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Summation Formulas on the Binary Digits

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Abstract: In this paper, $T(N)$ defines the difference among the arithmetic progression $a, a+3, a+6\cdots a+3n=N$ with $0\le a\le 2$. At the same time $T(N)$ shows the difference between numbers which have an even sum of binary digits and those which have an odd sum. Finally, we conclude that $T(3m)>0$, $T(3m+1)<0$, $T(3m+2)\le 0$ for all $m\ge 0$.

Keywords: Arithmetical progression; binary digits; sum of digits
Classification AMS(2000) 11A63.

1 Introduction

In 1969. D.J.Newman [4] proved L.Moser's conjecture, that is, the preponderance, among the multiples of 3, of numbers which have an even sum of binary digits over those which have an odd sum.

Let q be an odd integer greater than 2 and $S(q,N)$ denote the difference, among the arithmetic progression $i, i+q, i+2q, \cdots, i+nq=N$ where $0\le i<q$, between numbers which have an even sum of binary digits with those which have an odd sum. i.e.

$$S(q,N)=S_i(q,N)=\sum_{\substack{0\le k\le N\\ k\equiv N(\mathrm{mod}q)}}(-1)^{D(k)} \tag{1}$$

where $D(k)$ denote the sum of binary digits of k and $i\equiv N(\mathrm{mod}q)$. For convenience let $S_i(q,N)=0$ for $N<0$.

Newman [4] obtained the inequalities:

$$\frac{3^\alpha}{20}<\frac{S_0(3,3N)}{N^\alpha}<5\cdot 3^\alpha \qquad \text{with} \quad \alpha=\frac{\log 3}{\log 4}.$$

J.Coquet [1] obtained the more precise result in a different way:

$$\limsup_{N\to\infty}\frac{S_0(3,3N)}{(N+1)^\alpha}=\frac{55}{3}\left(\frac{3}{65}\right)^\alpha=1.601958421\cdots$$

$$\liminf_{N\to\infty}\frac{S_0(3,3N)}{(N+1)^\alpha}=\frac{2\sqrt{3}}{3}=1.154700538\cdots$$

In this paper we show that

(a) it is a definite preponderance, among the arithmetic progression $3k+1$ $(k=0,1,2,3,\cdots)$, of those containing an odd number of one digits over those containing an even number, that is, $S(3,3k+1)<0(k=0,1,2,\cdots)$;

(b) $S(3,3k+2)\le 0(k=0,1,2,\cdots)$;

For the other references one may refer to M.D.Mcilroy [3]. In this paper the following results are proved.

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Theorem 1 *(1) If $k \geq 0$ and $2^{2k} \leq m < 2^{2k+1}$, then*

$$2 \cdot 3^{k-1} \leq S_0(3,m) \leq 3^k,$$

$$-3^k \leq S_1(3,m) \leq -3^{k-1} - 1,$$

$$-3^{k-1} + 1 \leq S_2(3,m) \leq 0;$$

(2) if $k \geq 0$ and $2^{2k+1} \leq m < 2^{2k+2}$, then

$$3^k + 1 \leq S_0(3,m) \leq 2 \cdot 3^k,$$

$$-3^k \leq S_1(3,m) \leq -2 \cdot 3^{k-1},$$

$$-3^k \leq S_2(3,m) \leq -1.$$

Corollary 1 *For all $m \geq 0$, we have*

$$S_0(3,3m) > 0, \quad S_1(3,3m+1) < 0, \quad S_2(3,3m+2) \leq 0.$$

2 Preliminary Lemmas

Lemma 1 *Let $0 \leq n < 2^k$, and $t = q[n/q] + q - n$. Then*

$$S(q, 2^k + n) = S(q, 2^k - t) - S(q, n).$$

Proof. By (1) we have

$$\begin{aligned} S(q, 2^k + n) &= \sum_{0 \leq j \leq (2^k+n)/q} (-1)^{D(2^k+n-qj)} \\ &= \sum_{0 \leq j \leq n/q} (-1)^{D(2^k+n-qj)} + \sum_{0 \leq v \leq (2^k-t)/q} (-1)^{D(2^k-t-qv)} \\ &= \sum_{0 \leq v \leq (2^k-t)/q} (-1)^{D(2^k-t-qv)} - \sum_{0 \leq j \leq n/q} (-1)^{D(n-qj)} \\ &= S(q, 2^k - t) - S(q, n). \end{aligned}$$

