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# Some New Results on the Boundedness of Solutions of a Certain Nonlinear Differential Equation of Third Order

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**Abstract:**In this paper, the boundedness of solutions of a third-order nonlinear differential equation is investigated. Some criteria on the regularity and asymptotic behavior of solutions for the same equation are also given. The obtained results are further extensions that of Mehri&Shadman [1], and also include the theorems established there.

**Keywords:** boundedness; differential equations of third order; energy function
**AMS subject classification numbers:** 34C11, 34A99.

## 1 Introduction

It is well-known differential equations is an old but durable subject that remains alive and useful to a wide variety of engineers, scientists, and mathematicians. The study of differential equations began with the birth of calculus, which dates to the 1660s. Part of Newton's motivation in developing calculus was to solve problems that could be attacked with differential equations. Now, with over 300 years of history, the subject of differential equations represents a huge body of knowledge including many subfields and a vast array of applications in many disciplines. It is worth mentioning that principles of differential equations are largely about the qualitative theory of ordinary differential equations. Qualitative theory refers to the study of the behavior of solutions, for example the investigation of stability, instability, boundedness, oscillation, non-oscillation of solutions and etc., without determining explicit formulas for the solutions. It should be noted that, in particular, stability and boundedness problems of solutions of nonlinear differential equations, continue to attract the attentions of many specialists despite its long history. It is still one of most burning problems of control theory, dynamical systems, systems with time lag, power system analysis, time varying non-linear feedback systems and etc. because of the absence of its complete solution. In many works, the specialist dealt with the problems by using Lyapunov functions [2] or energy functions [1], and obtained criteria for the stability and boundedness of solutions. It is worth mentioning the opinions of some authors about the Lyapunov's method [15]: In 2000, Qian [3] expressed that "So far, the most effective method to study the stability of non-linear differential equations is still the Lyapunov's direct method". Next, in 2003, Iggidr and Sallet [4] stated that "The most efficient tool for the study of the stability of a given non-linear system is provided by Lyapunov theory", and so on. Now, to the best of our knowledge the boundedness properties of equations of the form (1.1) or of special cases of this equation provided $f(t,x,\dot{x},\ddot{x}) = f(x,\dot{x},\ddot{x})$, $b(t) = 1$ and $c(t) = 1$ in (1.1) have been studied extensively by several authors. For example, we refer to the book of Reissig, Sansone and Conti [5] as a survey and the papers of Avdzhyan [6], Bihari [7], Burton [8], Chukwu [9], El-Nahhas [10], Ezeilo ([11], [12]), Hara ([13], [14], [15], [16],), Hou and Sun [17], Nakashima [18], Qian [19], Swick ([20], [21], [22], [23]), Tejumola [24], Tunç ([25], [26]), Yamamoto [27], Zarghamee and Mehri [28] and the references registered therein. However, the results about the boundedness of solutions of nonlinear differential equations of the form (1.1), (in

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the case $f(t,x,\dot{x},\ddot{x}) \neq f(x,\dot{x},\ddot{x})$, $b(t) \neq 1$ and $c(t) \neq 1$), are relatively scarce. Perhaps, the possible difficulties raise this case is due to construction of Lyapunov functional for higher order nonlinear differential equations. In this direction, according to the our observations in the relevant literature, first in 1970, Swick [22] established some sufficient conditions for every solution of the third-order differential equation

$$\dddot{x} + p(t)\ddot{x} + q(t)g(\dot{x}) + h(x) = e(t)$$

to be bounded. Next, in 1971, Nakashima [18] proved that every solution of the equation

$$\dddot{x} + p(t)\ddot{x} + q(t)g(\dot{x}) + h(x) = e(t,x,\dot{x},\ddot{x})$$

satisfies the conditions
$x(t) \to 0, \quad \dot{x}(t) \to 0$ and $\ddot{x}(t) \to 0$ as $t \to \infty$, if $|e(t,x,y,z)| \leq \tilde{e}(t)$ for all $t$ and all $(x,y,z) \in \Re^3$, and $\int\limits_0^t \tilde{e}(s)ds < \infty$ for all $t$.

Afterward, in 1972, Hara [13] investigated the asymptotic behavior of solutions of the differential equation of the form

$$\dddot{x} + a(t)\ddot{x} + b(t)g(x,\dot{x}) + c(t)h(x) = p(t,x,\dot{x},\ddot{x})$$

and showed that all solutions of the equation are uniformly bounded and satisfy the conditions
$x(t) \to 0, \quad \dot{x}(t) \to 0$ and $\ddot{x}(t) \to 0$ as $t \to \infty$.

