdV-Burgers’ equation. While the viscous KdV-Burgers’ equation is transformed into a simpler linear equation, the performance index is transformed to a complicated rational expression. We show that a simpler performance index that retains the behavior of the original performance index near optimal values of the functional can be used.

Keywords: Viscous KdV-Burgers’ Equation; Optimal Control; Cole-Hopf transformation; Euler-Lagrange; Equivalent indices

1 Introduction

Many engineering and physical applications use the Navier-Stokes equation for modeling fluid flow. The complete system of equations is difficult to solve explicitly in most cases. Many approximation schemes have been suggested to solve the Navier-Stokes equation. Burgers equation is the simplest approximation that captures the nonlinear and non-planar aspects of the Navier-Stokes equation and was originally proposed as a model for turbulence by J.M. Burgers. Further investigation showed that the Burgers model was not useful to describe some central feature of turbulence. However, Burgers equation retains some key features of the Navier-Stokes equation and is an excellent model for the viscous structure of weak shock waves. Kriess and Lorentz ([1]) have given an existence theorem for Burgers’ equation. The viscous Burgers’ equation can be solved exactly using the Cole-Hopf transformation, which is a Backlund transformation between Burgers’ equation and linear heat equation. Ly, Mease and Titi ([2]) have obtained stability results for the viscous Burgers’ equation with distributed and boundary control for a variety of boundaries. Ram Vedantham ([3]) has studied the optimal control problem of the viscous Burgers’ equation using an equivalent index method. Compared with KdV equation, viscous KdV-Burgers Equation not only has dissipative item, but also has dispersive item. It can be used for explaining many physical phenomena, such as laser wave, water wave and other phenomena. Zhu, Tian, and Zhao studied the optimal control of KdV-Burgers equation and sufficient nonlinear KdV-Burgers equation ([4][5]). In this paper, we will investigate the optimal control of the viscous KdV-Burgers equation using an equivalent index method.

On the other hand, the optimal control which is an important component of modern control theories has a wider application in modern engineering. The optimal control theories with PDE are much more difficult to deal with. Especially, there are no united theories and methods for nonlinear control theories with PDE. Two methods are introduced to study the control theories with PDE: one is using low model method, and then changing into ODE model ([6]); the other is using quasi-optimal control method ([7]). No matter which one to choose, it's necessary to prove the existence of optimal solution according to the basic theories ([8]). In past decades, not only did the optimal control theories have many successful applications,
but it has also overrun traditional boundary of auto control. It had wider application in many fields and achieved remarkable effect. Due to above reasons, the study of the optimal control is carried out even further. The optimal control of distributed parameter system now has become much more active in academic field. Especially, the optimal control of nonlinear solitary wave equation which lies in the front of the intersection of math, engineering and computer science. Recently, S. Volkwein has studied the optimal control of Burgers equation using lagrangian-SQP method ([9]). S.Volkweins discussed the instantaneous control of Burgers equation ([10]). The distributed control problem for Burgers equation was studied in ([11]). Bernt Oksendal proved a sufficient maximum principle for the optimal control systems described by a quasi-linear stochastic heat equation ([12]). [13] considered the problem of boundary optimal control of a wave equation with boundary dissipation in the way of a time-domain decomposition of the corresponding optimality system. Omar Ghatts and Jai-Hyeong Bark studied the optimal control of two and three dimensional in compressible Navier-Stokes flows ([14]). Guangcao Ji and Clyde Martin studied the optimal boundary control of the heat equation with target function at terminal time ([15]).

In this paper, we consider only distributed control and, specifically, a single actuator at $x = x_a$. The method can be easily expanded to multiple actuators. The paper is organized as follows. First, we convert the nonlinear KdV-Burgers equation to a linear diffusion type equation using the Cole-Hopf transformation. However, the transformation converts the performance index $J [\phi[v], v]$ into a very complicated expression given by (2.13). Traditional methods are inapplicable with this performance index. We can overcome this by introducing a simpler performance index $C [\phi[v], v]$, given by (3.17). Using calculus of variations, we show in Section 3 that a minimum of performance index (3.17) implies the minimum of performance index (2.13). In other words, the alternate performance index preserves the integrity of the optimal control problem. Using this justification, we re-formulate the optimal control problem to use the simpler and newly defined performance index. In Section 4, we show the existence of the optimal control using the maximum principle and derive an exact expression for the optimal control. In Section 5, conclusions are obtained.

