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The VIM for solving a nonlinear inverse parabolic problem

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Abstract. In this paper, we will use the variational iteration method (VIM) for the determination of unknown coefficients in an inverse heat conduction problem (IHCP). The VIM, which is a modified general Lagrange multiplier method, has been attracted a lot of attention of the researchers for solving different problems. Applying this technique, a rapid convergent sequence to the exact solution is produced. Moreover, this method does not require any discretization, linearization or small perturbation. Therefore it can be considered as an efficient method to solve the various kinds of problems. To show the strength of the method, some examples are given.

Keywords: IHCPs, VIM, Convergent sequence, Lagrange multiplier, Exact solution.

1. Introduction

Inverse heat conduction problems (IHCPs) arise in many important scientific and technological fields. Hence analysis, design implementation and testing of inverse algorithms are also great scientific and technological interest [2]. Till now, various methods have been developed for the analysis of IHCPs [1-13]. When the radiation of heat from a solid is considered to pass through a nonparticipating media, the heat flux is often taken to be proportional to the difference of the boundary temperature of the solid to the fourth power and the temperature of the surrounding to the fourth power [14]. When the thermo-physical properties are independent of position and temperature, the heat transfer problem in this situation may be derived, in the dimensional space and time, as [1]:

$$u_t(x,t) = u_{xx}(x,t); \qquad 0 < x < 1, \quad 0 < t < T, \tag{1}$$

$$u(x,0) = r(x); \qquad 0 \le x \le 1, \tag{2}$$

$$u_x(0,t) = \phi(u(0,t)) + \zeta(t); \qquad 0 \le t \le T, \tag{3}$$

$$u_x(1,t) = \psi(u(1,t)) + \zeta(t); \qquad 0 \le t \le T, \tag{4}$$

with overspecified conditions:

$$u(x_i,t) = g_i(t); \qquad 0 < x_i < 1, \; i = 1,2, \; 0 \le t \le T, \tag{5}$$

where T is the final time, $r(x)$ is the initial temperature of solid, $x_i; i = 1,2$ are the locations of interior sensors recording the temperature measurements $g_i(t); i = 1,2$ and $\phi(u(0,t)) + \zeta(t)$ and $\psi(u(1,t)) + \eta(t)$ represent a general radiation law. In this context the functions $\zeta(t)$ and $\eta(t)$ are known heat fluxes arriving to the surfaces at $x = 0$ and $x = 1$, respectively, and the nonlinear terms $\phi(u(0,t))$ and $\psi(u(1,t))$ are unknown functions to be determined. The problem given by equations (1) and (2) is called the characteristic Chauchy problem and the problem given by equations (1), (3), (4) and (5) is called the non-characteristic Chauchy problem. The unique solvability of the problem (1)-(5) can be found in [1]. This problem has been solved by the finite difference method in [1].

In this work, we apply the VIM to construct a solution to the problem (1)-(5). The VIM was first suggested by Ji-Huan He [15-22]. This method is based on the use of Lagrange multipliers for the identification of optimal values of parameters in a functional. This method construct a rapidly convergent sequence to the exact solution. Moreover, VIM does not require any discretization, linearization or small perturbation. This method is effectively, convenience and accurate. Thus, it has been extensively applied to various kinds of linear and nonlinear problems [23-27].

The organization of the paper is as follows: In Section 2, analysis and application of VIM are presented. In Section 3, some examples are given. Section 4 ends this paper with a conclusion.

2. Analysis and application of VIM

Consider the general differential equation:

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$$Lu + Nu = g(t),$$

where L and N are linear and nonlinear operators, respectively, and $g(t)$ is an inhomogeneous term. According to VIM, we construct a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t,s)(Lu_n(s) + N\tilde{u}_n(s) - g(s))ds,$$

where λ is a Lagrange multiplier, which can be identified optimally via the variational theory, $\tilde{u}_n$ is a restricted variation, i.e. $\delta\tilde{u}_n = 0$ [15, 24]. Now, we need to determine the Lagrangian multiplier λ. Then by using the determined Lagrangian multiplier and an initial approximation $u_0(t)$, the successive approximations $u_{n+1}(t), n \geq 0,$ of the solutions $u(t)$ will be readily obtained. The convergence of the method is systematically discussed by Tatari and Dehghan [28].