Besides, in 1974, Hara [14] also considered the following third order differential equations

$$\dddot{x} + a(t)\ddot{x} + b(t)\dot{x} + c(t)x = p(t),$$

$$\dddot{x} + a(t)\ddot{x} + b(t)\dot{x} + c(t)h(x) = p(t,x,\dot{x},\ddot{x})$$

and

$$\dddot{x} + a(t)f(x,\dot{x})\ddot{x} + b(t)g(x,\dot{x})\dot{x} + c(t)h(x) = p(t,x,\dot{x},\ddot{x})$$

and established conditions under which all solutions of the above equations are uniform- bounded and tend to zero as $t \to \infty$. Finally, in 1975 and 1981 respectively, the same author ([15], [16]) took into consideration the third-order differential equations

$$\dddot{x} + a(t)f(x,\dot{x})\ddot{x} + b(t)g(x,\dot{x}) + c(t)h(x) = p(t)$$

and

$$\dddot{x} + a(t)f(x,\dot{x})\ddot{x} + b(t)g(x,\dot{x}) + c(t)h(x) = p(t,x,\dot{x},\ddot{x}),$$

and proved two results for the uniform ultimate boundedness of solutions of the equations above. Recently, the author in [26] also established some results for the uniformly ultimately boundedness of solutions of differential equations described by

$$\dddot{x} + a(t)f(x,\dot{x},\ddot{x})\ddot{x} + b(t)g(x,\dot{x}) + c(t)h(x) = p(t)$$

and

$$\dddot{x} + a(t)f(x,\dot{x},\ddot{x})\ddot{x} + b(t)g(x,\dot{x}) + c(t)h(x) = p(t,x,\dot{x},\ddot{x}).$$

Besides, in 1999, Mehri&Shadman [1] considered the third order nonlinear differential equation described as follows:

$$\dddot{x} + a(t)f(\ddot{x}) + b(t)g(\dot{x}) + c(t)h(x) = e(t).$$

The authors, Mehri&Shadman [1], presented sufficient conditions on the functions involved in the above equation, under which the solutions of this equation are bounded. Next, some results on the regularity and asymptotic behavior of the solutions of the above equation were also obtained by them. It should be noted that in spite of application of Lyapunov method [2] throughout all the papers mentioned above except that of Mehri&Shadman [1], Mehri&Shadman [1] constructed a very good energy function and used this function as a main tool through their established result. This case enables the investigation of boundedness of solutions of certain nonlinear differential equations of third order by way of a very simple function and under less restrictive conditions than that established in the literature (For this case, one can referee to the paper mentioned above).

Now, we consider the third-order nonlinear differential equation

$$\dddot{x} + f(t, x, \dot{x}, \ddot{x}) + b(t)g(x, \dot{x}) + c(t)h(x) = p(t, x, \dot{x}, \ddot{x}) \tag{1.1}$$

or its equivalent system

$$\begin{aligned} &\dot{x} = y, \quad \dot{y} = z, \\ &\dot{z} = -f(t, x, y, z) - b(t)g(x, y) - c(t)h(x) + p(t, x, y, z), \end{aligned} \tag{1.2}$$

in which the functions $b$, $c$, $f$, $g$, $h$ and $p$ depend only on the arguments displayed explicitly and the dots denote differentiation with respect to $t$. It is mainly assumed that the functions $b$ and $c$ are continuous on $\Re^+$, $\Re^+ = (0, \infty)$ and the functions $f$, $g$, $h$ and $p$ are continuous for all values their respective arguments on $\Re^+$ and $\Re$ . Besides, it is supposed that the derivatives $b'(t)$ and $g_x(x, y) \equiv \frac{\partial}{\partial x} g(x, y)$ exist and are continuous. The motivation for the present work inspired basically by the paper of Mehri&Shadman [1], and the papers mentioned above. Our aim is to extend the results verified by Mehri&Shadman [1] to the equation of the form (1.1). Further, it is worth mentioning that the technique used here is the same as that one used by Mehri&Shadman [1]. Meanwhile, our assumptions are completely different than those established in Hara ([13], [14], [15], [16]) and Tunç [26].

## 2 Main results

In this section, four main results are established.

**Theorem 1** *In addition to the basic assumptions imposed on functions* $a$ , $b$ , $c$ , $f$ , $g$ , $h$ *and* $p$ *we assume that the following conditions are satisfied:*

***(i)*** $b(t) > 0$ *and* $b'(t) > 0$ *for all* $t \in \Re^+$ .

***(ii)*** $zf(t, x, y, z) \geq 0$ *for all* $t \in \Re^+$ *and* $x$ , $y$ *and* $z \in \Re$ .

***(iii)*** $yg(x, y) \geq 0$ *and* $g_x(x, y) \leq 0$ *for all* $x$ *and* $y \in \Re$ ,

*and* $\lim\limits_{s \to \pm \infty} G(x, s) = +\infty$ , ( $G(x, s) = \int\limits_0^s g(x, \tau)d\tau$ ) .

***(iv)*** $|h(x)| \leq K\,|x|$ *for all* $x \in \Re$ , *where* $K$ *is a positive constant.*

***(v)*** $|p(t, x, y, z)| \leq |e(t)|$ *for all* $t \in \Re^+$ *and* $x$ , $y$ *and* $z \in \Re$ .

***(vi)*** *There are arbitrary continuous functions* $\alpha_0$ , *and on* $\Re^+ = (0, \infty)$ *such that*

$\alpha_0$ *and* $\alpha_1$ *are positive and decreasing and* $\beta$ *is positive and increasing for all* $t \in \Re^+$ , $\Re^+ = (0, \infty)$ *and*

$\frac{e(t)}{\sqrt{b(t)}}$, $\left(\frac{\alpha_0(t)}{\alpha_1(t)}\right)^{\frac{1}{2}}$, $\left(\frac{\alpha_1(t)b(t)}{\beta(t)}\right)^{\frac{1}{2}}$, $|c(t)| \left(\frac{\beta(t)}{\alpha_0(t)b(t)}\right)^{\frac{1}{2}} \in L^1(0, \infty)$, *where* $L^1(0, \infty)$ *is space of integrable Lebesgue functions.*

*Then, for every solution* $x(t)$ *of the equation (1.1),* $\frac{x}{\sqrt{\beta/\alpha_0}}$ , $\frac{x'}{\sqrt{\beta/\alpha_1}}$ *and* $\frac{x''}{\sqrt{b}}$ *are bounded for all* $t \in \Re^+$ .