2 Statement of the optimal control problem

Consider the following boundary and initial value problem.

$$y_t - vy_{xx} + y_{xx} + y_{xxx} = f(t),$$

(2.1)

$$y(0,t) = 0, \ y(1,t) = 0, \ y_x(0,t) = 0, \ y_x(1,t) = 0$$

(2.2)

$$y(x,0) = y_0(x)$$

(2.3)

where $(x,t) \in [0,1] \times [0,T]$ and $v$ is the viscous parameter. The source function $f$ is an arbitrary function of $x$ and $t$. We wish to guide this system to produce a desired response $y_d(x)$ at a desired time $T$. There are primarily two methods to control the natural evolution of $y(x,t)$ and steer it towards the desired output $y_d(x)$. The first method introduces control through the boundary while the second method introduces time dependent actuators at finite number of special points in the spatial domain. We will adopt the second method and introduce an actuator $v(t)$ at $x = x_a$.

We wish to obtain an expression for the control $v(t)$ that minimizes the performance index

$$J[y[v], v] = \frac{1}{2} \int_0^1 \int_0^T (y(x,t; v(t)) - y_d(x))^2 \, dt \, dx + \frac{\varepsilon}{2} \int_0^T v^2(t) \, dt,$$

(2.4)

where $y_d(x)$ is a specified target function at the terminal time $T$ and $0 < \varepsilon \ll 1$ is a known parameter. The first integral in (2.4) is the cumulative penalty of mismatch of the state variable $y(x,t)$ and the desired target function $y_d(x)$. The second integral is the contribution from introducing a control function during the evolution of the state function $\phi(x,t)$.

Remark:1. We assume that $f \equiv 0$. Through a change of variables, $f(t)$ can be incorporated into the state equation.

Remark:2. To simplify the performance index further, we make a change of variable given by $\tilde{y}(x,t) = y(x,t) - y_d(x)$.

Incorporating all the assumptions and after rewriting $\tilde{y}(x,t)$ as $y(x,t)$, the optimal control problem can be stated as follows:

IINS homepage: http://www.nonlinearscience.org.uk/
**Problem P.** Find the expression for \( v(t) \), \( 0 \leq t \leq T \), such that the solution \( y(x,t) \) of
\[
y_t - vy_{xx} + yy_x + y_{xxx} + (ya)_x + y_d(ya)_x - v(ya)_{xx} + (ya)_{xxx} = v(t) \delta(x-x_a),
\]
minimizes \( J[y(v),v] \) given by
\[
J[y(v),v] = \frac{1}{2} \int_0^1 \int_0^T y^2(x,t;v(t))dt dx + \frac{\varepsilon}{2} \int_0^T v^2(t)dt.
\]

2.1 Cole-hopf transformation

The nonlinear term in (2.5) prevents the usage of the adjoint method to find the optimal control. This can be overcome by using the nonlinear Cole-Hopf transformation to rewrite (2.5) into a linear diffusion type equation with source term.

Let
\[
y(x,t) = -2v \frac{\phi_x}{\phi} = -2v (\ln(\phi(x,t)))_x.
\]

Substituting (2.9) in (2.5) and integrating the resulting equation with respect to \( x \). Then, we can get
\[
\phi_t = \nu \phi_{xx} - \phi_{xxx} + 3 \frac{\phi_x^3}{\phi} - 2 \frac{\phi_x^3}{\phi^2} - y_d \phi_x + g(x) \phi + m(x) \phi(x) \phi,
\]
(2.10)
\[
\phi_x(0,t) = 0, \phi_x(1,t) = 0, \phi_{xx}(0,t) = 0, \phi_{xx}(1,t) = 0,
\]
(2.11)
\[
\phi(x,0) = \phi_0(x),
\]
(2.12)

where \( g(x) = \frac{1}{4v} y_d - \frac{1}{2} (ya)_x + \frac{1}{2v} (ya)_{xx}, m(x,t) = -\frac{1}{2v} H(x-x_a) v(t) \) and \( H(\cdot) \) denotes the Heaviside function. Using the Cole-Hopf transformation to the performance index, we have
\[
J[\phi[v],v] = \frac{1}{2} \int_0^1 \int_0^T (2 \nu (\ln(\phi(x,t)))_x^2) dt dx + \frac{\varepsilon}{2} \int_0^T v^2(t)dt.
\]
(2.13)

The transformed performance index, particularly the first integral, is not useful in obtaining the expression for optimal control \( v(t) \). In the next section, we introduce the idea of equivalent performance index.

