ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 9, No. 1, 2014, *pp.* 003-010

A reliable treatment for nonlinear Volterra integro-differential equations

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(*Received october 28, 2012, accepted June 20, 2013*)

Abstract. In this paper, we develop and modify Taylor-series expansion method to approximate a solution of nonlinear Volterra integro-differential equations (IDEs) as well as a solution of a system of nonlinear Volterra equations. By means of the *n*th-order Taylor-series expansion of an unknown function at an arbitrary point, a nonlinear Volterra equations can be converted approximately to a system of nonlinear equations for the unknown function itself and first *n* derivatives. The *n*th-order approximate solution is exact for a polynomial solution of degree equal to or less than *n*. Finally, error estimation of the proposed method is presented. Some numerical Examples are provided to illustrate the accuracy of the method.

Keywords: Integro-differential equations, Approximate solution, Nonlinear Volterra equations, Error estimation, Taylor-series expansion method.

1. Introduction

We consider the following Volterra IDEs of the form

$$D\left(x, y(x), y^{(1)}(x), \ldots, y^{(\alpha)}(x)\right) - \lambda \int_0^x \varphi\left(x, t, y(t), y^{(1)}(t), \ldots, y^{(\beta)}(t)\right) \mathrm{d}t = f(x), x, t \in \Gamma = [0, b], \quad (1)$$

with the initial conditions

$$\sum_{j=0}^{\alpha-1} B_{ij} y^{(j)}(0) = c_i, i = 1,2, \ldots, \alpha, \quad (2)$$

where D and φ are in the following forms

$$D(x, y(x), y^{(1)}(x), \ldots, y^{(\alpha)}(x)) = \sum_{i=0}^{\mu_1} (p_i(x) \prod_{j=0}^{\alpha} (y^{(j)}(x))^{\alpha_{ij}}),$$

$$\varphi(x, t, y(t), y^{(1)}(t), \ldots, y^{(\beta)}(t)) = \sum_{i=0}^{\mu_2} (k_i(x, t) \prod_{j=0}^{\beta} (y^{(j)}(t))^{\beta_{ij}}),$$

and $\alpha_{ij}, \beta_{ij} \in \mathbb{N} \cup \{0\}$.

Also assumed that the functions $k_i(x, t), p_i(x)$ and $f(x)$ are polynomials, otherwise they can be approximate by theirs truncated Taylor-series expansion respect to all arguments at origin.

Problems involving these equation arise frequently in many applied areas which include engineering mechanics, physics, chemistry, astronomy, biology, economics, potential theory, electrostatics, etc, [1- 12]. Taylor-series expansion method is a powerful technique for solving above problems. The Taylor-series expansion approach for solving Volterra integral equations has been presented by Kanwal and Liu [13] and then this method has been extended to Volterra integral equations and differential equations by Sezer [14,15]. A similar approach presented by Yalcinbas and Sezer has been used to solve linear Volterra-Fredholm IDEs, and it is extended to nonlinear Volterra-Fredholm integral equations by Yalcinbas[16], and to high-order linear Volterra-Fredholm differential equation by Yalcinbas and Sezer [17]. In [18,19], Taylor-series expansion approach is used to Abel integral equation. In this study, we will develop and modify Taylor-series expansion method [20] to solve nonlinear Volterra IDEs (1)-(2).

In the Taylor-series expansion method, the solution of (1)-(2) can be consider in the following form

$$y(x) \approx \sum_{j=0}^{N} \frac{y^{(j)}(0)}{j!} x^j, \quad (3)$$

where $y^{(j)}(0)$ are known from (2) for $j = 0,1, \ldots, \alpha - 1$ and for $j = \alpha, \alpha + 1, \ldots, N$, are unknown parameters which have to be determine. Substituting (3) into (1), we can obtain

$$\sum_{i=0}^{N-\alpha} (\Psi_i - \xi_i) x^i + Q(x^\tau) = 0,$$

where $Q(x^\tau)$ is a polynomial of degree greater than $N-\alpha$, Ψ_i is a nonlinear combination of $e_\alpha, e_{\alpha+1}, \dots, e_N$, and ξ_i is known constant. By neglecting $Q(x^\tau)$, the following nonlinear algebraic system is obtained.

$$\Psi_i = \xi_i, i = 0,1,\dots,N-\alpha. \quad (4)$$

By solving the above system, we obtain the unknown parameters $e_\alpha, e_{\alpha+1}, \dots, e_N$.

