On the Low Regularity Solutions for the Generalized Dullin-Gottwald-Holm Equation

Xiaolian Ai 1*, Lixin Tian2, Guilong Gui2
1 Department of Finance, Jiangsu University, 212013 Jiangsu, China
2 Nonlinear Scientific Research Center, Jiangsu University, Zhenjiang, Jiangsu, 212013, China
(Received 24 March 2008, accepted 19 October 2008)

Abstract: We investigate the low regularity solutions to the generalized Dullin-Gottwald-Holm equation, which describes the unidirectional propagation of two dimensional waves in shallow water over a flat bottom. By using the mollification method for the initial data, we obtain the energy estimate of the corresponding solutions, and then, we present a sufficient condition which guarantees the existence of the low regularity weak solutions.

Keywords: Generalized Dullin-Gottwald-Holm equation; bi-Hamiltonian structure; energy estimate; low regularity solution

Mathematics Subject Classification (2000): 35Q53, 35B35

1 Introduction

The nonlinear partial differential equation [1]

\[ m_t + c_0 u_x + u m_x + 2m u_x = -\gamma u_{xxx}, \quad x \in \mathbb{R}, \quad t \in \mathbb{R} \]  

(1.1)

in dimensionless time-space variables \((t, x)\) models the unidirectional propagation of two dimensional waves in shallow water over a flat bottom. In (1.1), \(u(t, x)\) represents the horizontal component of the fluid velocity, \(m = u - \alpha^2 u_{xx}\) is a momentum variable, the constants \(\alpha^2\) and \(\gamma/c_0\) are squares of length scales, and \(c_0 = \sqrt{gh}\) (where \(c_0 = 2\omega\)) is the linear wave speed for undisturbed water at rest at spatial infinity. In [1], Dullin, Gottwald and Holm derived the equation (1.1) by using asymptotic expansions directly in the Hamiltonian for Euler equations in the shallow water regime and thereby is shown to be bi-Hamiltonian and has a Lax pair formulation in [1]. Dullin-Gottwald-Holm equation (1.1) (DGH equation for short) combines the linear dispersion of Korteweg-de Vries (KdV) equation with the nonlinear/nonlocal dispersion of the Camassa-Holm (CH) equation, yet still preserves integrability via the inverse scattering transform (IST for short) method. This IST-integrable class of equations contains both the KdV equation and CH equation as limiting cases.

Using the notation \(m = u - \alpha^2 u_{xx}\), we rewrite the initial value problem of DGH equation as

\[ \begin{cases} u_t - \alpha^2 u_{xxt} + 2\omega u_x + 3u u_x + \gamma u_{xxx} = \alpha^2 (2u_x u_{xx} + uu_{xxx}), \\ u(0, x) = u_0(x), \end{cases} \quad t > 0, \quad x \in \mathbb{R}. \]  

(1.2)

Recently, the authors etc. [2–5] studied the well-posedness of Cauchy problem and the scattering problem for the DGH equation. Moreover, the issue of passing to the limit as the dispersive parameter tends to zero for the solution of DGH equation was investigated, and the scattering data of the scattering problem for the equation were explicitly expressed in [2]. Octavian G. Mustafa [6] investigated the low regularity conditions needed for the Cauchy problem of DGH equation via the semigroup approach of quasilinear hyperbolic
equations of evolution and the viscosity method. Y. Li and P. Olver [7] studied the well-posedness, blow-up and the low regular solutions for an integrable nonlinearly dispersive model wave equation. In [8], Adrian Constantin and Jonatan Lenells presented a simple algorithm for the inverse scattering approach to the Camassa-Holm equation. Yue Liu [9] investigated the problems of the existence of global solutions and the formation of singularities for the DGH equation. And the second author etc.[10] studied the limit behavior of the solutions to a class of nonlinear dispersive wave equations, which can be seen as some extension of DGH equation.

In this paper, we are interested in the Cauchy problem for the following generalized DGH equation

\[
\begin{aligned}
& u_t - \alpha^2 u_{xxt} + 2\omega u_x + \beta u^2 u_x + \gamma u_{xxx} = \alpha^2 (2u_x u_{xx} + uu_{xxx}) , \\
& u(0, x) = u_0(x), \\
& t > 0, \quad x \in \mathbb{R},
\end{aligned}
\]

(1.3)

where \( \omega, \gamma \in \mathbb{R} \) and \( \alpha, \beta \neq 0 \) are given. Observe that if \( \beta = 3 \) and \( \beta u^2 u_x \) is replaced by \( \beta uu_x \), then (1.3) is the classical DGH equation (1.2).

