Relative Order of Functions of Several Complex Variables
Analytic in the Unit Polydisc

Ratan Kumar Dutta
Department of Mathematics, Patulia High School, Patulia, Kolkata-700119, West Bengal, India
E-mail: ratan_3128@yahoo.com
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Abstract. Throughout the paper we consider relative order of functions of several complex variables analytic in the unit poly disc with respect to an entire function and after proving several theorems, we show that relative order of analytic function and its partial derivatives are same.

Keywords: Analytic function, entire function, relative order, unit polydisc, property (R).

1. Introduction

A function \( f \) analytic in the unit disc \( U : \{z : |z| < 1\} \), is said to be of finite Nevanlinna order [7] (Juneja and Kapoor 1985) if there exists a number \( \mu \) such that Nevanlinna characteristic function \( T(r, f) \) of \( f \) defined by

\[
T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f(re^{i\theta}) \right| d\theta
\]

satisfies

\[
T(r, f) = (1 - r)^{-\mu}
\]

for all \( r \) in \( 0 < r_0(\mu) < r < 1 \).

The greatest lower bound of all such numbers \( \mu \) is called Nevanlinna order of \( f \). Thus the Nevanlinna order \( \rho(f) \) of \( f \) is given by

\[
\rho(f) = \limsup_{r \to 1} \frac{\log T(r, f)}{-\log(1-r)}.
\]

In [1] Banerjee and Dutta introduced the idea of relative order of an entire function which as follows:

**Definition 1.1.** If \( f \) be analytic in \( U \) and \( g \) be entire, then the relative order of \( f \) with respect to \( g \), denoted by \( \rho_g(f) \) is defined by

\[
\rho_g(f) = \inf \{ \mu > 0 : T_f(r) < T_g \left( \frac{1}{1-r} \right)^{\mu} \text{ for all } 0 < r_0(\mu) < r < 1 \}.
\]

**Note 1.2.** When \( g(z) = \exp z \) then the Definition 1.1 coincides with the definition of Nevanlinna order of \( f \).

Also in [2] Banerjee and Dutta introduced the idea of relative order of an entire function of two complex variables which as follows:

**Definition 1.3.** Let \( f(z_1, z_2) \) be a non-constant analytic function of two complex variables \( z_1 \) and \( z_2 \) holomorphic in the closed unit poly disc \( P : \{(z_1, z_2) : |z_j| \leq 1; j = 1, 2\} \) and \( g(z_1, z_2) \) be an entire function then relative order of \( f \) with respect to \( g \) denoted by \( \rho_g(f) \) and is defined by
\[ \rho_g(f) = \inf \{ \mu > 0 : F(r_1, r_2) < G \left( \frac{1}{1 - r_1^{\mu}}, \frac{1}{1 - r_2^{\mu}} \right) \text{ for all } 0 < r_0(\mu) < r_1, r_2 < 1 \} . \]

In a recent paper [3] Dutta introduced the following definition.

**Definition 1.4.** Let \( f(z_1, z_2, \ldots, z_n) \) and \( g(z_1, z_2, \ldots, z_n) \) be two entire functions of \( n \) complex variables \( z_1, z_2, \ldots, z_n \), with maximum modulus functions \( F(r_1, r_2, \ldots, r_n) \) and \( G(r_1, r_2, \ldots, r_n) \) respectively, then relative order of \( f \) with respect to \( g \), denoted by \( \rho_g(f) \) and is defined by

\[ \rho_g(f) = \inf \{ \mu > 0 : F(r_1, r_2, \ldots, r_n) < G(r_1^{\mu}, r_2^{\mu}, \ldots, r_n^{\mu}) \text{ for all } r_i \geq R(\mu); i = 1, 2, \ldots, n \} . \]

Also in a paper [4] Dutta introduced the following definition.