As we know that the conventional Taylor-series approximation has the appropriate accuracy at the closed neighborhood $x=0$, but for the points far from origin this approximation has not appropriate accuracy, to overcome such difficultly in the following section, we modified Taylor-series expansion approximation at $x=h$.

2. Modified Taylor-series method

Let $\Delta = \{0 = x_0, x_1, \dots, x_n = b\}$, be an equidistance partition of $[0,b]$ where $s = x_{i+1} - x_i, i = 0,1,\dots,n-1$ is the discretization parameter of the partition. In modified Taylor series method, we need to prove the following theorem.

Theorem 1: If the known functions $k_i(x,t), p_i(x)$ and $f(x)$ in (1), are sufficiently differentiable on the interval $0 \le x, t \le b$, and also $k_i(x,t)$ is a separable function, then there exist a linear combination of the independent functions $\psi_i(X)$, so that satisfying in

$$D(X+h, Y(X), Y^{(1)}(X), \dots, Y^{(\alpha)}(X)) - \lambda \int_0^X \varphi(X+h, u+h, Y(u), Y^{(1)}(u), \dots, Y^{(\beta)}(u))\mathrm{d}u + \sum_{i=0}^{\gamma} c_i \psi_i(X) = f(X+h), \quad (5)$$

where $x = X+h, t = u+h$ and $Y(X) = y(X+h)$ is the exact solution.

Proof : For simplicity, the separable kernel $k_i(x,t)$ can be denote by

$$k_i(x,t) = v_i(x) w_i(t), i = 0,1,\dots,m. \quad (6)$$

Substitute (6) into (1) and using the change of variables $x = X+h$, $t = u+h$, we have

$$D(X+h, y(X+h), y^{(1)}(X+h), \dots, y^{(\alpha)}(X+h))\lambda \sum_{i=0}^{\mu_2} (v_i(X+h) \int_0^X w_i(u+h) \prod_{j=0}^{\beta} (y^{(j)}(u+h))^{\beta_{ij}} \mathrm{d}u) + \sum_{i=0}^{\mu_2} v_i(X+h) a_i = f(X+h) \quad (7)$$

where

$$a_i = \int_{-h}^{0} (w_i(u+h) \prod_{j=0}^{\beta} (y^{(j)}(u+h))^{\beta_{ij}}) \mathrm{d}u, i = 0,1,\dots,\mu_2, \quad (8)$$

are unknown constants.

By simplifying and classifying the last term on left hand side of (7), we can write

$$\sum_{i=0}^{m} v_i(X+h) a_i = \sum_{i=0}^{\gamma} c_i \psi_i(X), \quad (9)$$

where $\psi_i(X)$ are known linear independent functions and c_i are unknown constants. Using (9), and by denoting $Y(X) = y(X+h)$, we can rewrite (7), in the form of (5).

To determine c_i, by calculate ith derivation of relation (5) for $i = 0,1,\dots,\gamma-1$, and then by setting X=0, the following algebraic system is obtained

$$\begin{cases} \sum_{i=0}^{\gamma} c_i \psi_i(0) = f(h) - \chi(0), \\ \sum_{i=0}^{\gamma} c_i \psi_{i'}(0) = f'(h) - \chi'(0) + \lambda \phi(0), \\ \vdots \\ \sum_{i=0}^{(\gamma)} c_i \psi_i^{(\gamma-1)}(0) = f^{(\gamma-1)}(h) - \frac{d^{\gamma-1}\chi(X)}{dX^{\gamma-1}}|_{X=0} + \lambda \frac{d^{\gamma-2}\phi(X)}{dX^{\gamma-2}}|_{X=0}, \end{cases}$$

where

$\chi(X) = D(X+h, Y(X), Y^{(1)}(X), \dots, Y^{(\alpha)}$ and $\phi(X) = \varphi(X+h, X+h, Y, Y', \dots, Y^{(\beta)})$.