3 Equivalent Performance Index

In this section, we derive a performance index that is simpler in form and that can be used to compute the optimal cost to control the KdV-Burgers. The core of this idea is rooted in functional analysis under the guise of the equivalent metrics.

3.1 A class of equivalent functionals

**Definition 1** We define a function \( \phi^* \) to be \( P \)-optimal if \( P[\phi] \) attains an extremal value at \( \phi^* \). In this paper, we consider only the case when \( P[\phi] \) attains a minimum value at \( \phi^* \). That is
\[
P[\phi^*] = \min_{\phi} P[\phi].
\]
(3.14)

**Lemma 1** Let \( H[\phi] \) be a functional of the following form
\[
H[\phi] = \int_0^1 \int_0^T f(\phi,\phi_x) dt dx.
\]
(3.15)

Then, an extremal of \( H[\phi] \) satisfies the Euler-Lagrange equation given by
\[
f - \phi_x f_{\phi_x} = 0.
\]
(3.16)
An eloquent discussion of this topic can be found in [16]. The following theorem allows the replacement of a complicated performance index with a simpler performance index that still indicates optimal values if and when they exist.

**Theorem 1** Let \( J [\phi [v], v] \) be given by (2.8) and
\[
C [\phi [v], v] = \int_0^1 \int_0^T \frac{1}{2} \phi^2 (x, t; v (t)) dx dt + \frac{\varepsilon}{2} \int_0^T v^2 (t) dt.
\]

Let \( v^\ast \) be a fixed control function. Then, \( \phi^\ast \) is C-optimal implies \( \phi^\ast \) is J-optimal also. That is,
\[
C [\phi^\ast, v^\ast] = \min_{\phi} C [\phi, v^\ast] \Rightarrow J [\phi^\ast, v^\ast] = \min_{\phi} J [\phi, v^\ast].
\]

**Proof.** First, we assume that
\[
C [\phi [v], v] = \int_0^1 \int_0^T f (\phi, \phi_x) dx dt + \frac{\varepsilon}{2} \int_0^T v^2 (t) dt.
\]
We intend to show that the choice of
\[
f (\phi, \phi_x) = \frac{1}{2} \phi^2
\]
in (3.19) leads to the inequality
\[
0 \leq \frac{\delta J}{\delta \phi} \leq \frac{\delta C}{\delta \phi},
\]
and hence C-optimal will imply J-optimal. The proof of the theorem is complete if we show that choice of \( f \) given by (3.20) satisfies (3.21). Using Lemma 1, (3.21) can be proved if we can find an \( f (\phi, \phi_x) \) in (3.19) such that
\[
4 \varepsilon \phi^2 \frac{\phi_x}{\phi^2} \leq f - \phi_x f \phi_x.
\]

Both sides of the inequality results from computing the first functional approximation of the performance integral. By setting each to 0 individually, we obtain the Euler-Lagrange equations for the functionals \( J \) and \( C \). The solutions of the Euler-Lagrange equations provide the extremal solution of the functionals. Thus, (3.22) implies
\[
\phi_x f \phi_x - f + 4 \varepsilon \phi^2 \frac{\phi_x}{\phi^2} \leq 0 \Leftrightarrow \frac{1}{\phi_x} f \phi_x - \frac{1}{\phi_x} f + 4 \varepsilon \phi_x \frac{\phi_x}{\phi^2} \leq 0 \Leftrightarrow \left( \frac{1}{\phi_x} f + 4 \varepsilon \phi_x \frac{\phi_x}{\phi^2} \right) \phi_x \leq 0 \Leftrightarrow \frac{1}{\phi_x} f + 4 \varepsilon \phi_x \frac{\phi_x}{\phi^2} = G (\phi, \phi_x),
\]
where \( G (\phi, \phi_x) \) is a decreasing function of \( \phi_x \). One choice of such a function is
\[
G (\phi, \phi_x) = \frac{4 \varepsilon \phi_x}{\phi^2} + \frac{1}{2} \phi^2
\]
which implies \( f (\phi, \phi_x) = \frac{1}{2} \phi^2 \). This proves the theorem. ■

### 3.2 Re-formulation of the optimal control problem

In this subsection, we use the result from Theorem 1 to re-state an equivalent optimal control problem. The solution satisfies the (2.10)-(2.12) while minimizing the performance index functional (2.13).