**Definition 1.5.** Let \( f(z_1, z_2, \ldots, z_n) = \sum_{m_1, m_2, \ldots, m_n=0}^{\infty} c_{m_1, m_2, \ldots, m_n} z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n} \) be a function of \( n \) complex variables \( z_1, z_2, \ldots, z_n \), holomorphic in the unit polydisc

\[ P = \{(z_1, z_2, \ldots, z_n) : |z_j| \leq 1; j = 1, 2, \ldots, n \} \]

and

\[ F(r_1, r_2, \ldots, r_n) = \max \{| f(z_1, z_2, \ldots, z_n) | : |z_j| \leq r_j; j = 1, 2, \ldots, n \} , \]

be its maximum modulus. Then the order \( \rho \) and lower order \( \lambda \) are defined as

\[ \rho = \lim_{r_1, r_2, \ldots, r_n \to 1} \sup \log \log F(r_1, r_2, \ldots, r_n) \quad \text{inf} \frac{\log(1-r_1)(1-r_2)\ldots(1-r_n)}{\log(1-r_1^{\rho})(1-r_2^{\rho})\ldots(1-r_n^{\rho})} . \]

Now we introduce the following definition.

**Definition 1.6.** Let \( f(z_1, z_2, \ldots, z_n) \) be a non-constant analytic function of several complex variables \( z_1, z_2, \ldots, z_n \), holomorphic in the closed unit polydisc

\[ P : \{(z_1, z_2, \ldots, z_n) : |z_j| \leq 1; j = 1, 2, \ldots, n \} \]

and \( g(z_1, z_2, \ldots, z_n) \) be an entire function then relative order of \( f \) with respect to \( g \) denoted by \( \rho_g(f) \) and defined by

\[ \rho_g(f) = \inf \{ \mu > 0 : F(r_1, r_2, \ldots, r_n) < G(r_1^{\mu}, r_2^{\mu}, \ldots, r_n^{\mu}) \text{ for all } 0 < r_0(\mu) < r_1, r_2 \ldots, r_n < 1 \} \]

where \( G(r_1, r_2, \ldots, r_n) = \max \{| g(z_1, z_2, \ldots, z_n) | : |z_j| = r_j; j = 1, 2, \ldots, n \} \).

**Note 1.7.** When \( g(z_1, z_2, \ldots, z_n) = e^{z_1^2 + \cdots + z_n^2} \) then Definition 1.6 coincides with the Definition 1.5 and if \( n=2 \) then coincide with Definition 1.3.

We require the following definition.

**Definition 1.8.** An entire function \( g(z_1, z_2, \ldots, z_n) \) is said to have the property (R) if for any \( \sigma > 1, \lambda > 0 \) and for all \( r_i \) sufficiently close to 1; \( i = 1, 2, \ldots, n, \)

\[ \left[ G \left( \frac{1}{(1-r_1)^{\sigma}}, \frac{1}{(1-r_2)^{\sigma}}, \ldots, \frac{1}{(1-r_n)^{\sigma}} \right) \right]^2 < G \left( \frac{1}{(1-r_1)^{\lambda}}, \frac{1}{(1-r_2)^{\lambda}}, \ldots, \frac{1}{(1-r_n)^{\lambda}} \right) \]

**Note 1.9.** The function \( g(z_1, z_2, \ldots, z_n) = e^{z_1^2 + \cdots + z_n^2} \) has the property (R) but \( g(z_1, z_2, \ldots, z_n) = z_1z_2 \ldots z_n \) has not.
Throughout we shall assume that \( f, f_1, f_2 \) etc, to be functions analytic in \( P \) and \( g, g_1, g_2 \) etc, are non-constant entire functions of several complex variables. We do no explain standard notations and definitions of analytic functions those are available in [5] and [6].

2. Lemmas
We require the following lemmas.

**Lemma 2.1.** Let \( g(z_1, z_2, \ldots, z_n) \) be an entire function which has the property (R). Then for any positive integer \( n \) and for all \( \sigma > 1, \lambda > 0 \),

\[
\left( G \frac{1}{(1-r_1)^\lambda}, \frac{1}{(1-r_2)^\lambda}, \ldots, \frac{1}{(1-r_n)^\lambda} \right)^\sigma < G \left( \frac{1}{(1-r_1)^\sigma}, \frac{1}{(1-r_2)^\sigma}, \ldots, \frac{1}{(1-r_n)^\sigma} \right)
\]

holds for all \( r_i, 0 < r_i < 1 \) sufficiently close to 1; \( i = 1, 2, \ldots, n \).