By solving above algebraic system, the unknown constants can be determined as follows

$$c_i = \xi_i(h, y(h), y^{(1)}(h), \dots, y^{(\alpha+\gamma-1)}(h)), i = 0,1,\dots,\gamma. \quad (10)$$

For solving nonlinear Volterra IDEs (1)-(2) by modified Taylor-series expansion method, let the solution of the converted nonlinear Volterra IDEs (5) is as follows

$$Y(X) \approx \sum_{j=0}^{N} \frac{Y^{(j)}(0)}{j!} X^j, \quad (11)$$

where $Y^{(j)}(0) = y^{(j)}(h)$ are known for $j = 0,1,\dots,\alpha-1$, and for $j = \alpha, \alpha+1, \dots, N$, are unknown parameters which have to be determine. Substituting (11) into (5), we obtain a nonlinear algebraic system. By solving derived system, the unknown parameters (11) is obtained as follows

$$Y^{(\alpha+j)}(0) = \phi_j(h, c_0, c_1, \dots, c_\gamma, y(h), y'(h), \dots, y^{(\alpha+j-1)}(h)), j = 0,1,\dots,N-\alpha. \quad (12)$$

At the first step, let $h = x_0$, then $c_i = 0, i = 0,1,\dots,\gamma$, by (8) and $Y^{(j)}(0) = y^{(j)}(0)$ are known by the initial conditions (2) for $j = 0,1,\dots,\alpha-1$., therefore by using (12) we obtain

$$y(x) \approx \sum_{j=0}^{N} \frac{y^{(j)}(0)}{j!} x^j, \quad (13)$$

where is Taylor-series expansion approximate solution (1)-(2) at $x_0 = 0$.
At the next step, let $h = x_1$, from (13), we have

$$Y^{(j)}(0) = y^{(j)}(x_1), j = 0,1,\dots,\alpha+\gamma-1. \quad (14)$$

Now by using (14) and (10), we obtain the unknown parameters in (12), therefore we have

$$y(x) \approx \sum_{j=0}^{N} \frac{y^{(j)}(x_1)}{j!} (x-x_1)^j, \quad (15)$$

where is Taylor approximate solution (1)-(2) at $x = x_1$.
By repeating the above step for $i = 2,3,\dots,n-1$, we can obtain Taylor-series expansion approximate (1)-(2) at $h = x_i$.

3. Error estimation

In this section, an error function is obtained for the approximate solution of (1)-(2). Let $e(x) = y(x) - y_N(x)$ denoted as the error function of the conventional Taylor-series approximate $y_N(x)$, to approximate the exact solution $y(x)$. Substituting $y(x) = y_N(x) + e(x)$, in (1)-(2) , then we have

$$\sum_{i=0}^{\mu_1} p_i(x) \prod_{j=0}^{\alpha} (y_N^{(j)}(x) + e^{(j)}(x))^{\alpha_{ij}} - \lambda \int_0^x \sum_{i=0}^{\mu_2} k_i(x,t) \prod_{j=0}^{\beta} (y_N^{(j)}(t) + e^{(j)}(t))^{\beta_{ij}} \mathrm{d}t = f(x), \quad (16)$$

and

$$e(0) = 0, e'(0) = 0, \dots, e^{(\alpha-1)}(0) = 0. \quad (17)$$

By using

$$(y_N(t) + e(t))^p = \sum_{k=0}^{p} \binom{p}{k} (y_N(t))^{p-k} (e(t))^k,$$

in Eq. (16), we obtain

$$\sum_{i=0}^{\mu_1} p_i(x) Q_i(e(x), e^{(1)}(x), \dots, e^{(\alpha)}(x), y_N(x), y_N^{(1)}(x), \dots, y_N^{(\alpha)}(x)) - \lambda \int_0^x \sum_{i=0}^{\mu_2} (k_i(x,t) R_i(e(t), e^{(1)}(t), \dots, e^{(\beta)}(t), y_N(t), y_N^{(1)}(t), \dots, y_N^{(\beta)}(t))) \mathrm{d}t = f(x), \quad (18)$$

where

$$Q_i(e(x), e^{(1)}(x), \dots, e^{(\alpha)}(x), y_N(x), y_N^{(1)}(x), \dots, y_N^{(\alpha)}(x)) = \prod_{j=0}^{\alpha} \varphi_{ij}(x, \alpha_{ij}),$$
$$R_i(e(t), e^{(1)}(t), \dots, e^{(\beta)}(t), y_N(t), y_N^{(1)}(t), \dots, y_N^{(\beta)}(t)) = \prod_{j=0}^{\beta} \varphi_{ij}(t, \beta_{ij}),$$

and

$$\varphi_{ij}(z,\mu) = \sum_{k=1}^{\mu} \binom{\mu}{k} (y_N(z))^{\mu-k} (e(z))^k.$$

