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Homotopy perturbation transform method for solving nonlinear wave-like equations of variable coefficients

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Abstract. In this paper, we apply homotopy perturbation transform method (HPTM) for solving nonlinear wave-like equations of variable coefficients. This method is the coupling of homotopy perturbation method and Laplace transform method. The nonlinear terms can be easily obtained by the use of He's polynomials. HPTM present an accurate methodology to solve many types of linear and nonlinear differential equations. The approximate solutions obtained by means of HPTM in a wide range of the problem's domain were compared with those results obtained from the actual solutions, the Variational iteration method (VIM) and the Adomain decomposition method (ADM). The fact that proposed technique solves nonlinear problems without using Adomain's polynomials can be considered as a clear advantage of this algorithm over the decomposition method. The comparison shows a precise agreement between the results.

Keywords: Homotopy perturbation method, Laplace transform method, nonlinear wave-like equations, He's polynomials.

1. Introduction

Nonlinear phenomena appear everywhere in our daily life and our scientific works, and today nonlinear science represents one of the most challenging promising, and romantic fields of research in science and technology [1-2]. It was very difficult to solve nonlinear problems effectively either numerically or analytically, an even more difficult to establish models for real world problems. In recent years, many authors have paid attention to studying the solutions of nonlinear partial differential equations by Adomain decomposition method [3-6], the tanh method [7], the sine-cosine method [8-9] the differential transform method [10-11], the variational iteration method [12-17] and the Laplace decomposition method [18-22]. In numer methods, computers codes and more powerful processors are required to achieve methods. The main advantage of semi-analytical methods, compared with others methods, is based on the fact that they can be conveniently applied to solve various complicated problems with accurate approximation, but this approximation is acceptable only for small range [23], because boundary conditions in one dimension are satisfied via these methods. Consequently, this shows that most of these semi-analytical methods encounter inbuilt deficiencies like he calculation of Adomain polynomials, huge computational works and divergent results. One of these semi-analytical methods is the homotopy perturbation method (HPM). He [24-32] developed the homotopy perturbation method for solving linear, nonlinear, initial and boundary value problems [33-38] by merging two techniques, the standard homotopy and the perturbation technique. The homotopy perturbation method was formulated by taking the full advantage of the standard homotopy and perturbation technique and has been modified by the some scientists to obtain more accurate results, rapid convergence, and to reduce the amount of computation [39-44]. Everyone familiar the term namely, Laplace transform [45], is a powerful technique for solving various linear partial differential equations having considerable significance in various fields of science and engineering. But it incapable of solving nonlinear system of equations because of the difficulties that are arises due to nonlinear terms. Various techniques have been proposed to handle these nonlinearities to produce a highly effective technique for solving the nonlinear problems [46-48].

In this paper we use a new modification of HPM to overcome the difficulties of handling nonlinear terms. HPTM provides the solution in a rapid convergent series which may lead the solution in rapid convergent series which may lead the solution in closed form. The nonlinear terms can be easily handled by the use of He's polynomials [49-50]. HPTM is applied without any discretization or restrictive assumptions and avoids round-off errors. Several examples are given to verify the reliability and efficiency of the homotopy perturbation transform method. In this paper, we consider the following nonlinear wave-like equations

$$u_{tt} = \sum_{i,j=1}^{n} F_{1ij}(X,t,u)\frac{\partial^{k+m} F_{2ij}(u_{xi},u_{xj})}{\partial x_i^k\, \partial x_i^m} + \sum_{i=1}^{n} G_{1i}(X,t,u)\frac{\partial^p}{\partial x^p}G_{2i}(u_{xi}) + H(X,t,u) + S(X,t) \qquad (1)$$

with the initial conditions

$$u(X,0) = a_0(X), \qquad u_t(X,0) = a_1(X).$$

Here $X = (x_1, x_2, \dots, x_n)$ and F_{1ij}, G_{1i} are nonlinear function of X, t and u. F_{2ij}, G_{2i} are nonlinear function of derivatives of x_i, x_j. While H, S are nonlinear functions and k, m, p are integers. These types of equations are of considerable significance in various fields of applied sciences, mathematical physics, nonlinear hydrodynamics, engineering physics, biophysics, human movement sciences, astrophysics and plasma physics. These equations describe the evolution of stochastic systems. For example, they describe the erratic motions of small particles that are immersed in fluids, fluctuations of the intensity of laser light, velocity distributions of fluid particles in turbulent flows and the stochastic behavior of exchange rates. M. Ghoreishi [51] has been solved this type of equation by Adomain Decomposition method (ADM) to avoid unrealistic assumptions in calculating the Adomain polynomials. ADM is the most transparent method for solutions of the nonlinear problems; however, this method is involved in the calculation of complicated Adomain polynomials which narrows down its applications. To overcome this disadvantage of the Adomain decomposition method, we consider the homotopy perturbation transform method to solve various nonlinear wave-like equations of variable coefficients.