The Lemma 2.1 follows from Lemma 2.1 in [3] on replacing \( r_i \) by \( 1 - (1-r_i)^\lambda \), where \( i = 1, 2, \ldots, n \).

**Lemma 2.2.** Let \( g(z_1, z_2, \ldots, z_n) \) be an entire and \( \alpha > 1, 0 < \beta < \alpha \) then

\[
G \left( \frac{\alpha}{(1-r_1)^\lambda}, \frac{\alpha}{(1-r_2)^\lambda}, \ldots, \frac{\alpha}{(1-r_n)^\lambda} \right) > \beta G \left( \frac{1}{(1-r_1)^\lambda}, \frac{1}{(1-r_2)^\lambda}, \ldots, \frac{1}{(1-r_n)^\lambda} \right)
\]

for all \( r_i, 0 < r_i < 1 \) sufficiently close to 1; \( i = 1, 2, \ldots, n \).

The Lemma 2.2 follows from Lemma 2.2 in [3] on replacing \( r_i \) by \( 1 - (1-r_i)^\lambda \), where \( i = 1, 2, \ldots, n \).

3. Sum and Product Theorems

**Theorem 3.1.** Let \( f_1(z_1, z_2, \ldots, z_n) \) and \( f_2(z_1, z_2, \ldots, z_n) \) be analytic in the unit polydisc \( P \) having relative order \( \rho_g(f_1) \) and \( \rho_g(f_2) \) respectively, where \( g(z_1, z_2, \ldots, z_n) \) is an entire function having the property (R). Then

(a) \( \rho_g(f_1 + f_2) \leq \max(\rho_g(f_1), \rho_g(f_2)) \)

and

(b) \( \rho_g(f_1 f_2) \leq \min(\rho_g(f_1), \rho_g(f_2)) \).

The same inequality holds for the quotient. The equality holds in (a) if \( \rho_g(f_1) \neq \rho_g(f_2) \).

**Proof.** We may assume that \( \rho_g(f_1) \) and \( \rho_g(f_2) \) both are finite, because if one of them or both are infinite then inequalities are evident.

Let \( f = f_1 + f_2, \rho_1 = \rho_g(f_1), \rho_2 = \rho_g(f_2) \) and \( \rho_1 \leq \rho_2 \).

For arbitrary \( \varepsilon > 0 \) and for all \( r_i, 0 < r_i < 1 \); \( i = 1, 2, \ldots, n \), sufficiently close to 1, we have

\[
F_1(r_1, r_2, \ldots, r_n) < G \left( \frac{1}{(1-r_1)^{\rho_1+\varepsilon}}, \frac{1}{(1-r_2)^{\rho_1+\varepsilon}}, \ldots, \frac{1}{(1-r_n)^{\rho_1+\varepsilon}} \right)
\]

\[
\leq G \left( \frac{1}{(1-r_1)^{\rho_2+\varepsilon}}, \frac{1}{(1-r_2)^{\rho_2+\varepsilon}}, \ldots, \frac{1}{(1-r_n)^{\rho_2+\varepsilon}} \right)
\]

and

\[
F_2(r_1, r_2, \ldots, r_n) < G \left( \frac{1}{(1-r_1)^{\rho_1+\varepsilon}}, \frac{1}{(1-r_2)^{\rho_1+\varepsilon}}, \ldots, \frac{1}{(1-r_n)^{\rho_1+\varepsilon}} \right).
\]