**Remark 1** *It should be noted that the theorem stated above includes and improves the first theorem established in Mehri&Shadman [Theorem 1, 1]. Because our result, Theorem 1, is proved here without the assumption* $a(t) \geq 0$ *constituted in Mehri&Shadman [Theorem 1, 1] and the equation (1.1) also includes the equation considered in Mehri&Shadman [1].*

**Remark 2** *It should also be point out that for the special case* $p(t, x, y, z) \equiv 0$ *in (1.1) the conclusion of Theorem 1 also remains valid.*

Now, throughout all the main results established here, our main tool is the continuous differentiable energy function $E = E(t, x, y, z)$ defined by:

$$E := \quad \frac{\alpha_0(t)}{\beta(t)} x^2 + \frac{\alpha_1(t)}{\beta(t)} y^2 + \frac{1}{b(t)} z^2 + 2 \int\limits_0^y g(x, \eta) d\eta, \tag{2.1}$$

where $\alpha_0$ , $\alpha_1$ , $\beta$ and $b$ are positive functions, and both $\alpha_0$ and $\alpha_1$ and $\beta$ and $b$ , respectively, are decreasing and increasing functions for all $t \in \Re^+$ , $\Re^+ = (0, \infty)$ . It should be noted that the special case of this function was constituted in Mehri&Shadman [1].

**Proof.** Since the coefficients $\alpha_0$ , $\alpha_1$ , $\beta$ and $b$ in (2.1) are positive; $\alpha_0(t) > 0$ , $\alpha_1(t) > 0$ and $\beta(t) > 0$ , $b(t) > 0$ , and $yg(x, y) > 0$ , then it is clear that the function $E$ defined by (2.1) is positive definite. Now, let $(x, y, z) = (x(t), y(t), z(t))$ be an arbitrary solution of the system (1.2). Now, differentiating the function $E = E(t, x, y, z)$ given by (2.1) along the solution $(x, y, z)$ of the system (1.2), it can easily be followed that

$$\begin{aligned}\dot{E} \equiv \frac{d}{dt}E(t,x,y,z) &= \frac{\partial E}{\partial x}y + \frac{\partial E}{\partial y}z + \frac{\partial E}{\partial z}\dot{z} + \frac{\partial E}{\partial t} \\ &= \left(\frac{\alpha_0'(t)}{\beta(t)} - \frac{\alpha_0(t)\beta'(t)}{\beta^2(t)}\right)x^2 + \left(\frac{\alpha_1'(t)}{\beta(t)} - \frac{\alpha_1(t)\beta'(t)}{\beta^2(t)}\right)y^2 \\ &+ \frac{2\alpha_0(t)}{\beta(t)}xy + \frac{2\alpha_1(t)}{\beta(t)}yz - \frac{b'(t)}{b^2(t)}z^2 - \frac{2}{b(t)}zf(t,x,y,z) \\ &- \frac{2c(t)}{b(t)}zh(x) + \frac{2}{b(t)}zp(t,x,y,z) + 2y\int_0^y g_x(x,\eta)d\eta\end{aligned} \tag{2.2}$$

Now, clearly, the assumptions imposed on the functions $\alpha_0$ , $\alpha_1$ , $\beta$ and $b$ show that

$$\left(\frac{\alpha_0'(t)}{\beta(t)} - \frac{\alpha_0(t)\beta'(t)}{\beta^2(t)}\right) < 0, \quad \left(\frac{\alpha_1'(t)}{\beta(t)} - \frac{\alpha_1(t)\beta'(t)}{\beta^2(t)}\right) < 0,$$

$$-\frac{2}{b(t)}zf(t,x,y,z) \le 0, -\frac{b'(t)}{b^2(t)} < 0$$

and

$$2y\int_0^y g_x(x,\eta)d\eta \le 0.$$

Hence, we obtain from (2.2) that

$$\dot{E} \le \frac{2\alpha_0(t)}{\beta(t)}xy + \frac{2\alpha_1(t)}{\beta(t)}yz - \frac{2c(t)}{b(t)}zh(x) + \frac{2}{b(t)}zp(t,x,y,z).$$

Now, in view of assumptions (i)-(v) of Theorem 1 and the inequality obtained above, it can be written that

$$\begin{aligned}\dot{E} &\le \frac{2\alpha_0(t)}{\beta(t)}|x|\;|y| + \frac{2\alpha_1(t)}{\beta(t)}|y|\;|z| + \frac{2K|c(t)|}{b(t)}|z|\;|x| + 2\frac{|p(t,x,y,z)|}{b(t)}|z| \\ &\le \frac{2\alpha_0(t)}{\beta(t)}|x|\;|y| + \frac{2\alpha_1(t)}{\beta(t)}|y|\;|z| + \frac{2K|c(t)|}{b(t)}|z|\;|x| + 2\frac{|e(t)|}{b(t)}|z|.\end{aligned} \tag{2.3}$$