Now suppose that $k_i(x,t) = v_i(x) w_i(t)$, by using theorem 1. the problem (18) converted to following nonlinear Volterra IDEs

$$\sum_{i=0}^{\mu_1} p_i(X+h) Q_i(E(X), E^{(1)}(X), \dots, E^{(\alpha)}(X), Y_N(X), Y_N^{(1)}(X), \dots, Y_N^{(\alpha)}(X))$$
$$-\lambda \int_0^X \sum_{i=0}^{\mu_2} (k_i(X+h, u+h) R_i(E(u), E^{(1)}(u), \dots, E^{(\beta)}(u), \mathrm{Y}_N(u), Y_N^{(1)}(u), \dots, Y_N^{(\beta)}(u)) \mathrm{d}u + \sum_{i=0}^{\gamma} d_i \psi_i(X) = f(X+h), \quad (19)$$

where $E(X) = e(X+h)$. Now, for determining the unknown constants $d_0, \dots, d_\gamma$, we need to take ξth derivative of relation (19) for $\xi = 0, \dots, \gamma-1$, and then by setting X=0, we will obtain the following algebraic system

$$\begin{cases} \sum_{i=0}^{\gamma} d_i \psi_i(0) = f(h) - \sum_{i=0}^{\mu_1} \chi_i^{(1)}(0), \\ \sum_{i=0}^{\gamma} d_i \psi_i^{(1)}(0) = f^{(1)}(h) - \sum_{i=0}^{\mu_1} \frac{d\chi_i^{(1)}(0)}{dX}|_{X=0} + \lambda \sum_{i=0}^{\mu_2} \chi_i^{(2)}(0), \\ \vdots \\ \sum_{i=0}^{\gamma} d_i \psi_i^{(\gamma-1)}(0) = f^{(\gamma-1)}(h) - \sum_{i=0}^{\mu_1} \frac{d^{\gamma-1}\chi_i^{(1)}(X)}{dX^{\gamma-1}}|_{X=0} + \sum_{i=0}^{\mu_2} \frac{d^{\gamma-2}\chi_i^{(2)}(X)}{dX^{\gamma-2}}|_{X=0}, \end{cases} \quad (20)$$

where

$$\chi_i^{(1)}(X) = p_i(X+h) Q_i(E(X), E^{(1)}(X), \dots, E^{(\alpha)}(X), Y_N(X), Y_N^{(1)}(X), \dots, Y_N^{(\alpha)}(X)),$$
$$\chi_i^{(2)}(X) =, k_i(X+h, X+h) R_i(E(X), E^{(1)}(X), \dots, E^{(\beta)}(X), Y_N(X), Y_N^{(1)}(X), \dots, Y_N^{(\beta)}(X)).$$

By solving the above algebraic system, the unknown constants can be determined as follows

$$d_i = \zeta_i(h, e(h), e^{(1)}(h), \dots, e^{(\alpha+\gamma-1)}(h), y(h), y^{(1)}(h), \dots, y^{(\alpha+\gamma-1)}(h)), i = 0,1,\dots,\gamma. \quad (21)$$

The nonlinear Volterra IDEs (19), can be solved by modified Taylor-series expansion method introduced in section 2., therefore Taylor-series expansion of the error function $e(x)$ can be determined at $x = x_i, i = 0,1,\dots,n-1$.