Now for all \( r_i, 0 < r_i < 1 \); \( i = 1, 2, \ldots, n \) sufficiently close to 1,
\[ F(r_1, r_2, \ldots, r_n) = F_1(r_1, r_2, \ldots, r_n) + F_2(r_1, r_2, \ldots, r_n) \]
\[ \leq 2G \left( \frac{1}{(1-r_1)^{\rho_1+\varepsilon}}, \frac{1}{(1-r_2)^{\rho_2+\varepsilon}}, \ldots, \frac{1}{(1-r_n)^{\rho_n+\varepsilon}} \right) \]
\[ \leq G \left( \frac{3}{(1-r_1)^{\rho_1+\varepsilon}}, \frac{3}{(1-r_2)^{\rho_2+\varepsilon}}, \ldots, \frac{3}{(1-r_n)^{\rho_n+\varepsilon}} \right) \text{ by Lemma 2.2} \]
\[ \leq G \left( \frac{1}{(1-r_1)^{\rho_1+3\varepsilon}}, \frac{1}{(1-r_2)^{\rho_2+3\varepsilon}}, \ldots, \frac{1}{(1-r_n)^{\rho_n+3\varepsilon}} \right). \]

Therefore \( \rho \leq \rho_2 - 3\varepsilon. \)

Since \( \varepsilon > 0 \) is arbitrary, so \( \rho \leq \rho_2. \)

Therefore

\[ \rho_2 (f_1 + f_2) \leq \rho_2 = \max (\rho_2(f_1), \rho_2(f_2)) \]

which proves (a).

Next let \( \rho_i < \rho_2 \) and suppose \( \rho_i < \mu < \lambda < \rho_2. \)

Then for all \( r_i, 0 < r_i < 1; i = 1, 2, \ldots, n, \) sufficiently close to 1, we have

\[ F_1(r_1, r_2, \ldots, r_n) < G \left( \frac{1}{(1-r_1)^\mu}, \frac{1}{(1-r_2)^\mu}, \ldots, \frac{1}{(1-r_n)^\mu} \right) \]

and there exist non-decreasing sequence \( \{r_{ik}\}; r_{ik} \to 1_- \text{ as } k \to \infty \) such that

\[ F_2(r_1, r_2, \ldots, r_n) > G \left( \frac{1}{(1-r_{ik})^\xi}, \frac{1}{(1-r_{2k})^\xi}, \ldots, \frac{1}{(1-r_{nk})^\xi} \right) \]

for \( k = 1, 2, \ldots. \)

We see that

\[ G \left( \frac{1}{(1-r_1)^\xi}, \frac{1}{(1-r_2)^\xi}, \ldots, \frac{1}{(1-r_n)^\xi} \right) > 2G \left( \frac{1}{(1-r_1)^\mu}, \frac{1}{(1-r_2)^\mu}, \ldots, \frac{1}{(1-r_n)^\mu} \right) \]

for all \( r_i, 0 < r_i < 1; i = 1, 2, \ldots, n, \) sufficiently close to 1.

From (1), (2) and (3) we get

\[ F_2(r_{ik}, r_{2k}, \ldots, r_{nk}) > 2F_1(r_{ik}, r_{2k}, \ldots, r_{nk}) \]

for \( k = 1, 2, \ldots. \)

Therefore

\[ F(r_{ik}, r_{2k}, \ldots, r_{nk}) \geq F_2(r_{ik}, r_{2k}, \ldots, r_{nk}) - F_1(r_{ik}, r_{2k}, \ldots, r_{nk}) \]
\[ > \frac{1}{2} F_2(r_{ik}, r_{2k}, \ldots, r_{nk}) \]
\[ > \frac{1}{2} G \left( \frac{1}{(1-r_{ik})^\xi}, \frac{1}{(1-r_{2k})^\xi}, \ldots, \frac{1}{(1-r_{nk})^\xi} \right) \text{ from (2)} \]
\[ > G \left( \frac{1}{3(1-r_{ik})^\xi}, \frac{1}{3(1-r_{2k})^\xi}, \ldots, \frac{1}{3(1-r_{nk})^\xi} \right) \text{ for all large } k \text{ and by Lemma 2.2} \]
\[ > G \left( \frac{1}{(1-r_{ik})^{\lambda+\varepsilon}}, \frac{1}{(1-r_{2k})^{\lambda+\varepsilon}}, \ldots, \frac{1}{(1-r_{nk})^{\lambda+\varepsilon}} \right) \]

where \( \varepsilon > 0 \) is arbitrary.