Now, for equation (1), the correction functional can be expressed as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(t,s)(u_{n_s}(x,s) - \tilde{u}_{n_{xx}}(x,t))ds,$$

where $\tilde{u}_n$ is a restricted variation and λ is the Lagrange multiplier.

To find the optimal value of λ, we have:

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^t \lambda(t,s)(u_{n_s}(x,s) - \tilde{u}_{n_{xx}}(x,t))ds = 0,$$

After some calculation, we obtain the following stationary conditions:

$$\lambda'(t,s) = 0,$$

$$1 + \lambda(t,s)|_{s=t} = 0.$$

So, we have:

$$\lambda(t,s) = -1.$$

Therefore, we obtain the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t (u_{n_s}(x,s) - u_{n_{xx}}(x,t))ds. \tag{6}$$

Now, taking $u_0(x,t) = u(x,0)$ as an initial value, we can find the n-order approximate solution $u_n(x,t)$ of (1).

Consequently, for approximating ϕ and ψ, using (6), we can find the solution of (3) and (4) as a convergent sequences, respectively.

3. Test examples

In this Section, to justify the accuracy of the method, some examples are given. These examples are chosen from [1] to demonstrate that the present method is remarkably effective. All the computations are performed on the PC (pentium(R) 4 CPU 3.20 GHz).

Example 1. Consider the following problem:

$$u_t(x,t) = u_{xx}(x,t); \qquad 0 < x < 1, \quad 0 < t < 1, \tag{7}$$

$$u(x,0) = x^2; \qquad 0 \leq x \leq 1, \tag{8}$$

$$u_x(0,t) = \phi(u(0,t)) - \sin(4t^2 + 1); \qquad 0 \leq t \leq 1, \tag{9}$$

$$u_x(1,t) = \psi(u(1,t)) + 2 - \cos(1 + 2t + (1+2t)^2); \qquad 0 \leq t \leq 1, \tag{10}$$

with two overspecified conditions:

$$u(0.4,t) = 0.16 + 2t, \qquad u(0.6,t) = 0.36 + 2t. \tag{11}$$

The exact solution of this problem is in [1].

To solve this problem, we use the equation (6). Starting with $u_0 = x^2$. Substituting u_0 into equation (6), we compute:

$$u_1(x,t) = x^2 + 2t, \tag{12}$$

which is the exact solution. Therefore, from (9) and (12), we obtain:

$$\phi(u_1(0,t)) = \sin(4t^2 + 1),$$

and from (10) and (12), we have:

$$\psi(u_1(1,t)) = \cos(1+2t+(1+2t)^2)\,.$$

Tables 1 and 2, show the comparison between the exact, the VIM solution and the solution of the methods, regularization method (RM) and pseudoinversion method (PM), in reference 1 (tables 1 and 2 in Ref.1 with noiseless data) for $\phi(t)$ and $\psi(t)$, respectively. In tables 1 and 2, n is the iteration number in VIM. Figure 1 presents the comparison between the $|\phi_{Exact} - \phi_{VIM}|$, $|\phi_{Exact} - \phi_{RM}|$, $|\phi_{Exact} - \phi_{PM}|$, $|\psi_{Exact} - \psi$ |, $|\psi_{Exact} - \psi_{RM}|$ and $|\psi_{Exact} - \psi_{PM}|$. In figure 1 , $RMS(\phi)$ [3] and $RMS(\psi)$ for VIM and Ref.1 are given.