4. Solution system of nonlinear Volterra IDEs via Taylor-series method

In this section, we apply the modified Taylor-series method to a system of nonlinear Volterra IDEs

$$\sum_{s=1}^{\mu_1} D_{rs}(x, y_1(x), y_1^{(1)}(x), \dots, y_1^{(\alpha_{rs1})}(x), \dots, y_m(x), y_m^{(1)}(x), \dots, y_m^{(\alpha_{rsm})}(x)) \quad (22)$$
$$\lambda_r \sum_{s=1}^{\mu_2} \int_0^x \varphi_{rs}(x, t, y_1(t), y_1^{(1)}(t), \dots, y_1^{(\beta_{rs1})}(t), \dots, y_m(t), y_m^{(1)}(t), \dots, y_m^{(\beta_{rsm})}(t))\mathrm{d}t = f_r(x), x, t \in \Gamma = [0, b],$$

for $r = 1,2,\dots,m$, with the initial conditions

$$\sum_{k=1}^{m} \sum_{j=0}^{\alpha_k-1} B_{ijk} y_k^{(j)}(0) = c_i, i = 1,2,\dots,\alpha, \quad (23)$$

where α_k is order of $y_k(x)$, $\alpha = \sum_{k=1}^{m} \alpha_k$ and

$$D_{rs}(x, y_1(x), y_1^{(1)}(x), \dots, y_1^{(\alpha_{rs1})}(x), \dots, y_m(x), y_m^{(1)}(x), \dots, y_m^{(\alpha_{rsm})}(x)) = p_{rs}(x) \prod_{i=1}^{m} (\prod_{j=0}^{\alpha_{rsi}} (y_i^{(j)}(x))^{\alpha_{rsij}}),$$
$$\varphi_{rs}(x, t, y_1(t), y_1^{(1)}(t), \dots, y_1^{(\beta_{rs1})}(t), \dots, y_m(t), y_m^{(1)}(t), \dots, y_m^{(\beta_{rsm})}(t)) = k_{rs}(x, t) \prod_{i=1}^{m} (\prod_{j=0}^{\beta_{rsi}} (y_i^{(j)}(t))^{\beta_{rsij}}).$$

For obtain the Taylor-series approximate for (22)-(23), we need to the following theorem.

Theorem 2: Let, the functions $f_i(x), k_{rs}(x,t), p_{rs}(x)$, in (22) are sufficiently differentiable on the interval $0 \le x, t \le b$, and $k_{rs}(x,t)$ is a separable function, then there exist linear independent functions $\psi_0(X), \dots, \psi_\gamma(X)$ and constants $c_0, \dots, c_\gamma$ such that, $Y_1(X), Y_2(X), \dots, Y_m(X)$ is the exact solution of following nonlinear Volterra IDEs system

$$\sum_{s=1}^{\mu_1} D_{rs}(X+h, Y_1(X), Y_1^{(1)}(X), \dots, Y_1^{(\alpha_{rs1})}(X), \dots, Y_m(X), Y_m^{(1)}(X), \dots, Y_m^{(\alpha_{rsm})}(X)) + \sum_{k=0}^{\gamma} \delta_{rk}\varphi_k(X) -$$
$$\lambda_r \sum_{s=1}^{\mu_2} \int_0^X \varphi_{rs}(X+h, u+h, Y_1(u), Y_1^{(1)}(u), \dots, Y_1^{(\beta_{rs1})}(u), \dots, Y_m(u), Y_m^{(1)}(u), \dots, Y_m^{(\beta_{rsm})}(u))\mathrm{d}u = f_r(X+h), x, t \in \Gamma = [0, b], \quad (24)$$

for $r = 1,2,\dots,m$, where $Y_i(X) = y_i(X+h)$, $x = X+h$ and δ_{ij}, is equal 0, 1, or -1.

Proof : It's similar to prove theorem 1. Here, to obtain the Taylor-series expansion approximate by proposed method, we suppose that the solution of the converted system of nonlinear Volterra IDEs (24), is as follows

$$Y_i(X) \approx \sum_{j=0}^{N} \frac{Y_i^{(j)}(0)}{j!} X^j, i = 1,2,\dots,m, \quad (25)$$

where for $i = 1,2,\dots,m$, $Y_i^{(j)}(0) = y_i^{(j)}(h), j = 0,1,\dots,\alpha_i - 1$ are known and $Y_i^{(j)}(0), j = \alpha_i, \alpha_i+1, \dots, N$ are unknown parameters which have to be determine. Substituting (25) into (24), we can obtain

$$\sum_{j=0}^{N-\sigma} (\Psi_{ij} - \xi_{ij})x^j + Q(x^{\tau_i}) = 0, i = 1,2,\dots,m, \quad (26)$$

where $\sigma = \min\alpha_j, j = 1,2,\dots,m$ and $Q(x^{\tau_i})$ is a polynomial of degree greater than $N - \alpha_i$, Ψ_{ij} is a nonlinear combination of $Y_i^{(\alpha_i)}(0), Y_i^{(\alpha_i+1)}(0), \dots, Y_i^{(\alpha_i+j)}(0)$ for $i = 1,2,\dots,m$, and ξ_{ij} is known constant. By neglecting $Q(x^{\tau_i})$, the following nonlinear algebraic system is obtained.