Because of properties of the functions $\alpha_0$ , $\alpha_1$ , $\beta$ , $b$ and the assumption $yg(x, y) > 0$ of the theorem, it is clear from (2.1) that

$$|x| \le \left(\frac{\beta(t)}{\alpha_0(t)}\right)^{\frac{1}{2}} E^{\frac{1}{2}} \quad ,$$

$$|y| \le \left(\frac{\beta(t)}{\alpha_1(t)}\right)^{\frac{1}{2}} E^{\frac{1}{2}}$$

and

$$|z| \le b^{\frac{1}{2}}(t)E^{\frac{1}{2}} \le b^{\frac{1}{2}}(t)\left(\frac{1}{2} + \frac{E}{2}\right).$$

Hence, we get that

$$\begin{aligned}&\frac{2\alpha_0(t)}{\beta(t)}\left|x\right|\ \left|y\right|\le 2\left(\frac{\alpha_0(t)}{\alpha_1(t)}\right)^{\frac{1}{2}}E,\\&\frac{2\alpha_1(t)}{\beta(t)}\left|y\right|\ \left|z\right|\le 2\left(\frac{\alpha_1(t)b(t)}{\beta(t)}\right)^{\frac{1}{2}}E,\\&2\frac{\left|e(t)\right|}{b(t)}\left|z\right|\le\frac{\left|e(t)\right|}{(b(t))^{\frac{1}{2}}}+\frac{\left|e(t)\right|}{(b(t))^{\frac{1}{2}}}E,\\&\frac{2K\left|c(t)\right|}{b(t)}\left|z\right|\ \left|x\right|\le 2K\left|c(t)\right|\ \left(\frac{\beta(t)}{\alpha_0(t)b(t)}\right)^{\frac{1}{2}}E.\end{aligned}\tag{2.4}$$

Therefore, the estimates (2.3) and (2.4) together imply that

$$\begin{aligned}\dot{E}(t)\le{}&2\left(\frac{\alpha_0(t)}{\alpha_1(t)}\right)^{\frac{1}{2}}E(t)+2\left(\frac{\alpha_1(t)b(t)}{\beta(t)}\right)^{\frac{1}{2}}E(t)\\&+\frac{\left|e(t)\right|}{(b(t))^{\frac{1}{2}}}+\frac{\left|e(t)\right|}{(b(t))^{\frac{1}{2}}}E(t)+2K\left|c(t)\right|\ \left(\frac{\beta(t)}{\alpha_0(t)b(t)}\right)^{\frac{1}{2}}E(t)\end{aligned}\tag{2.5}$$

Let

$$\Phi(t)=2\left[\left(\frac{\alpha_0(t)}{\alpha_1(t)}\right)^{\frac{1}{2}}+\left(\frac{\alpha_1(t)b(t)}{\beta(t)}\right)^{\frac{1}{2}}+\frac{\left|e(t)\right|}{2b^{\frac{1}{2}}(t)}+K\left|c(t)\right|\ \left(\frac{\beta(t)}{\alpha_0(t)b(t)}\right)^{\frac{1}{2}}\right].\tag{2.6}$$

Now, making use of expression (2.5) and (2.6), we have that

$$\dot{E}(t)\le\left(\frac{\left|e(t)\right|}{b^{\frac{1}{2}}(t)}\right)+\Phi(t)E(t).\tag{2.7}$$

Next, integrating (2.7) from 0 to $t$ , we obtain

$$E(t)-E(0)=\int\limits_0^t\frac{\left|e(s)\right|}{\sqrt{b(s)}}ds+\int\limits_0^t E(s)\Phi(s)ds.$$

By using assumption (vi) of Theorem 1 and the Gronwall-Reid-Bellman inequality, (see Rao [29]), we finally find that

$$E(t)\le A\exp(\int\limits_0^t\Phi(s)ds)$$

for some positive constant $A$ , $A=E(0)+\int\limits_0^t\frac{\left|e(s)\right|}{\sqrt{b(s)}}ds$ . Eventually, assumption (vi) of Theorem 1 yields $\Phi\in L^1(0,\infty)$ , which implies the boundedness of the function $E$ . That is, we can easily conclude that $\frac{\alpha_0}{\beta}x^2$ , $\frac{\alpha_1}{\beta}y^2$ and $\frac{1}{b}z^2$ are bounded, and hence this result guarantees the boundedness of $\frac{x}{\sqrt{\beta/\alpha_0}}$ , $\frac{x'}{\sqrt{\beta/\alpha_1}}$ and $\frac{x''}{\sqrt{b}}$ , which proves Theorem 1. ■

**Example 1:** Consider the equation

$$\dddot{x}+(1+t^2+x^2+(\dot{x})^2+(\ddot{x})^2)\ddot{x}+(t^2+1)\,\dot{x}+\frac{1}{(t^2+1)^4}x=e(t),$$