t	ϕ_{Exact}	ϕ_{VIM}	$\phi_{\mathrm{Re}f.1}$	$\phi_{\mathrm{Re}f.1}$
	-	$n=1$	RM	PM
0.01	0.8416870	0.8416870	0.831947	0.841687
0.02	0.8423344	0.8423344	0.833978	0.859458
0.03	0.8434106	0.8434106	0.837059	0.899116
0.04	0.8449116	0.8449116	0.840768	0.951001
0.05	0.8468319	0.8468319	0.844373	1.012033
0.06	0.8491638	0.8491638	0.848423	1.069035
0.07	0.8518986	0.8518986	0.852030	0.852030
0.08	0.8550255	0.8550255	0.855993	1.152094
0.09	0.8585321	0.8585321	0.859655	1.171462
0.10	0.8624042	0.8624042	0.863476	1.174237

Table 1.The comparison between exact, VIM and Ref.1 solutions for $\phi(t)$.

t	ϕ_{Exact}	ϕ_{VIM}	$\phi_{\mathrm{Re}f.1}$	$\phi_{\mathrm{Re}f.1}$
	-	$n=1$	RM	PM
0.01	-0.4702761	-0.4702761	-0.472109	-0.470276
0.02	-0.5233722	-0.5751632	-0.524033	-0.520628
0.03	-0.5751632	-0.5751632	-0.577216	-0.571307
0.04	-0.6253685	-0.6253685	-0.626706	-0.618657
0.05	-0.6736996	-0.6736996	-0.675809	-0.665354
0.06	-0.7198621	-0.7198621	-0.721152	-0.709127
0.07	-0.7635570	-0.7635570	-0.765603	-0.751899
0.08	-0.8044825	-0.8044825	-0.805527	-0.791129
0.09	-0.8423364	-0.8423364	-0.843925	-0.828908
0.10	-0.8768179	-0.8768179	-0.877754	-0.862740

Table 2.The comparison between exact, VIM and Ref.1 solutions for $\psi(t)$.

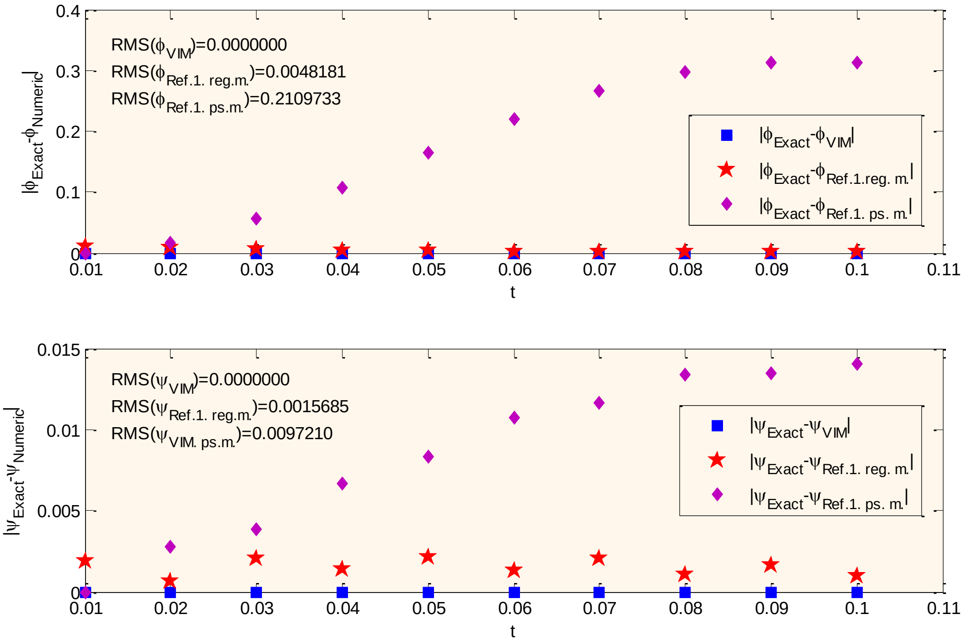

Figure1. The comparison between $|\phi_{Exact}-\phi_{VIM}|$, $|\phi_{Exact}-\phi_{RM}|$ and $|\phi_{Exact}-\phi_{PM}|$ and also between $|\psi_{Exact}-\psi_{VIM}|$, $|\psi_{Exact}-\psi_{RM}|$ and $|\psi_{Exact}-\psi_{PM}|$.