$$\begin{cases} \Psi_{10} = \xi_{10}, \\ \vdots \\ \Psi_{m0} = \xi_{m0}, \\ \vdots \\ \Psi_{1(N-\sigma)} = \xi_{1(N-\sigma)}, \\ \vdots \\ \Psi_{m(N-\sigma)} = \xi_{m(N-\sigma)}. \end{cases} \quad (27)$$

By solving above system, the unknown parameters $Y_i^{(j)}(0), j = \alpha_i, \alpha_i+1, \dots, N, i = 1,2,\dots,m$, in (25) respect to $h, c_k, k = 0,1,\dots,\gamma$ and $Y_i^{(j)}(0), j = 0,1,\dots,\alpha_i - 1, i = 1,2,\dots,m$ is obtained.

In consequently, we can obtain the Taylor-series expansion approximate (22)-(23) at $x = x_i, i = 0,1,\dots,n-1$ by similar procedure which introduced in the section 2. Also we can obtain the error estimate of our method by similar manner which introduced in section 3.

5. Application

In order to illustrate the performance and accuracy of the proposed methods in the solution of nonlinear Volterra IDEs and also system of nonlinear IDEs, we consider the following Examples.

Example 1. Consider the following Volterra IDEs

$$y'''(x) + x^2 y'(x) + \sin(x) y(x) = f(x) + \int_0^x \left(\sin(x) y(t) + (2x+t) y'(t) + (t^2 - x) y''(t)\right) dt, x, t \in \Gamma = [0,5], \quad (28)$$

under the initial conditions $y(0) = y'(0) = y''(0) = 1$, that $f(x) = x + 1 + \sin(x)$, and $y(x) = e^x$ is the exact solution.

By using theorem 1. nonlinear Volterra IDEs (28) converts into the following form

$$Y'''(X) + (X+h)^2 Y'(X) + \sin(X+h) Y(X) = f(X+h) + c_0 \sin(X+h) + c_1 X + c_2 \quad (29)$$
$$+ \int_0^X (\sin(X+h) Y(u) + (2X + u + h)) Y'(u) + ((u+h)^2 - (X+h)) Y''(u)) \mathrm{d}u,$$

where c_0, c_1, c_2 are as follows

$$\begin{cases} c_0 = \csc(h)(-\sin(h) - (2\cos(h) + \sin(h)) y(h) + (-3 + 2\cos(h) \\ 0.7cm - \sin(h)) y'(h) + (2 - h + \sin(h)) y''(h) + h y^{(3)}(h) + y^{(5)}(h)), \\ \\ c_1 = -1 - \cos(h) - c_1 \cos(h) + (\cos(h) - \sin(h)) y(h) \\ 4.0cm + (-h + \sin(h)) y'(h) + h y''(h) + y^{(4)}(h), \\ \\ c_2 = -1 - h - \sin(h) - c_1 \sin(h)) + \sin(h) y(h) + h^2 y'(h) + y^{(3)}(h). \end{cases} \quad (30)$$

First of all, we expand $\sin(X+h)$ with Taylor's expansion at $X = 0$ in (29). In the same procedure given in the section (2),(4) for $N = 15, s = 0.1$, we obtain the approximate solution and error estimation in the interval [0,10]. The numerical results are given in Table 1.

Table 1. Results for Example 1

x_i	Absolute error	Error estimation
1.0	5.424823E-21	5.424821E-21
2.0	4.939478E-19	4.939475E-19
3.0	15.945844E-18	5.945841E-18
4.0	3.176782E-17	3.176780E-17
5.0	1.134140E-16	1.134140E-16
6.0	3.540182E-16	3.540180E-16
7.0	1.083947E-15	1.083946E-15
8.0	1.83.311980E-15	3.311988E-15
9.0	1.082753E-14	1.082753E-14
10 2	3.463834E-14	3.463832E-14

Results show that our modified Taylor-series methods and theirs error estimation can be used for broad intervals.