Clearly, the above equation is a special case of the equation (1.1), and this equation is equivalent to the system

$$\begin{aligned}&\dot{x}=y,\quad\dot{y}=z,\\&\dot{z}=-\,(1+t^2+x^2+y^2+z^2)z-(t^2+1)y-\frac{1}{(t^2+1)^4}x+e(t).\end{aligned}$$

Now, let

$$\alpha_0(t)=\frac{1}{(t^2+1)^3},\quad \alpha_1(t)=\frac{1}{t^2+1},\quad \beta(t)=(t^2+1)^4.$$

Clearly, $\alpha_0$ and $\alpha_1$ are positive and decreasing functions and $\beta$ is positive-definite and an increasing function on $(0,\infty)$ , and

$$\left(\frac{\alpha_0(t)}{\alpha_1(t)}\right)^{\frac{1}{2}}=\frac{1}{t^2+1}\in L^1(0,\infty),$$

$$\left(\frac{\alpha_1(t)b(t)}{\beta(t)}\right)^{\frac{1}{2}}=\frac{1}{(t^2+1)^2}\in L^1(0,\infty),$$

$$|c(t)|\left(\frac{\beta(t)}{\alpha_0(t)b(t)}\right)^{\frac{1}{2}}=\frac{1}{t^2+1}\in L^1(0,\infty).$$

In fact, $\int\limits_0^\infty \frac{1}{t^2+1}dt=\frac{\pi}{2}$ and $\int\limits_0^\infty \frac{1}{(t^2+1)^2}dt=\frac{\pi}{4}$ .

Now, if we choose $e(t)$ such that $\frac{e(t)}{\sqrt{b(t)}}=\frac{e(t)}{(t^2+1)^{\frac{1}{2}}}\in L^1(0,\infty)$ , then, for every solution $x(t)$ of equation (1.1), one can reach the following conclusion: $\frac{x(t)}{(t^2+1)^{\frac{7}{2}}}$ , $\frac{x'(t)}{(t^2+1)^{\frac{5}{2}}}$ and $\frac{x''(t)}{(t^2+1)^{\frac{1}{2}}}$ are bounded for all $t\ge 0$ . In view of the above choice, we have that

$$\begin{aligned}E=\quad E(t,x,y,z)&=\frac{\alpha_0(t)}{\beta(t)}x^2+\frac{\alpha_1(t)}{\beta(t)}y^2+\frac{1}{b(t)}z^2+2\int\limits_0^y g(s)ds\\ &=\frac{1}{(t^2+1)^7}x^2+\frac{1}{(t^2+1)^5}y^2+\frac{1}{(t^2+1)}z^2+2\int\limits_0^y sds\\ &=\frac{1}{(t^2+1)^7}x^2+\frac{1}{(t^2+1)^5}y^2+\frac{1}{(t^2+1)}z^2+y^2\end{aligned}$$

It is clear that the function $E=E(t,x,y,z)$ is a positive definite function. Now, let $(x,y,z)=(x(t),y(t),z(t))$ be any arbitrary solution of the above differential system and the function $v=v(t)$ be defined by $v(t)=E(t,x(t),y(t),z(t))$ . Differentiating the function $E$ along the above system, we obtain that

$$\begin{aligned}\dot v=\quad &\frac{d}{dt}E(t,x,y,z)=\quad -\frac{14t}{(t^2+1)^8}x^2-\frac{10t}{(t^2+1)^6}y^2-\frac{2t}{(t^2+1)^2}z^2\\ &+\frac{2}{(t^2+1)^7}xy+\frac{2}{(t^2+1)^5}yz-\frac{2(1+t^2+x^2+y^2+z^2)}{(t^2+1)}z^2\\ &-\frac{2}{(t^2+1)^5}xz+\frac{2e(t)}{(t^2+1)}z\end{aligned}$$

Because of $t>0$ and $1+t^2+x^2+y^2+z^2>0$ , it follows from $\dot v$ that

$$\begin{aligned}\dot v\le\quad &\frac{2}{(t^2+1)^7}xy+\frac{2}{(t^2+1)^5}yz-\frac{2}{(t^2+1)^5}xz+\frac{2e(t)}{t^2+1}z\\ \le\ &\frac{2}{(t^2+1)^7}|x|\ \ |y|+\frac{2}{(t^2+1)^5}|y|\ \ |z|+\frac{2}{(t^2+1)^5}|x|\ \ |z|+\frac{2e(t)}{t^2+1}|z|\quad .\end{aligned}$$

Now, when we consider the energy function constructed for the equation in Example 1, it is clear that

$$\begin{aligned}\dot v&\le 2\left[\frac{2}{t^2+1}+\frac{1}{(t^2+1)^2}+\frac{e(t)}{2(t^2+1)^{1/2}}\right]v(t)\quad+\frac{e(t)}{(t^2+1)^{1/2}}\\ &=\phi(t)v(t)+\frac{e(t)}{(t^2+1)^{1/2}},\end{aligned}$$

where

$$\phi(t) = 2\left[\frac{2}{t^2+1} + \frac{1}{(t^2+1)^2} + \frac{e(t)}{2(t^2+1)^{1/2}}\right].$$