Example 2. Now, consider the problem:

$$u_t(x,t)=u_{xx}(x,t); \qquad 0<x<1, \quad 0<t<1, \tag{13}$$

$$u(x,0)=\cos(x); \qquad 0\le x\le 1, \tag{14}$$

$$u_x(0,t)=\phi(u(0,t))-\exp(-2t); \qquad 0\le t\le 1, \tag{15}$$

$$u_x(1,t)=\psi(u(1,t))-0.2919\exp(-2t)-1.3818\exp(-t); \qquad 0\le t\le 1, \tag{16}$$

with two overspecified conditions:

$$u(0.4,t)=0.9210610\exp(-t), \qquad u(0.6,t)=0.8253356\exp(-t). \tag{17}$$

The exact solution of this problem is in [1].

Consider $u_0(x,t)=(1-t)\cos(x)$, substituting $u_0(x,t)$ into equation (6), we obtain:

$$u_1(x,t)=(1-t)\cos(x).$$

In the same way, we compute:

$$u_2(x,t)=(1-t+\frac{t^2}{2!})\cos(x),$$

$$u_3(x,t)=(1-t+\frac{t^2}{2!}-\frac{t^3}{3!})\cos(x),$$

$$\vdots$$

Therefore

$$u_n(x,t)=\sum_{k=0}^{n}\frac{(-t)^k}{k!}\cos(x). \tag{18}$$

So from (15) and (18), we obtain:

$$\phi_n(u_n(0,t))=\exp(-2t),$$

and from (16) and (18), we have:

$$\psi_n(u_n(1,t))=-0.8415\sum_{k=0}^{n}\frac{(-t)^k}{k!}+0.2919\exp(-2t)+1.3818\exp(-t),$$

where n represent the iteration number.

Tables 3 and 4, show the comparison between the exact, VIM solution and the solution of the methods, regularization method (RM) and pseudoinversion method (PM), in reference 1 (tables 5 and 6 in Ref.1 with noiseless data) for ϕ and ψ , respectively. In tables 3 and 4, n is the iteration number in VIM. Figure 2 presents the comparison between the $|\phi_{Exact}-\phi_{VIM}|$, $|\phi_{Exact}-\phi_{RM}|$, $|\phi_{Exact}-\phi_{PM}|$, $|\psi_{Exact}-\psi \quad |$, $|\psi_{Exact}-\psi_{RM}|$ and $|\psi_{Exact}-\psi_{PM}|$. In figure 1 , $RMS(\phi)$ and $RMS(\psi)$ for VIM and Ref.1 are given.

t	ϕ_{Exact}	ϕ_{VIM}			$\phi_{\mathrm{Ref.1}}$	$\phi_{\mathrm{Ref.1}}$
	-	$n=3$	$n=5$	$n=10$	RM	PM
0.025	0.9512295	0.9512295	0.9512295	0.9512295	0.969150	1.057538
0.050	0.9048374	0.9048374	0.9048374	0.9048374	0.923894	1.079427
0.075	0.8607080	0.8607080	0.8607080	0.8607080	0.879669	1.049798
0.100	0.8187308	0.8187308	0.8187308	0.8187308	0.836813	0.980539
0.125	0.7788008	0.7788008	0.7788008	0.7788008	0.795375	0.887018
0.150	0.7408182	0.7408182	0.7408182	0.7408182	0.754961	0.781291
0.175	0.7046881	0.7046881	0.7046881	0.7046881	0.712522	0.669601
0.200	0.6703200	0.6703200	0.6703200	0.6703200	0.676927	0.601222
0.225	0.6376281	0.6376281	0.6376281	0.6376281	0.643878	0.561693
0.250	0.6065307	0.6065307	0.6065307	0.6065307	0.612630	0.532510

Table 3.The comparison between exact, VIM and Ref.1 solutions for $\phi(t)$.