This completes the proof of Lemma 1. ■

Lemma 2 *Let $\zeta = e^{2\pi i/q}$ be a primitive qth-root of unity and $1 \leq t \leq q$, $w \geq 1$. Then*

$$S(q, 2^w - t) = \sum_{j=1}^{q-1} \frac{1}{q \cdot \zeta^{j(2^w-t)}} \prod_{v=0}^{w-1} (1 - \zeta^{j \cdot 2^v}).$$

Proof. It is clear that

$$\prod_{v=0}^{w-1} (1 - x^{2^v}) = \sum_{n=0}^{2^w-1} (-1)^{D(n)} \cdot x^n.$$

Hence, for $1 \leq t \leq q$, we have

$$S(q, 2^w - t) = \frac{1}{2\pi i} \int_{|x|=1/2} \frac{\prod_{v=0}^{w-1} (1 - x^{2^v})}{1 - x^q} \cdot \frac{dx}{x^{2^w-t+1}}.$$

Thus if we move the contour to a large circle $|x| = R$, record the residues at the qth roots ζ^j (j=1, 2, $\cdots$, q-1), and let $R \to \infty$, then by the integrand being $O(\frac{1}{x^2})$ at ∞, we obtain

$$\begin{aligned} S(q, 2^w - t) &= -\sum_{j=1}^{q-1} \text{Res}\left(\frac{1}{(1-x^q)x^{2^w-t+1}} \cdot \prod_{v=0}^{w-1}(1-x^{2^v}), \zeta^j\right) \\ &= \sum_{j=1}^{q-1} \frac{1}{q\cdot\zeta^{j(2^w-t)}} \cdot \prod_{v=0}^{w-1}(1-\zeta^{j\cdot 2^v}) \end{aligned}$$

This completes the proof of Lemma 2. ■

Lemma 3 *Let p be an odd prime, 2 a primitive root of $q = p^\alpha \quad (\alpha \geq 1)$, and $1 \leq t \leq p^\alpha$, $k \geq 0$, $r \geq 1$. Then*

$$S(p^\alpha, 2^{\phi(p^\alpha)k+r} - t) = p^k \cdot S(p^\alpha, 2^r - t)$$

where $\phi(n)$ is the Euler totient function.

Proof. By Lemma 2, we have

$$S(p^\alpha, 2^{\phi(p^\alpha)k+r} - t) = \sum_{j=1}^{q-1} \frac{1}{q\cdot\zeta^{j(2^r-t)}} \cdot \left(\prod_{v=0}^{\phi(p^\alpha)-1}(1-\zeta^{2^v})\right)^k \cdot \prod_{v=0}^{r-1}(1-\zeta^{j\cdot 2^v})$$

$$= \left(\prod_{\substack{1\leq v\leq p^\alpha \\ (v,p)=1}}(1-\zeta^v)\right)^k \cdot S(p^\alpha, 2^r - t) = p^k \cdot S(p^\alpha, 2^r - t).$$

This completes the proof of Lemma 3. ■

Lemma 4 *Let p be an odd prime and 2 a primitive root of $p^\alpha (\alpha \geq 1)$. Then for any $k \geq 1$ we have,*

$$S(p^\alpha, 2^{\phi(p^\alpha)k} - t) = \begin{cases} p^k - p^{k-\alpha} & \textit{if} \quad t = 1, \\ -p^{k-\alpha} & \textit{if} \quad 2 \leq t \leq p^\alpha. \end{cases}$$

Proof. It is clear from Lemma 2.

■

3 3. q=3, Proofs of Theorem 1 and Corollary 1

The proof of Theorem 1 By induction on k.