Integrating $\dot{v}$ from 0 to $t$ , we obtain

$$v(t) - v(0) = \int_0^t v(s)\phi(s)ds + \int_0^t \frac{e(s)}{(s^2+1)^{1/2}}ds.$$

By using the assumption $\frac{e(t)}{\sqrt{b(t)}} = \frac{e(t)}{(t^2+1)^{1/2}} \in L^1(0,\infty)$ and Gronwall-Reid-Bellman inequality, (see Rao [29]), we finally find

$$v(t) \leq D\exp(\int_0^t \phi(s)ds),$$

where $D = v(0) + \int_0^t \frac{e(s)}{(s^2+1)^{1/2}}ds$ . Thus, $\phi \in L^1(0,\infty)$ implies the boundedness of $v$ , and hence the boundedness of $\frac{x}{\sqrt{\beta/\alpha_0}}$ , $\frac{x'}{\sqrt{\beta/\alpha_1}}$ and $\frac{x''}{\sqrt{b}}$ . This shows applicability of Theorem 1.

The following theorem, Theorem 2, is about the regularity of solutions of the equation (1.1).

**Theorem 2** *Let us replace conditions (i), (ii) and (vi) in Theorem 1 by the conditions as follows:*

$(i)'b(t) > 0$ *for all* $t \in \Re^+$ .

$(ii)'$ *There exist a positive constant* $M$ *such that* $\frac{f(t,x,y,z)}{z} \geq M$ *for all* $t \in \Re^+$ *and* $x$ , $y$ *and* $z \neq 0 \in \Re$ *, and* $b'(t) + 2Mb(t) > 0$ *for all* $t \in \Re^+$ .

$(vi)'$ *There are arbitrary continuous functions* $\alpha_0$ *, and on* $\Re^+ = (0,\infty)$ *such that*

$\alpha_0$ *and* $\alpha_1$ *are positive and decreasing and* $\beta$ *is positive and increasing for all* $t \in \Re^+$ , $\Re^+ = (0,\infty)$ *and*

$$\frac{e^2(t)}{b'(t)+2Mb(t)},\quad \frac{e(t)}{\sqrt{b(t)}},\quad \left(\frac{\alpha_0(t)}{\alpha_1(t)}\right)^{\frac{1}{2}},\quad \left(\frac{\alpha_1(t)b(t)}{\beta(t)}\right)^{\frac{1}{2}},\quad |c(t)|\left(\frac{\beta(t)}{\alpha_0(t)b(t)}\right)^{\frac{1}{2}} \in L^1(0,\infty).$$

*Then the conclusion of Theorem 1 holds.*

**Remark 3** *Theorem 2 includes and improves the second theorem of Mehri&Shadman [1]. Namely, Theorem 2 is proved here without the assumption* $a(t) \geq 0$ *in Mehri&Shadman [1] and the assumptions* $b'(t) + 2Mb(t) > 0$ *and* $\frac{e^2(t)}{b'(t)+2Mb(t)} \in L^1(0,\infty)$ *of Theorem 2 are less restrictive than that* $b'(t)+2Ma(t)b(t) > 0$ *and* $\frac{e^2(t)}{b'(t)+2Ma(t)b(t)} \in L^1(0,\infty)$ *established in Mehri&Shadman [Theorem 2, 1] ,and the equation (1.1) includes the equation considered therein*

**Remark 4** *According to our observations assumption* $(ii)'$ *of Theorem 2 in Mehri&Shadman [Theorem 2, 1], "* $0 < sf(s) \leq Ms^2$ , $s \in \Re$ *and* $M$ *is a positive constant", needs a revision; because this assumption does not imply the inequality*

$$-\frac{2a(t)}{b(t)}\left(\frac{f(z)}{z}\right)z^2 \leq -\frac{2a(t)}{b(t)}Mz^2,$$

*which has been constructed in Mehri & Shadman [Theorem 2, 1]. Therefore, our result, Theorem 2, revise the second theorem given there.*

**Proof.** As known, the function $E$ defined in (2.1) is positive definite. Now, subject to the assumptions of Theorem 2, an easy calculation from (2.1) and (1.2) shows that