t	ψ_{Exact}	ψ_{VIM}			$\psi_{\mathrm{Ref.1}}$	$\psi_{\mathrm{Ref.1}}$
	-	$n=3$	$n=5$	$n=10$	RM	PM
0.025	0.8046513	0.8046514	0.8046513	0.8046513	0.806012	0.802910
0.050	0.7780976	0.7780977	0.7780976	0.7780976	0.777164	0.773592
0.075	0.7525255	0.7525266	0.7525255	0.7525255	0.751056	0.747507
0.100	0.7278950	0.7278985	0.7278950	0.7278950	0.726276	0.722918
0.125	0.7041678	0.7041761	0.7041678	0.7041678	0.702502	0.699571
0.150	0.6813070	0.6813242	0.6813071	0.6813070	0.679201	0.676407
0.175	0.6592777	0.6593095	0.6592778	0.6592777	0.660201	0.656673
0.200	0.6380463	0.6381003	0.6380464	0.6380463	0.639507	0.634502
0.225	0.6175808	0.6176667	0.6175809	0.6175808	0.618471	0.613228
0.250	0.5978503	0.5979807	0.5978506	0.5978503	0.598578	0.593464

Table 4.The comparison between exact, VIM and Ref.1 solutions for $\psi(t)$.

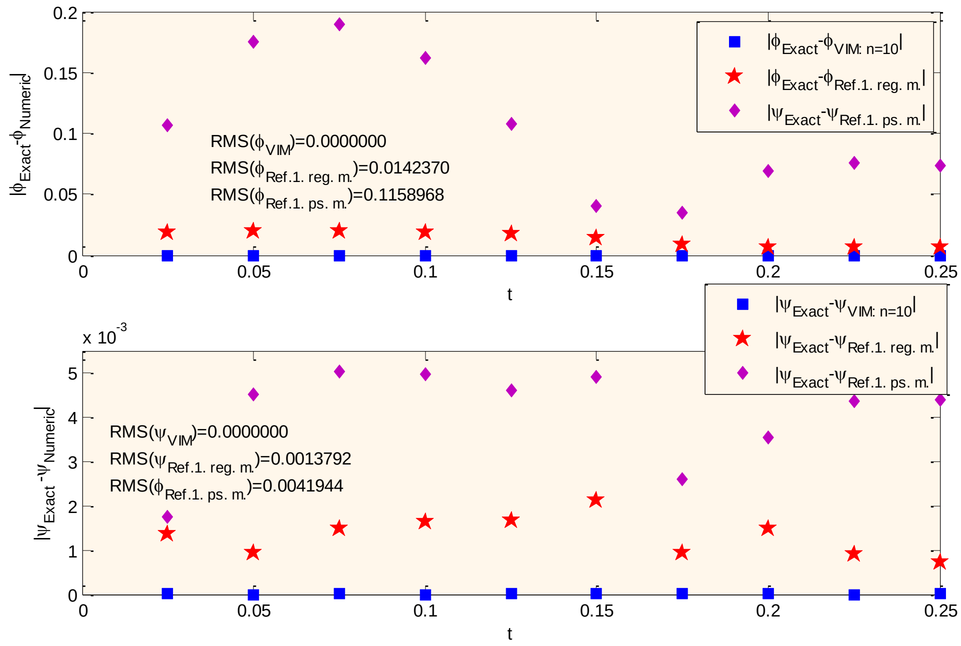

Figure2. The comparison between $|\phi_{Exact}-\phi_{VIM}|$, $|\phi_{Exact}-\phi_{RM}|$ and $|\phi_{Exact}-\phi_{PM}|$ and also between $|\psi_{Exact}-\psi_{VIM}|$, $|\psi_{Exact}-\psi_{RM}|$ and $|\psi_{Exact}-\psi_{PM}|$.

4. Conclusions

In this paper, the variational iteration method was successfully applied to solve the inverse heat conduction problem. This method solves the problem without any discretization of variables. Thus, it is not affected by rounding errors in the computational process. Application of VIM is very easy and straightforward.

Using the VIM, a function series is obtained which converges to the exact solution of the discussed problem. In comparison with the methods in Ref.1, the numerical results show that the VIM is more accurate.

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