(I) $k = 1$, it is easy to see that: $S_1(3,4) = -2$, $S_2(3,5) = 0$, $S_0(3,6) = 3$, $S_1(3,7) = -3$; $S_2(3,8) = -1$, $S_0(3,9) = 4$, $S_1(3,10) = -2$, $S_2(3,11) = -2$, $S_0(3,12) = 5$, $S_1(3,13) = -3$, $S_2(3,14) = -3$, $S_0(3,15) = 6$. So the inequalities of Theorem 1 are true.

(II) Suppose that for any $\leq k$, the inequalities of Theorem 1 are true.

(III) Now we consider the case $k+1$, and we will prove the following results:

(1) If $2^{2(k+1)} \leq m < 2^{2(k+1)+1}$, then (1.1) $2\cdot 3^k \leq T_0(m) \leq 3^{k+1}$; (1.2) $-3^{k+1} \leq T_1(m) \leq -3^k - 1$; (1.3) $-3^k + 1 \leq T_2(m) \leq 0$.

(2) If $2^{2(k+1)+1} \leq m < 2^{2(k+1)+2}$, then (2.1) $3^{k+1} + 1 \leq T_0(m) \leq 2\cdot 3^{k+1}$; (2.2) $-3^{k+1} \leq T_1(m) \leq -2\cdot 3^k$; (2.3) $-3^{k+1} \leq T_2(m) \leq -1$.

First we prove the case (1), suppose that $2^{2(k+1)} \leq m < 2^{2(k+1)+1}$, and

$$m = 2^{2(k+1)} + n, \quad 0 \leq n < 2^{2(k+1)}.$$

(A) If $n = 0$, then $S_1(3,m) = S(3, 2^{2(k+1)} - 3) - S(3,0) = -3^k - 1$. So the inequality of (1.2) is true.

(B) If $n = 1$, then $S_2(3,m) = S(3, 2^{2(k+1)} - 2) - S(3,1) = -3^k + 1$. So the inequality of (1.3) is true.

(C) If $n = 2$, then $S_0(3, m) = S(3, 2^{2(k+1)} - 1) - S(3, 2) = 3^{k+1} - 3^k + 1$. So the inequality of (1.1) is true.

(D) If $n \geq 3$, then $S(3, 2^{2(k+1)} + n) = S(3, 2^{2(k+1)} - t) - S(3, n)$.

If $n \equiv 0 (\bmod 3)$, then $S(3, m) = S_1(3, m) = S_1(3, 2^{2(k+1)} - 3) - S_0(3, n)$. Using $2 \leq S_0(3, n) \leq 2 \cdot 3^k$ and $S_1(3, 2^{2(k+1)} - 3) = -3^k$, we have

$$-3^k - 2 \cdot 3^k \leq S_1(3, m) \leq -3^k - 1,$$

so the inequality of (1.2) is true.

If $n \equiv 1 (\bmod 3)$, then $S(3, m) = S_2(3, m) = S_2(3, 2^{2(k+1)} - 2) - S_1(3, n)$. Using $-3^k \leq S_1(3, n) \leq -1$ and $S_2(3, 2^{2(k+1)} - 2) = -3^k$, we have

$$-3^k \leq S_2(3, m) \leq -3^k - (-3^k),$$

so the inequality of (1.3) is true.

If $n \equiv 2 (\bmod 3)$, then $S(3, m) = S_0(3, m) = S_0(3, 2^{2(k+1)} - 1) - S_2(3, n)$. Using $-3^k \leq S_2(3, n) \leq 0$ and $S_0(3, 2^{2(k+1)} - 1) = 3^{k+1} - 3^k$, we have

$$2 \cdot 3^k \leq S_0(3, m) \leq 3^{k+1},$$

so the inequality of (1.1) is true.

Secondly, the proof of case (2) is similar to the proof of case (1). This completes the proof of Theorem 1. ■

The proof of Corollary 1 This follows from Theorem 1. ■

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