$$\begin{aligned}
\dot{E} = \quad \frac{dE}{dt} = \quad & \left(\frac{\alpha_0'(t)}{\beta(t)} - \frac{\alpha_0(t)\beta'(t)}{\beta^2(t)}\right)x^2 + \left(\frac{\alpha_1'(t)}{\beta(t)} - \frac{\alpha_1(t)\beta'(t)}{\beta^2(t)}\right)y^2 \\
& + \frac{2\alpha_0(t)}{\beta(t)}xy + \frac{2\alpha_1(t)}{\beta(t)}yz - \frac{b'(t)}{b^2(t)}z^2 - \frac{2}{b(t)}zf(t,x,y,z) - \frac{2c(t)}{b(t)}zh(x) + \frac{2}{b(t)}zp(t,x,y,z) + 2y\int\limits_0^y g_x(x,\eta)d\eta \\
& \le \frac{2\alpha_0(t)}{\beta(t)}xy + \frac{2\alpha_1(t)}{\beta(t)}yz - \frac{b'(t)}{b^2(t)}z^2 - \frac{2}{b(t)}\left(\frac{f(t,x,y,z)}{z}\right)z^2 - \frac{2c(t)}{b(t)}zh(x) + \frac{2}{b(t)}zp(t,x,y,z) \\
& \le - \quad 2M\frac{1}{b(t)}z^2 - \frac{b'(t)}{b^2(t)}z^2 + 2\frac{|e(t)|}{b(t)}|z| + \frac{2\alpha_0(t)}{\beta(t)}|x||y| + \frac{2\alpha_1(t)}{\beta(t)}|y||z| + 2K\frac{|c(t)|}{b(t)}|z||x| \\
& = -(b'(t)+2Mb(t))\left(\frac{|z|}{b(t)} - \frac{|e(t)|}{b'(t)+2Mb(t)}\right)^2 + \frac{e^2(t)}{b'(t)+2Mb(t)} \\
& + \frac{2\alpha_0(t)}{\beta(t)}|x||y| + \frac{2\alpha_1(t)}{\beta(t)}|y||z| + 2K\frac{|c(t)|}{b(t)}|z||x| \\
& \le \quad - \quad (b'(t)+2Mb(t))\left(\frac{|z|}{b(t)} - \frac{|e(t)|}{b'(t)+2Mb(t)}\right)^2 + \frac{e^2(t)}{b'(t)+2Mb(t)} \\
& + 2\left(\frac{\alpha_0(t)}{\alpha_1(t)}\right)^{\frac{1}{2}}E + 2\left(\frac{\alpha_1(t)b(t)}{\beta(t)}\right)^{\frac{1}{2}}E + 2K|c|\left(\frac{\beta(t)}{\alpha_0(t)b(t)}\right)^{\frac{1}{2}}E \\
& = -(b'(t)+2Mb(t))\left(\frac{|z|}{b(t)} - \frac{|e(t)|}{b'(t)+2Mb(t)}\right)^2 + \frac{e^2(t)}{b'(t)+2Mb(t)} \\
& + 2\left[\left(\frac{\alpha_0(t)}{\alpha_1(t)}\right)^{\frac{1}{2}} + \left(\frac{\alpha_1(t)b(t)}{\beta(t)}\right)^{\frac{1}{2}} + K|c|\left(\frac{\beta(t)}{\alpha_0(t)b(t)}\right)^{\frac{1}{2}}\right]E.
\end{aligned}$$

This implies that

$$\dot{E}(t) \le \frac{e^2(t)}{b'(t)+2Mb(t)} + \left[\Phi(t) - \frac{|e(t)|}{b^{\frac{1}{2}}(t)}\right]E \tag{2.8}$$

where $\Phi(t)$ is the same as in (2.6). Similarly, as in the proof of Theorem 1, integrating (2.8) from 0 to $t$, later using assumption $(vi)'$ of Theorem 2 and the Gronwall-Reid-Bellman inequality, one can easily deduce the boundedness of the function $E$. The proof of this theorem is now complete. ∎

**Theorem 3** *Let all the conditions of Theorem 2 are hold. Then, every solution of the equation (1.1) satisfies* $\left(\frac{|\alpha_0'(t)|}{\beta(t)}\right)^{\frac{1}{2}} x \in L^2(0,\infty)$ *and* $\left(\frac{|\alpha_1'(t)|}{\beta(t)}\right)^{\frac{1}{2}} x' \in L^2(0,\infty)$.

*If, in addition, we assume*

$$l.\ u.\ b.\ \frac{b^2(t)}{b'(t)+2Mb(t)} = k < \infty, \quad t \ge 0,$$

*then*

$$x'' \in L^2(0,\infty).$$

**Remark 5** *Theorem 3 improves the third theorem proved in Mehri&Shadman [Theorem 3, 16] because our assumptions less restrictive than the assumptions* $a(t) \ge 0$ *and* $l.\ u.\ b.\ \frac{b^2(t)}{b'(t)+2Ma(t)b(t)} = k < \infty$ *in [Theorem 3, 1] and our result, Theorem 3, includes the same result proved there.*

**Proof.** The proof of this theorem follows the lines indicated in Theorem 1 and Theorem 2 above, except for some minor modification. Therefore, we omit the details. However, following the procedure as indicated above, one can easily obtain

$$\begin{aligned}\frac{dE}{dt} &\le \left(\frac{\alpha_0'(t)}{\beta(t)}-\frac{\alpha_0(t)\beta'(t)}{\beta^2(t)}\right)x^2+\left(\frac{\alpha_1'(t)}{\beta(t)}-\frac{\alpha_1(t)\beta'(t)}{\beta^2(t)}\right)y^2-\frac{b'(t)}{b^2(t)}z^2-\frac{2M}{b(t)}z^2+\frac{|e(t)|}{\sqrt{b(t)}}\\ &+2\left[\left(\frac{\alpha_0(t)}{\alpha_1(t)}\right)^{\frac{1}{2}}+\left(\frac{\alpha_1(t)b(t)}{\beta(t)}\right)^{\frac{1}{2}}+\frac{|e(t)|}{2b^{\frac{1}{2}}(t)}+K\,|c|\left(\frac{\beta(t)}{\alpha_0(t)b(t)}\right)^{\frac{1}{2}}\right]E(t)\\ &=\left(\frac{\alpha_0'(t)}{\beta(t)}-\frac{\alpha_0(t)\beta'(t)}{\beta^2(t)}\right)x^2+\left(\frac{\alpha_1'(t)}{\beta(t)}-\frac{\alpha_1(t)\beta'(t)}{\beta^2(t)}\right)y^2\\ &+\Phi(t)E(t)-\frac{b'(t)}{b^2(t)}z^2-\frac{2M}{b(t)}z^2+\frac{|e(t)|}{\sqrt{b(t)}},\end{aligned}$$

where $\Phi$ is given by the equation (2.6).