This gives \( \rho \geq \lambda - \varepsilon \) and since \( \rho_i < \mu < \lambda < \rho_2 \) and \( \varepsilon > 0 \) is arbitrary, we get \( \rho \geq \rho_2. \)

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Therefore
\[ \rho \left( f_1 + f_2 \right) = \rho_2 = \max (\rho_1 (f_1), \rho_2 (f_2)). \]

For (b), we consider \( f = f_1, f_2, \rho = \rho_1 (f) \) and \( \rho_1 \leq \rho_2 \).

Then for any arbitrary \( \varepsilon > 0 \),
\[
F \left( r_1, r_2, \ldots, r_n \right) = F_1 \left( r_1, r_2, \ldots, r_n \right) \cdot F_2 \left( r_1, r_2, \ldots, r_n \right)
\]
\[
\leq G \left( \frac{1}{(1-r_1)^{\rho_1 + \varepsilon}}, \frac{1}{(1-r_2)^{\rho_1 + \varepsilon}}, \ldots, \frac{1}{(1-r_n)^{\rho_1 + \varepsilon}} \right)^2
\]
\[
\leq G \left( \frac{1}{(1-r_1)^{\sigma \rho_1 + \varepsilon}}, \frac{1}{(1-r_2)^{\sigma \rho_1 + \varepsilon}}, \ldots, \frac{1}{(1-r_n)^{\sigma \rho_1 + \varepsilon}} \right) \text{ by Lemma 2.1,}
\]
for every \( \sigma > 1 \).

So
\[ \rho \leq \sigma (\rho_2 + \varepsilon). \]

Since \( \varepsilon > 0 \) is arbitrary, we obtain by letting \( \sigma \to 1^+ \),
\[ \rho \leq \rho_2. \]

Therefore
\[ \rho \left( f_1, f_2 \right) \leq \max (\rho_1 (f_1), \rho_2 (f_2)). \]

This proves the theorem.

4. Asymptotic Behavior

**Definition 4.1.** Two entire functions \( g_1 \) and \( g_2 \) are said to be asymptotic equivalent in the unit polydisc \( P \) if there exists \( l, 0 < l < \infty \) such that
\[
G_1 \left( \frac{1}{(1-r_1)^{d}}, \frac{1}{(1-r_2)^{d}}, \ldots, \frac{1}{(1-r_n)^{d}} \right) \to l \text{ as } r_i \to 1, \quad i = 1, 2, \ldots, n,
\]
\[
G_2 \left( \frac{1}{(1-r_1)^{d}}, \frac{1}{(1-r_2)^{d}}, \ldots, \frac{1}{(1-r_n)^{d}} \right)
\]
where \( \lambda > 0 \) is any number and in this case we write \( g_1 \sim g_2 \).

**Note 4.2.** If \( g_1 \sim g_2 \) then clearly \( g_2 \sim g_1 \).

**Theorem 4.3.** Let \( g_1 \) and \( g_2 \) be entire functions having property (R) and \( g_1 \sim g_2 \) then \( \rho \left( f \right) = \rho \left( g, f \right) \),
where \( f \) is analytic in \( P \).

**Proof.** Let \( \varepsilon > 0 \) any arbitrary number and for \( r_i, \ 0 < r_i < 1; \ i = 1, 2, \ldots, n \), sufficiently close to 1, we have
\[
G_1 \left( \frac{1}{(1-r_1)^{d}}, \frac{1}{(1-r_2)^{d}}, \ldots, \frac{1}{(1-r_n)^{d}} \right) \leq (l + \varepsilon) G_2 \left( \frac{1}{(1-r_1)^{\alpha}}, \frac{1}{(1-r_2)^{\alpha}}, \ldots, \frac{1}{(1-r_n)^{\alpha}} \right)
\]
\[
\leq G_2 \left( \frac{\alpha}{(1-r_1)^{\alpha}}, \frac{\alpha}{(1-r_2)^{\alpha}}, \ldots, \frac{\alpha}{(1-r_n)^{\alpha}} \right)
\]
where \( \lambda > 0 \) and \( \alpha > 1 \) is such that \( l + \varepsilon < \alpha \).