Example 2. Consider the following nonlinear Volterra IDEs [21, 22, 23]

$$y'(x) + \int_0^x 3\cos(x-t) y^2(t) \mathrm{d}t = 2\sin x \cos x, \quad (31)$$

with the initial condition $y(0) = 1$, and the exact solution $y(x) = \cos x$.

By using Theorem 1., we can convert (31) to the following nonlinear Volterra IDEs

$$Y'(X) + c_0 \cos X + c_1 \sin X + \int_0^X 3\cos(X-u) Y^2(u) \mathrm{d}u = 2\sin(X+h)\cos(X+h), 0.5cm \quad (32)$$

Where

$$\begin{cases} c_0 = 2\sin(h)\cos(h) - y'(h), \\ \\ c_1 = 3\cos(2h) - y''(h) - 3y^2(h). \end{cases} \quad (33)$$

Table 2. shows exact error for some points using our method for $s = 0.1$, $N = 8$ and compare the results with Adomian's method, BPFs method, and also by method given in [23]. Table 2. show that our results are considerable accurate.

Table 2. Results for Example 2.

x_i	Our method	Adomian's method	BPFs method(m=16)	Method in [23]
0.1	0	5.32E-15	1.37E-4	1.16E-3
0.2	2.00E-15	2.36E-4	4.28E-3	1.57E-3
0.3	2.01E-1	3.49E-4	5.05E-3	1.23E-3
0.4	8.78E-15	1.04E-4	2.62E-3	2.20E-4
0.5	2.55E-13	5.35E-4	1.54E-2	1.34E-3
0.6	5.82E-13	2.74E-3	3.63E-3	7.34E-4
0.7	1.14E-12	9.51E-3	1.19E-2	7.66E-4
0.8	2.01E-12	2.90E-2	1.37E-2	1.30E-3
0.9	2.28E-12	7.44E-2	4.38E-3	2.18E-3
1.0	5.03E-12	1.76E-1	2.66E-2	2.19E-3

Example 3. Consider the following nonlinear system of IDE's

$$\begin{cases} y''_1(x) + \frac{1}{2}y_2^2(x) - \frac{1}{2}\int_0^x (y_1^2(t) + y_2^2(t))dt = 1 - \frac{1}{3}x^3, 5.5cm \\ y''_2(x) + xy_1(x) - \frac{1}{4}\int_0^x (y_1^2(t) - y_2^2(t))dt = -1 + x^2, 5.5cm \end{cases} \quad (34)$$

with the initial conditions $y_1(0) = 1, y'_1(0) = 2, y_2(0) = -1, y'_2(0) = 0$. The exact solution of this problem is given in Ref.[24] as $(y_1(x), y_2(x)) = (x + e^x, x - e^x)$. Now, by using Theorem 2., we can convert system (34) into the following nonlinear Volterra IDE's system,

$$\begin{cases} Y''_1(X) + \frac{1}{2}Y_2^2(X) + c_1 - \frac{1}{2}\int_0^X (Y_1^2(u) + Y_2^2(u))dt = 1 - \frac{1}{3}(X+h)^3, 5.5cm \\ Y''_2(X) + (X+h)Y_1(X) + c_2 - \frac{1}{4}\int_0^X (Y_1^2(u) - y_2^2(u))dt = -1 + (X+h)^2, 5.5cm \end{cases} \quad (35)$$

where

$$\begin{cases} c_0 = 1 - \frac{1}{3}h^3 - y''_1(h) - \frac{1}{2}y'^2_2(h), \\ c_1 = -1 + h^2 - hy_1(h) - y''_2(h). \end{cases} \quad (36)$$

By using proposed methods, we obtain the approximate solution and error estimation in the interval [0,1]. The numerical results are given in the Table 3. for $s = 0.1$, and $N = 9$.