**Problem Q:** Find an optimal control \( v (t) \) that minimizes the performance index
\[
C [\phi [v], v] = \int_0^1 \int_0^T \frac{1}{2} \phi^2 (x, t; v (t)) dx dt + \frac{\varepsilon}{2} \int_0^T v^2 (t) dt,
\]
where \( \phi (x, t) \) satisfies
\[
\phi_t = u \phi_{xx} - \phi_{xxx} + n (\phi, \phi_x) - y_d \phi_x + g (x) \phi + m (x, t) \phi
\]
\[
\phi_x (0, t) = 0, \phi_x (1, t) = 0, \phi (0, t) = 0, \phi (1, t) = 0
\]
where \( g (x) = \frac{1}{4 \tau y_d^2} - \frac{1}{2} (y_d)_{xx} + \frac{1}{2 \tau} (y_d)_{xx} \), \( m (x, t) = -\frac{1}{2 \tau} H (x - x_a) v (t), n (\phi, \phi_x) = 3 \phi_x \phi_{xx} \phi - 2 \phi_x \phi^2 \).

---

A. Gao, C. Shen, X. Fan: Optimal Control of the Viscous KdV-Burgers Equation ··· 315

IJNS homepage: http://www.nonlinearscience.org.uk/
4 Optimal control of the viscous KdV-Burgers equation

In this section, we will derive the expression for an optimal control for the redefined problem (3.24)-(3.27) by using the adjoint problem. In the following subsection, we prove the existence of a minimum value of the performance index $C[\phi,v]$ defined by (3.24). First, we define a Hamiltonian $H[x,t;\phi,\psi,v]$ that corresponds to the performance index given by

$$H[x,t;\phi^*,\psi,v] = -\varepsilon \frac{1}{2} \int_0^T v^2 dt + \int_0^1 \int_0^T H(x-x_a)\phi^*\psi v dt dx.$$  \hfill (4.28)

We justify this expression for the Hamiltonian by the following argument: if the performance index can be interpreted as lagrangian, then the Hamiltonian can be obtained by applying the Legendre transform on the Lagrangian.

4.1 Existence of the optimal control

**Theorem 2** Let $v$ and $v^*$ be elements in the space of admissible controls denoted by $F_{ad}$ with corresponding dependent state variables $\phi$ and $\phi^*$ respectively that satisfy Equations (3.25)-(3.27). Also, let $\psi$ and $\psi^*$ be the corresponding adjoint state variables that satisfy Equations (4.31)-(4.33). Assume that $v^*$ satisfies

$$H[x,t;\phi^*,\psi,v^*] = \max_{v \in F_{ad}} H[x,t;\phi,\psi,v],$$  \hfill (4.29)

where $H[x,t;\psi,v]$ is defined by (4.28). Then

$$C[\phi^*[v^*],v^*] \leq C[\phi[v],v] \quad \forall v \in F_{ad}.$$  \hfill (4.30)

**Proof.** Let $\psi(x,t)$ be the adjoint of $\phi(x,t)$ that satisfies

$$-\psi_t = v\psi_{xx} - \psi_{xxx} + n^*(\psi,\psi_x) - y_d\psi_x + g(x)\psi + m(x,t)\psi - \phi$$  \hfill (4.31)

$$\psi_x(0,t) = 0, \psi_x(1,t) = 0, \psi_{xx}(0,t) = 0, \psi_{xx}(1,t) = 0,$$  \hfill (4.32)

$$\psi(x,T) = 0.$$  \hfill (4.33)

First, we find the implication of the adjoint problem and use it later in the proof. Let $\Delta \phi = \phi(x,t) - \phi^*(x,t)$, then $\Delta \phi$ satisfies the equation,

$$\Delta \phi_t = v\Delta \phi_{xx} - \Delta \phi_{xxx} + n(\phi,\phi_x;\phi^*,\phi^*_x) - y_d\Delta \phi_x + g(x)\Delta \phi + \Delta (m(x,t)\phi)$$  \hfill (4.34)

$$\Delta \phi_x(0,t) = 0, \Delta \phi_x(1,t) = 0, \Delta \phi_{xx}(0,t) = 0, \Delta \phi_{xx}(1,t) = 0,$$  \hfill (4.35)

$$\Delta \phi(x,0) = 0.$$  \hfill (4.36)