Next let \( \rho \left( g_1, f \right) = \rho_1 \) and \( \rho \left( g_2, f \right) = \rho_2 \).
Then
\[
F(r_1, r_2, \ldots, r_n) \leq G_1 \left( \frac{1}{(1-r_1)^{\beta_1+\varepsilon}} \cdot \frac{1}{(1-r_2)^{\beta_2+\varepsilon}} \cdot \cdots \cdot \frac{1}{(1-r_n)^{\beta_n+\varepsilon}} \right) \leq G_2 \left( \frac{\alpha}{(1-r_1)^{\beta_1+\varepsilon}} \cdot \frac{\alpha}{(1-r_2)^{\beta_2+\varepsilon}} \cdot \cdots \cdot \frac{\alpha}{(1-r_n)^{\beta_n+\varepsilon}} \right) \leq G_2 \left( \frac{1}{(1-r_1)^{\beta_1+2\varepsilon}} \cdot \frac{1}{(1-r_2)^{\beta_2+2\varepsilon}} \cdot \cdots \cdot \frac{1}{(1-r_n)^{\beta_n+2\varepsilon}} \right).
\]

Therefore
\[
\rho_2 \leq \rho_1 + 2\varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, so \( \rho_2 \leq \rho_1 \).

Therefore
\[
\rho_{g_1}(f) \leq \rho_{g_2}(f).
\]

Also from \( g_2 \sqsubset g_1 \), we obtain
\[
\rho_{g_1}(f) \leq \rho_{g_2}(f).
\]

This proves the theorem.

**Note 4.4.** The converse of the above theorem is not always true.

**Example 4.5.** Consider the functions
\[
g_1(z_1, z_2, \ldots, z_n) = e^{x_1 z_1 - x_2 z_2}, \quad g_2(z_1, z_2, \ldots, z_n) = e^{x_2 z_2 - x_1 z_1}
\]
and \( f(z_1, z_2, \ldots, z_n) = e^{x_1 z_1 - x_2 z_2} \). Then \( g_1 \) is not asymptotic equivalent to \( g_2 \) but \( \rho_{g_1}(f) = \rho_{g_2}(f) \).

5. **Relative Order of the Partial Derivatives**

**Theorem 5.1:** If \( f \) is analytic in the unit polydisc \( P \) and \( g \) be transcendental entire having the property (R), then
\[
\rho_g \left( \frac{\partial f}{\partial z_i} \right) = \rho_g(f).
\]

To prove the theorem we require the following lemma.

**Lemma 5.2.** Let \( f(z_1, z_2, \ldots, z_n) \) be a transcendental entire function then
\[
\frac{F(r_1, r_2, \ldots, r_n)}{r_1} \leq \frac{F(2r_1, r_2, \ldots, r_n)}{r_1} \leq \frac{F(2r_1, r_2, \ldots, r_n)}{r_1}
\]
where
\[
\overline{F}(r_1, r_2, \ldots, r_n) = \max_{|z_j| = r_j, j = 1, 2, \ldots, n} \frac{|\partial f(z_1, z_2, \ldots, z_n)|}{|\partial z_i|}.
\]

**Proof.** Let \( (z'_1, z'_2, \ldots, z'_n) \) be such that
\[
|f(z'_1, z'_2, \ldots, z'_n)| = \max \{|f(z_1, z_2, \ldots, z_n)| : |z_j| = r_j, j = 1, 2, \ldots, n\}.
\]

With out loss of generality we may assume that \( f(0, z'_2, \ldots, z'_n) = 0 \). Otherwise we set
\[
h(z_1, z_2, \ldots, z_n) = z_1 f(z_1, z_2, \ldots, z_n).
\]

Then \( h(0, z'_2, \ldots, z'_n) = 0 \) and \( \rho_g(f) = \rho_g(h) \).

We may write for fixed \( z_i \) on \( |z| = r_i, i = 2, \ldots, n \)
\[
f(z_1, z_2, \ldots, z_n) = \int_0^{r_1} \frac{\partial f(t, z_2, \ldots, z_n)}{\partial t} dt,
\]

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where the line of integration is the segment from \( z = 0 \) to \( z = re^{i\theta}, r > 0 \).