Equation (1.3) was investigated in [11] and [12], where the local well-posedness problem was studied with the initial data \( u_0 \in H^s(\mathbb{R}) \) for \( s > \frac{3}{2} \).

Motivated by [2] and [7], the purpose of this paper is to study the low regularity solutions for the generalized Dullin-Gottwald-Holm Equation (1.3). For the sake of convenience, we always assume \( \alpha = 1 \) in the remainder of the paper.

**Notations.** We shall use the standard notation \( \| \cdot \|_{L^p} \) for the norm of the space \( L^p(\mathbb{R}) \), \( 1 \leq p < \infty \), i.e., \( \| f \|_{L^p} = \left( \int_{\mathbb{R}} |f|^p dx \right)^{1/p} \). The space \( L^{\infty} = L^{\infty}(\mathbb{R}) \) consists of all essentially bounded, Lebesgue measurable functions with the standard norm

\[
\| f \|_{L^{\infty}} = \inf_{m(\varepsilon) = 0} \sup_{x \in \mathbb{R}\setminus\varepsilon} |f(x)|.
\]

And we denote the norm in the Sobolev space \( H^s = H^s(\mathbb{R}) \) by

\[
\| f \|_{H^s} = \left( \int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2}
\]

for \( s \in \mathbb{R} \), where \( \hat{f}(\xi) \) is the Fourier transform of \( f(x) \). We also define the operator \( \Lambda^s_\alpha \) for any \( \alpha, \ s \in \mathbb{R} \) by

\[
\Lambda^s_\alpha = \left( 1 - \alpha^2 \partial_x^2 \right)^{s/2}, \quad (\text{resp. } \Lambda^s := \Lambda^s_1 = \left( 1 - \partial_x^2 \right)^{s/2}, \ \Lambda = \Lambda^1_1).
\]

The remainder of the paper is organized as follows. In Section 2, we present some results of the regularized generalized DGH equations. The low regularity solutions for the generalized Dullin-Gottwald-Holm Equation (1.3) is studied in Section 3.

### 2 The regularized generalized DGH equation

In order to give a sufficient condition for the solution of the generalized DGH equation (1.3) which exists in the low order Sobolev space \( H^s(\mathbb{R}) \) for some \( 1 < s \leq \frac{3}{2} \), we use the following regularized equation (2.1) to estimate norms of its solutions, showing that they are bounded when \( \varepsilon \) is sufficiently small.

\[
\begin{aligned}
& u_t - u_{xxt} + \varepsilon u_{xxtxt} + (2\omega u + \frac{\beta}{3} u^3)_x + \gamma u_{xxx} = 2u_x u_{xx} + uu_{xxx} , \\
& u(0, x) = u_0(x), \\
& t > 0, \quad x \in \mathbb{R}.
\end{aligned}
\]

(2.1)

A standard application of the contraction mapping principle leads to the following existence result.

**Theorem 2.1** Suppose that \( u_0 \in H^s(\mathbb{R}) \) with \( s \geq 1 \), there exists a \( T > 0 \) depending only on \( \| u_0 \|_{H^s} \) such that there corresponds a unique solution \( u(t, x) \in C([0, T]; H^s) \) of the equation (2.1) in the sense of distribution. If \( s \geq 2 \), the solution \( u(t, x) \in C([0, \infty); H^s) \) exists globally in the time. In particular, for \( s \geq 4 \), the corresponding solution is a classical globally defined solution of (2.1).

IJNS email for contribution: editor@nonlinearscience.org.uk
A standard energy estimate yields

**Theorem 2.2** Let $s \geq 4$ and the function $u(x, t)$ be a solution of the regularized equation with the initial data $u_0(x) \in H^s$. Then the following inequalities hold

$$ \|u(t)\|_{H^1} \leq c \int_{\mathbb{R}} (u^2 + u_x^2 + \varepsilon u_{xx}^2) \, dx = c \int_{\mathbb{R}} (u_0^2 + u_{0x}^2 + \varepsilon u_{0xx}^2) \, dx. $$

(2.2)