Multiplying (4.34) by $\psi$ and multiplying (4.31) by $\Delta \phi$ and subtracting one from the other, we can get

$$\psi \Delta \phi_t + \Delta \phi \psi_t = v[\psi \Delta \phi_{xx} - \Delta \phi \psi_{xx}] - [\psi \Delta \phi_{xxx} - \Delta \phi \psi_{xxx}] + [\psi n(\phi,\phi_x;\phi^*,\phi^*_x) - \Delta \phi n^*(\psi,\psi_x)]$$

$$- [y_d \Delta \phi_x - y_d \Delta \phi \psi_x] + \psi \Delta (m\phi) - \Delta \phi (m\psi) + \phi \Delta \phi$$  \hfill (4.37)

Integrating both sides with respect $x$ and $t$ and applying appropriate boundary conditions, we obtain

$$\int_0^1 (\psi \Delta \phi) T^0 dx = \int_0^1 \int_0^T (\psi \Delta (m\phi) - \Delta \phi (m\psi)) dt dx + \int_0^1 \int_0^T \phi \Delta \phi dt dx.$$  \hfill (4.38)

Applying the terminal condition of the adjoint problem, we obtain

$$\int_0^1 \int_0^T \phi(x,t) \Delta \phi(x,t) dt dx = -\int_0^1 \int_0^T (\psi \Delta (m\phi) - \Delta \phi (m\psi)) dt dx.$$  \hfill (4.39)
Equation (4.38) summarizes mathematically the effect of the adjoint problem. We use this result in the proof of the theorem. Observe that, from (3.24),

\[
C[\phi[v], v] = \frac{1}{2} \int_0^1 \int_0^T \phi^2(x, t) dt dx + \frac{\varepsilon}{2} \int_0^T v^2(t) dt.
\]  

(4.39)

We define \( \Delta C = C[\phi[v], v] - C[\phi^*[v^*], v^*] \). The convexity property of a quadratic function \( f(\phi) = \frac{1}{2}\phi^2 \) provides the inequality given by

\[
f(\phi) - f(\phi^*) \geq f_\phi(\phi - \phi^*) = \phi \Delta \phi.
\]  

(4.40)

Using (4.40), we have

\[
\Delta C \geq \int_0^1 \int_0^T H(x - x_a) \psi \phi^* v^* dt dx - \frac{\varepsilon}{2} \int_0^T (v^*)^2 dt - \int_0^1 \int_0^T H(x - x_a) \psi \phi^* v dt dx - \frac{\varepsilon}{2} \int_0^T v^2 dt.
\]  

(4.41)

Using (4.29), we can get \( C[\phi[v], v] \geq C[\phi[v^*], v^*], \forall v \in F_{\text{ad}} \). This proves the existence of a minimum value of performance index and also helps find the expression for the optimal control \( v^*(t) \). Using the same observations made earlier, we will now derive an expression for the optimal control. ■

4.2 An expression for the optimal control

In this subsection, we use elementary calculus to derive an expression for \( v^*(t) \). From (4.29), we know that the Hamiltonian achieves its maximum at \( v = v^* \). This implies that the variational derivative of the Hamiltonian with respect to \( v \) should equal zero. That is,

\[
\frac{\delta H}{\delta v} = 0 \Rightarrow -\varepsilon v^*(t) + \int_0^1 H(x - x_a) \phi \psi dx = 0 \Rightarrow v^*(t) = \frac{1}{\varepsilon} \int_0^1 H(x - x_a) \phi \psi dx.
\]  

(4.42)

5 Conclusion

In this paper, we have used an equivalent performance index which indicates extremal values of the given performance index. This has enabled us to use the adjoint problem to derive a maximum principle for a nonlinear optimal control problem that has been solved, hitherto, by using either numerical schemes or linearization. The idea of equivalent performance index can be extended even to linear optimal control problems with complicated performance index. The study of the optimal control of the viscous KdV-Burgers equation provides a theoretical basis for further studying and application in engineering field.

Acknowledgements

The author would like to express great gratitude to professor Linxin Tian of Nonlinear Scientific Research center of Jiangsu University for useful discussions and valuable suggestions. Research is supported by the Post-doctoral Foundation of China(No. 20080441071), the Post-doctoral Foundation of Jiangsu Province(No.0802073c) and the High- level Talented Person Special Subsidizes of Jiangsu University (No.08JDG013).

References