Table 3. Results for Example 3.

x_i	$e(y_1(x_i))$	$Ee(y_1(x_i))$	$e(y_2(x_i))$	$Ee(y_2(x_i))$
0.1	4.44×10^{-16}	2.53×10^{-19}	2.22×10^{-16}	2.53×10^{-16}
0.2	2.22×10^{-16}	1.71×10^{-17}	2.22×10^{-16}	1.77×10^{-17}
0.3	4.44×10^{-16}	7.94×10^{-17}	2.22×10^{-16}	8.32×10^{-17}
0.4	4.44×10^{-16}	2.15×10^{-16}	4.44×10^{-16}	2.29×10^{-16}
0.5	1.33×10^{-15}	4.47×10^{-16}	6.66×10^{-16}	4.92×10^{-16}
0.6	1.78×10^{-15}	7.89×10^{-16}	1.11×10^{-15}	9.12×10^{-16}
0.7	2.66×10^{-15}	1.24×10^{-15}	1.77×10^{-15}	1.53×10^{-15}
0.8	3.11×10^{-15}	1.75×10^{-15}	2.66×10^{-15}	2.42×10^{-15}
0.9	4.44×10^{-15}	2.24×10^{-15}	3.77×10^{-15}	3.61×10^{-15}
1.0	6.22×10^{-15}	2.54×10^{-15}	5.55×10^{-15}	5.19×10^{-15}

Example 4. Consider the following nonlinear system of IDE's

$$\begin{cases} y'''_1(x) = x - y'_1(x) - \int_0^x (y''^2_1(t) + y''_2(t))dt, 5.5cm \\ y'''_2(x) = \sin x(1 + \frac{1}{2}\sin x) + \int_0^x (y''_1(t)y_2(t))dt, 5.5cm \end{cases} \quad (37)$$

with the initial conditions $y_1(0) = y''_1(0) = 0, y_2(0) = 1$ and $y_2(0) = 1, y'_2(0) = 0, y''_2(0) = -1$. The exact solution of this problem is given in Ref.[25] as $(y_1(\chi), y_2(\chi)) = (\sinh(x), \cosh(x))$.
By using theorem 2. we have

$$\begin{cases} Y'''_1(X) = X - Y'_1(X) - \int_0^X (Y''^2_1(u) + Y''_2(u))du, 5.5cm \\ Y'''_2(X) = \sin X(1 + \frac{1}{2}\sin X + \int_0^X (Y''_1(u)y_2(u))du, 5.5cm \end{cases} \quad (38)$$

where

$$\begin{cases} c_0 = y'''_1(h) - h + y'_1(h), \\ c_1 = y'''_2(h) - \sinh(1 + \frac{1}{2}\sinh). \end{cases}$$

The numerical results in the Table 3. are given for $s = 0.1$, and $N = 10$ is obtain. Table 4. shows that proposed method can be used for broad intervals.

Table 4. Results for Example 4.

x_i	$e(y_1(x_i))$	$Ee(y_1(x_i))$	$e(y_2(x_i))$	$Ee(y_2(x_i))$
0.5	0	1.71×10^{-18}	1.11×10^{-16}	1.64×10^{-19}
1.0	2.22×10^{-16}	2.93×10^{-17}	2.22×10^{-16}	9.71×10^{-19}
1.5	1.11×10^{-16}	1.37×10^{-16}	5.55×10^{-16}	9.81×10^{-18}
2.0	1.11×10^{-16}	4.10×10^{-16}	9.44×10^{-16}	3.57×10^{-17}
2.5	9.99×10^{-16}	1.00×10^{-15}	1.55×10^{-15}	6.13×10^{-17}
3.0	2.80×10^{-15}	2.16×10^{-15}	1.10×10^{-15}	9.96×10^{-17}
3.5	6.22×10^{-15}	4.17×10^{-15}	2.22×10^{-15}	1.85×10^{-16}
4.0	1.12×10^{-14}	7.23×10^{-15}	9.99×10^{-16}	9.77×10^{-16}
4.5	1.80×10^{-14}	1.13×10^{-14}	2.47×10^{-15}	2.94×10^{-15}
5.0	2.62×10^{-14}	1.60×10^{-14}	9.27×10^{-15}	6.91×10^{-15}

6. Conclusion

In this research, Taylor-series method has been developed for approximating the solution of nonlinear Volterra IDEs. The major advantages of the modified Taylor-series method are simplification and easy-to-apply in programming, and applicable to high order of nonlinear Volterra IDEs, moreover the proposed method is a powerful procedure for solving nonlinear Volterra IDEs on broad intervals. The computed results show that our method is much accurate in comparison with Adomian's method, BPFs method, and the given method in [23].

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