For any $q \in (0, s - 1]$, there is a constant $c_0$ depending only on $q$ and $s$ such that

$$ \|u(t)\|_{H^{q+1}} \leq \int_{\mathbb{R}} \left( \Lambda^{q+1} u_0 \right)^2 + \varepsilon(\Lambda^q u_{0xx})^2 + c_0 \|u(t)\|_{L^\infty} \|u(t)\|_{L^\infty} \|u(t)\|_{H^{q+1}} \, dt. $$

(2.3)

For any $q \in (\frac{1}{2}, s - 1]$ and any $r \in (\frac{1}{2}, q]$, there is a constant $c$ depending only on $r$, $q$ and $s$ such that

$$ \|u(t)\|_{H^{q+1}} \leq \int_{\mathbb{R}} \left( \Lambda^{q+1} u_0 \right)^2 + \varepsilon(\Lambda^q u_{0xx})^2 + c \|u(t)\|_{H^{r+1}} \|u(t)\|_{H^r} \|u(t)\|_{H^{q+1}} \, dt. $$

(2.4)

For any $q \in [0, s - 1]$ and any $r \in (\frac{1}{2}, q]$, there is a constant $c_0$ independent of $\varepsilon$, such that

$$ (1 - 2\varepsilon) \|u(t)\|_{H^q} \leq c_0 \|u(t)\|_{H^{q+1}} \left( 1 + \|u(t)\|_{L^\infty} \|u(t)\|_{H^1} \right). $$

(2.5)

**Proof.** The proof can be similarly treated as in [7] or [10], and we omit it. ■

The following lemma is critical to the proof of our main Theorem.

**Lemma 2.3** ([7]) Let $u_{\varepsilon} = \phi_\varepsilon * u_0$ be the convolution of $u_{\varepsilon} = \phi_\varepsilon * u_0$ of the functions $\phi_\varepsilon(x) = \varepsilon^{-1/4}\phi(\varepsilon^{-1/4}x)$ and $u_0$ such that the Fourier transform $\hat{\phi}$ of $\phi$ satisfies $\hat{\phi} \in C_c^\infty(\mathbb{R})$, $\hat{\phi}(\xi) \geq 0$, and $\hat{\phi}(\xi) = 1$ for any $\xi \in (-1, 1)$, the following estimates hold for any $\varepsilon$ with $0 < \varepsilon < 1/4$:

$$ \|u_{\varepsilon,0}\|_{H^q} \leq c, \quad (q \leq s) $$

(2.6)

$$ \|u_{\varepsilon,0}\|_{H^q} \leq \varepsilon\varepsilon^{-\frac{q}{4}}, \quad (q > s) $$

(2.7)

$$ \|u_{\varepsilon,0} - u_0\|_{H^q} \leq \varepsilon\varepsilon^{-\frac{q}{4}}, \quad (q \leq s) $$

(2.8)

$$ \|u_{\varepsilon,0} - u_0\|_{H^q} = o(1). $$

(2.9)

## 3 The existence of low regularity solution to the generalized DGH equation

From the results in section 2, the following property holds for the weak solution given in Theorem 2.1.

**Theorem 3.1** Suppose that $u_0 \in H^s(\mathbb{R})$ for some $1 \leq s \leq \frac{3}{2}$ such that $\|u_{0x}\|_{L^\infty} < \infty$. Let $u_{\varepsilon,0}$ be defined as in Lemma 2.3. Then there exist constants $T > 0$ and $c > 0$ independent of $\varepsilon$ such that the corresponding solution $u_{\varepsilon}$ of (2.1) satisfies $\|u_{\varepsilon,0}\|_{L^\infty} \leq c$.

**Proof.** Let us start from the regularized equation (2.1) with $u = u_{\varepsilon}$. Differentiating with respect to $x$ on both sides of the first equation of (2.1), one can see

$$ (1 - \varepsilon)u_{xt} - \varepsilon u_{xxt} - \gamma u_{xx} - (\gamma + 2\omega)u - \beta^2 u^3 + \frac{1}{2} u_x^2 + \frac{1}{2} u^2 = 0. $$

(3.1)