Now

\[
F(r_1, r_2, \ldots, r_n) = \int_0^1 \left| f(z_1, z_2, \ldots, z_n) \right| dt
\]

\[
= \left| \int_0^1 \frac{\partial f(t, z_1, z_2, \ldots, z_n)}{\partial t} dt \right|
\]

\[
\leq r_1 \max_{|z_1| = r_1} \left| \frac{\partial f(z_1, z_2, \ldots, z_n)}{\partial z_1} \right|
\]

\[
= r_1 F(r_1, r_2, \ldots, r_n).
\]  

(4)

Let \((z_1^n, z_2^n, \ldots, z_n^n)\) be such that

\[
\left| \frac{\partial f(z_1^n, z_2^n, \ldots, z_n^n)}{\partial z_1} \right| = \max_{|z_1| = r_1} \left| \frac{\partial f(z_1, z_2, \ldots, z_n)}{\partial z_1} \right|
\]

Let C denote the circle \(|t - z_1^n| = r_1\).

So,

\[
\int_{C} \frac{f(t, z_2^n, \ldots, z_n^n)}{(t - z_1^n)^2} dt
\]

\[
= \frac{1}{2\pi r_1^2} \left( F(2r_1, r_2, \ldots, r_n) \right)
\]

\[
= \frac{F(2r_1, r_2, \ldots, r_n)}{r_1^2} 2\pi r_1
\]

(5)

From (4) and (5) we obtain

\[
\frac{F(r_1, r_2, \ldots, r_n)}{r_1} \leq \int_{C} \frac{f(t, z_2^n, \ldots, z_n^n)}{(t - z_1^n)^2} dt \leq \frac{F(2r_1, r_2, \ldots, r_n)}{r_1}
\]

This proves the lemma.

**Proof of the theorem 5.1:** Let us consider any arbitrary \( \varepsilon > 0 \) then from definition of \( \rho_\varepsilon \left( \frac{\partial f}{\partial z_1} \right) \), we have for all \( r_i, 0 < r_i < 1; i = 1, 2, \ldots, n \) sufficiently close to 1,

\[
\int_{C} \frac{f(t, z_2^n, \ldots, z_n^n)}{(t - z_1^n)^2} dt \leq G \left( \frac{1}{(1 - r_1)} \rho_\varepsilon \left( \frac{\partial f}{\partial z_1} \right)^e \right)^e \left( \frac{1}{(1 - r_2)} \rho_\varepsilon \left( \frac{\partial f}{\partial z_1} \right)^e \right)^e \ldots \left( \frac{1}{(1 - r_n)} \rho_\varepsilon \left( \frac{\partial f}{\partial z_1} \right)^e \right)^e
\]

Now by Lemma 5.2
\[ F(r_1, r_2, \ldots, r_n) \leq r_1 F(r_1, r_2, \ldots, r_n) \]

\[
\leq G \left( \frac{1}{(1-r_1)^{\sigma \rho g} + \varepsilon}, \frac{1}{(1-r_2)^{\sigma \rho g} + \varepsilon}, \ldots, \frac{1}{(1-r_n)^{\sigma \rho g} + \varepsilon} \right)^2
\]

by Lemma 2.1 and for any \( \sigma > 1 \), since \( g \) has the property (R).

So,

\[
\rho_g(f) \leq \sigma \left( \rho_g \left( \frac{\partial f}{\partial z_1} \right) + \varepsilon \right).
\]

Letting \( \sigma \to 1_+ \), and since \( \varepsilon > 0 \) is arbitrary

\[
\rho_g(f) \leq \rho_g \left( \frac{\partial f}{\partial z_1} \right).
\]

Using (5) we obtain similarly

\[
\rho_g \left( \frac{\partial f}{\partial z_1} \right) \leq \rho_g(f).
\]

So,

\[
\rho_g \left( \frac{\partial f}{\partial z_1} \right) = \rho_g(f).
\]

This proves the theorem.

Note 5.3. Similar result hold for other partial derivatives.

6. References