Hence, it follows that

$$\begin{aligned}&\left(\frac{\alpha_0(t)\beta'(t)}{\beta^2(t)}-\frac{\alpha_0'(t)}{\beta(t)}\right)x^2+\left(\frac{\alpha_1(t)\beta'(t)}{\beta^2(t)}-\frac{\alpha_1'(t)}{\beta(t)}\right)y^2+\frac{b'(t)+2Mb(t)}{b^2(t)}z^2\\ &\le -\frac{dE}{dt}+\frac{|e(t)|}{\sqrt{b(t)}}+\Phi(t)E(t).\end{aligned}$$

Now, integrating both sides of the above expression from $0$ to $t$, we obtain that

$$\begin{aligned}&\int_0^t\left[\left(\frac{\alpha_0(s)\beta'(s)}{\beta^2(s)}-\frac{\alpha_0'(s)}{\beta(s)}\right)x^2+\left(\frac{\alpha_1(s)\beta'(s)}{\beta^2(s)}-\frac{\alpha_1'(s)}{\beta(s)}\right)y^2+\frac{b'(s)+2Mb(s)}{b^2(s)}z^2\right]ds\\ &\le E(0)-E(t)+\int_0^t\frac{|e(s)|}{\sqrt{b(s)}}ds+K_1\int_0^t\Phi(s)ds,\end{aligned}$$

where it is assumed that $E(t)\le K_1$ , $t>0$ . In view of the assumptions of Theorem 3, and the boundedness of the function $E(t)$ , we can easily conclude that

$$\int_0^t\frac{|\alpha_0'(s)|}{\beta(s)}x^2ds<\infty,\quad \int_0^t\frac{|\alpha_1'(s)|}{\beta(s)}y^2ds<\infty,\quad \int_0^t z^2ds<\infty,\quad t\ge 0.$$

The proof of Theorem 3 is now complete. ∎

**Theorem 4** *Let all the conditions of Theorem 3 be satisfied except* $\frac{f(t,x,y,z)}{z}\ge M$ *, and besides, we assume that* $b(t)$ *,* $c(t)$ *,* $p(t,x,y,z)$ *and* $\frac{\beta(t)}{\alpha_0(t)}$ *are bounded for all* $t\ge 0$ *, and* $\frac{f(t,x,y,z)}{z}\le M_1$ *for all* $t\in\Re^+$ *and* $x$ *,* $y$ *and* $z\ne 0\in\Re$ *, where* $M_1$ *is a positive constant. Then the solutions of the equation (1.1), for which* $x''$ *is bounded, satisfy*

$$\lim_{t\to\infty}x''(t)=0.$$

**Remark 6** *Theorem 4 includes and improves the fourth theorem proved in Mehri&Shadman [Theorem 4, 1].*

**Proof.** It can easily be written from (1.2) that

$$\left|zz'\right|\le|z|\,|f(t,x,y,z)|+b(t)\,|z|\,|g(x,y)|+c(t)\,|z|\,|h(x)|+|z|\,|p(t,x,y,z)|\,.$$

Since the function $g$ is continuous and $x$ and $y$ are bounded on $\Re^+$ , the function $g$ is bounded on $\Re^+$ , and by using the assumptions $(v)$ of Theorem 1, $(ii)'$ of Theorem 2 and the boundedness of the function $g$ , say $|g(x,y)| \leq K_1$ ( $K_1$ - a positive constant) which we now assume, we have

$$\left|zz'\right| \leq M_1 b(t)\frac{z^2}{b(t)} + K_1 b^{\frac{3}{2}}(t)\frac{|z|}{\sqrt{b(t)}}$$
$$+K\left|c(t)\right|\sqrt{b(t)}\sqrt{\frac{\beta(t)}{\alpha_0(t)}}\frac{|z|}{\sqrt{b(t)}}\frac{|x|}{\sqrt{\beta(t)/\alpha_0(t)}} + \left|e(t)\right|\sqrt{b(t)}\frac{|z|}{\sqrt{b(t)}}.$$

Hence the expression on the right hand is bounded. Thus

$$\lim_{t\to\infty} z(t) = 0,$$

(see Bihari [7], (p.291)).

Similar results can be obtained for $x$ and $y$ by requiring more restrictive conditions on the equation (1.1). For example, let $l.u.b.\frac{\beta}{\sqrt{\alpha_1'}} \leq c_1$ . Then, by Theorem 3, we have $y \in L^2(0,\infty)$.

The relation $\dot{y} = z$ necessary implies that $\dot{y}$ is also bounded. Hence

$$\lim_{t\to\infty} y(t) = 0,$$

(see Lefschetz [30] and Mehri&Shadman [1]). ■

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