Let $p > 0$ be an integer. Then, multiplying equation (3.1) by $(u_{\varepsilon})^{2p+1}$ to integrate with respect to $x$ yields the equality

$$ \frac{1 - \varepsilon}{2p + 2} \int_{\mathbb{R}} (u_{\varepsilon})^{2p+2} \, dx + \int_{\mathbb{R}} (u_{\varepsilon})^{2p+1}(-\gamma u - 2\omega u - \beta^2 u^3 + \frac{1}{2} u_x^2) \, dx $$

$$ + \frac{p}{2p + 2} \int_{\mathbb{R}} (u_{\varepsilon})^{2p+3} \, dx - \int_{\mathbb{R}} (u_{\varepsilon})^{2p+1} u_{xxt} \, dx - \int_{\mathbb{R}} (u_{\varepsilon})^{2p+1} u_{xx} \, dx $$

$$ = - \int_{\mathbb{R}} (u_{\varepsilon})^{2p+1} \Lambda^{-2} \varepsilon u_{x} + (2\omega + \gamma)u + \beta^2 u^3 + \frac{1}{2} u_x^2 - \frac{1}{2} u^2 \, dx. $$

(3.2)
From integration by parts and the Hölder’s inequality, we get

\[
\frac{1 - \varepsilon}{2p + 2} \frac{d}{dt} \|u_x\|_{L^{2p+2}}^{2p+2} \\
\leq \|u_x\|_{L^{2p+2}}^{2p+1} \left( \varepsilon \|u_{xxx}\|_{L^{2p+2}} + (\gamma + 2\omega) \right) + \frac{\beta}{3} \|u^3\|_{L^{2p+2}} + \frac{1}{2} \|u^2\|_{L^{2p+2}} + \|g\|_{L^{2p+2}} \right) + \frac{p}{2p + 2} \|u_x\|_{L^\infty} \|u_x\|_{L^{2p+2}},
\]

where

\[
g = \Lambda^{-2} [\varepsilon u_xt + (2\omega + \gamma)u + \frac{\beta}{3} u^3 + \frac{u^2}{2} - \frac{1}{2} u^2].
\]

Because \(\|f\|_{L^p} \to \|f\|_{L^\infty}\) as \(p \to \infty\) for any \(f \in L^\infty \cap L^2\), integrating (3.3) with respect to \(t\) and taking the limit as \(p \to \infty\) leads to the estimate

\[
(1 - \varepsilon) \|u_x\|_{L^\infty}(t) \\
\leq (1 - \varepsilon) \|u_0\|_{L^\infty} + c \int_0^t \left[ \varepsilon \|u_{xxx}\|_{L^\infty} + \|u\|_{L^\infty} + \|u^3\|_{L^\infty} \\
+ \|u^2\|_{L^\infty} + \|g\|_{L^\infty} + \|u_x\|_{L^\infty}^2 \right] \, d\tau.
\]

The Sobolev imbedding theorem and the algebraic property of Sobolev space \(H^s(\mathbb{R})\) with \(s > \frac{1}{2}\) yield

\[
\|u^3\|_{L^\infty} \leq c \|u^3\|_{H^1} \leq c \|u\|_{H^1}^3 \leq c_1,
\]
\[
\|u^2\|_{L^\infty} \leq c \|u^2\|_{H^1} \leq c \|u\|_{H^1}^2 \leq c_1,
\]

and

\[
\|g\|_{L^\infty} \leq \|\Lambda^{-2} [(2\omega + \gamma)u + \frac{\beta}{3} u^3 - \frac{1}{2} u^2]\|_{L^\infty} + \|\Lambda^{-2} (\varepsilon u_xt + \frac{u^2}{2})\|_{H^{\frac{1}{2}+}} \\
\leq c_1 + c_2 \left( \|u_t\|_{L^2} + \|u_x\|_{L^\infty} \|u_x\|_{L^2} \right) \\
\leq c_1 + c_2 \left( \|u_t\|_{L^2} + \|u_x\|_{L^\infty}^3 \right),
\]

where constants \(c_1\) and \(c_2\) are independent of \(\varepsilon\) when \(\varepsilon\) is sufficiently small. By (2.2), (2.5) and (2.6), we have

\[
\|g\|_{L^\infty} \leq c_1 (c_2 + \|u_0\|_{H^1}) \leq c_3,
\]

where the constant \(c_3\) is independent of \(\varepsilon\). Moreover, for any fixed \(r \in (\frac{1}{2}, 1)\), there exists a constant \(c_r\) such that \(\|u_{xxx}\|_{L^\infty} \leq c_r \|u_{xxx}\|_{H^r} \leq c_r \|u\|_{H^{r+3}}.\) Using Theorem 2.2 yields

\[
\|u_{xxx}\|_{L^\infty} \leq c_r \|u\|_{H^{r+4}}.
\]

Applying the Gronwall’s inequality to (2.3) with \(q = r + 3\) and \(u = u_x\), we obtain

\[
\|u\|_{H^{r+4}}^2 \leq \int_{\mathbb{R}} \left( \Lambda^{r+4} u_0 \right)^2 + \varepsilon (\Lambda^{r+3} u_{0xx})^2 \, dx \exp(c \int_0^t \|u_x\|_{L^\infty} \, d\tau).
\]

Therefore, from (2.7), one has

\[
\|u_{xxx}\|_{L^\infty} \leq c \varepsilon^{\frac{-r-4}{r}} \exp(c \int_0^t \|u_x\|_{L^\infty} \, d\tau)
\]

for some constant \(c\). For \(\varepsilon < \frac{1}{4}\), we get the inequality from (3.4),(3.5),(3.6) and (3.8)

\[
\|u_x\|_{L^\infty}(t) \leq \|u_0\|_{L^\infty} + c \int_0^t \left[ \varepsilon^{\frac{-r}{r+4}} \exp(c \int_0^\tau \|u_x\|_{L^\infty} \, ds) + \frac{1}{2} \|u_x\|_{L^\infty}^2 + 1 \right] \, d\tau.
\]
Applying the contraction mapping principle yields that there exists a $T > 0$ such that the integral equation

$$f(t) = \|u_{0x}\|_{L^\infty} + c \int_0^T \left[ e^{\frac{-\tau}{T}} \exp(c \int_0^\tau f(s) ds) + \frac{1}{2} f^2(\tau) + 1 \right] d\tau$$

has a unique solution $f(t) \in C([0, T])$. The comparison principle leads to the estimate $\|u_x\|_{L^\infty} \leq f(t)$ for any $t \in [0, T]$, which implies the conclusion of Theorem 3.1. ■

As a direct result of Theorem 3.1, the existence of a weak solution to the Cauchy problem (1.3) can be obtained as follows.

**Theorem 3.2** Suppose that $u_0 \in H^s(\mathbb{R})$ for some $1 \leq s \leq \frac{3}{2}$ such that $\|u_{0x}\|_{L^\infty} < \infty$. Then there exists a constants $T > 0$ such that the Cauchy problem (1.3) has a solution $u(x, t) \in L^2([0, T]; H^s)$ in the sense of distribution and $u_x \in L^\infty([0, T] \times \mathbb{R})$.

**Proof.** It follows from Theorem 3.1 that $\{u_{\varepsilon_n x}\}$ is bounded in the space $L^\infty$ with $\varepsilon_n \to 0$. As a result, the sequences $\{u_{\varepsilon_n}^2\}$ and $\{u_{\varepsilon_n}^2\}$ are weakly convergent to $u^2$ and $u^2$ in $L^2([0, T]; H^r(-R, R))$ for any $r \in [0, s - 1)$, respectively. Hence, for any $f \in C_0^\infty$, $u$ satisfies the equation

$$\int_0^T \int_\mathbb{R} u(f_1 - f_{xx}) dx dt = \int_0^T \int_\mathbb{R} \left( u^2 \frac{u}{2} f_{xxx} - \gamma u f_{xxx} - (2\omega u + \frac{\beta}{3} u^3 + \frac{1}{2} u^2) f_x \right) dx dt,$$

with $u(x, 0) = u_0$. Since $X = L^1([0, T] \times \mathbb{R})$ is a separable Banach space and $\{u_{\varepsilon_n x}\}$ is a bounded sequence in the dual space $X^* = L^\infty([0, T] \times \mathbb{R})$ of $X$, there exists a subsequence of $\{u_{\varepsilon_n x}\}$, still denoted by $\{u_{\varepsilon_n x}\}$, weakly convergent to a function $v$ in $L^\infty([0, T] \times \mathbb{R})$. Therefore, $u_x = v$ almost everywhere since $\{u_{\varepsilon_n x}\}$ is also weakly convergent to $u_x$ in $L^2([0, T] \times \mathbb{R})$. Hence, $u_x \in L^\infty([0, T] \times \mathbb{R})$ which completes the proof of Theorem 3.2. ■

**Acknowledgments**

The authors would like to thank the referee for valuable comments. The authors are supported by the National Nature Science Foundation of China (No: 90610031), the Outstanding Personnel Program in Six Fields of Jiangsu(No:6-A-029) and the Teaching and Research Award Program for Outstanding Young Teachers in Higher Education Institutions of POE, China(No:2002-383).

**References**


IINS homepage: http://www.nonlinearscience.org.uk/