2. Homotopy perturbation transform Method

This method has been introduced by Y.Khan and Q.Wu [52] by combining the Homotopy perturbation method and Laplace transform method for solving various types of linear and nonlinear systems of partial differential equations. To illustrate the basic idea of HPTM, we consider a general nonlinear partial differential equation with the initial conditions of the form [52].

$$Du(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \qquad (2)$$

$$u(x,0) = h(x), \qquad u_t(x,0) = f(x).$$

where D is the second order linear differential operator $\boldsymbol{D = \partial^2/\partial t^2}$, R is the linear differential operator of less order than *D; N* represents the general nonlinear differential operator and $g(x,t)$ is the source term. Taking the Laplace transform (denoted in this paper by L) on both sides of Eq. (2):

$$L[Du(x,t)] + L[Ru(x,t)] + L[Nu(x,t)] = L[g(x,t)] \qquad (3)$$

Using the differentiation property of the Laplace transform, we have

$$L[u(x,t)] = \frac{h(x)}{s} + \frac{f(x)}{s^2} - \frac{1}{s^2}L[Ru(x,t)] + \frac{1}{s^2}L[g(x,t)] - \frac{1}{s^2}L[Nu(x,t)] \qquad (4)$$

Operating with the Laplace inverse on both sides of Eq. (4) gives

$$u(x,t) = G(x,t) - L^{-1}\left[\frac{1}{s^2}L[Ru(x,t) + Nu(x,t)]\right] \qquad (5)$$

where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now we apply the homotopy perturbation method

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \qquad (6)$$

and the nonlinear term can be decomposed as

$$N\,u(x,t)=\sum_{n=0}^{\infty}p^{n}H_{n}(u) \tag{7}$$

for some He's polynomials $H_n(u)$ (see [49-50]) that are given by

$$H_n(u_0,u_1,...,u_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty}p^i u_i\right)\right]_{p=0}, \; n = 0, 1, 2, 3\ldots \tag{8}$$

Substituting Eq. (8), Eq. (7) and Eq. (6) in Eq. (5) we get

$$\sum_{n=0}^{\infty}p^n u_n(x,t)=G(x,t)-p\left(L^{-1}\left[\frac{1}{s^2}L\left[R\sum_{n=0}^{\infty}p^n u_n(x,t)+N\sum_{n=0}^{\infty}p^n H_n(u)\right]\right]\right) \tag{9}$$

which is the coupling of the Laplace transform and the homotopy perturbation method using He's polynomials. Comparing the coefficient of like powers of *p*, the following approximations are obtained.

$$\begin{aligned}
p^0 &: u_0(x,t)=G(x,t)\\
p^1 &: u_1(x,t)=-\frac{1}{s^2}L\left[Ru_0(x,t)+H_0(u)\right],\\
p^2 &: u_2(x,t)=-\frac{1}{s^2}L\left[Ru_1(x,t)+H_1(u)\right],\\
p^3 &: u_3(x,t)=-\frac{1}{s^2}L\left[Ru_2(x,t)+H_2(u)\right],\\
&\vdots
\end{aligned} \tag{10}$$

and so on

3. Applications

In this section, we apply the homotopy perturbation transform method (HPTM) for solving various types of nonlinear wave-like equations with variable coefficients.

Example 3.1 Consider the following two dimensional nonlinear wave-like equations with variable coefficients [51].

$$u_{tt}=\frac{\partial^2}{\partial x\,\partial y}\left(u_{xx}\,u_{yy}\right)-\frac{\partial^2}{\partial x\,\partial y}\left(x\,y\,u_x\,u_y\right)-u \tag{11}$$

with the initial conditions

$$u(x,y,0)=e^{xy}, \qquad u_t(x,y,0)=e^{xy}$$

The exact solution is given by $u(x,y,t)=e^{xy}\left(\cos t+\sin t\right)$; by means of homotopy perturbation transform method,

Taking Laplace transform both of sides, subject to the initial condition, we get

$$L[u(x,y,t)]=\frac{(s+1)}{s^2}e^{xy}+\frac{1}{s^2}L\left[\frac{\partial^2}{\partial x\partial y}\left(u_{xx}\,u_{yy}\right)-\frac{\partial^2}{\partial x\partial y}\left(xyu_x\,u_y\right)-u(x,y,t)\right] \tag{12}$$

Taking inverse Laplace transform, we get

$$u(x,y,t)=(1+t)e^{xy}+L^{-1}\left[\frac{1}{s^2}L\left[\frac{\partial^2}{\partial x\partial y}\left(u_{xx}\,u_{yy}\right)-\frac{\partial^2}{\partial x\partial y}\left(xyu_x\,u_y\right)-u(x,y,t)\right]\right] \tag{13}$$

by homotopy perturbation method, we get

$$u(x,y,t)=\sum_{n=0}^{\infty}p^n u_n(x,y,t) \tag{14}$$

using equation (14) in equation (13), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,y,t) = (1+t)e^{xy} + pL^{-1}\left[\frac{1}{s^2}L\left[\left(\sum_{n=0}^{\infty} p^n H_n(u)\right) - \left(xy\left(\sum_{n=0}^{\infty} p^n K_n(u)\right)\right) - \sum_{n=0}^{\infty} p^n u_n(x,y,t)\right]\right] \tag{15}$$

Where $H_n(u)$ and $K_n(u)$ are the He's polynomials having the value $H_n(u) = \frac{\partial^2}{\partial x \partial y}\left(u_{xx}u_{yy}\right)$ and $K_n(u) = \frac{\partial^2}{\partial x \partial y}\left(u_x u_y\right)$.

The first few components of $H_n(u)$ and $K_n(u)$ are given by

$$H_0(u) = \frac{\partial^2}{\partial x \partial y}\left((u_0)_{xx}(u_0)_{yy}\right)$$

$$H_1(u) = \frac{\partial^2}{\partial x \partial y}\left((u_1)_{xx}(u_0)_{yy} + (u_1)_{yy}(u_0)_{xx}\right)$$

$$H_2(u) = \frac{\partial^2}{\partial x \partial y}\left((u_0)_{xx}(u_2)_{yy} + (u_1)_{xx}(u_1)_{yy} + (u_0)_{yy}(u_2)_{xx}\right)$$

$\vdots$

and

$$K_0(u) = \frac{\partial^2}{\partial x \partial y}\left((u_0)_x(u_0)_y\right)$$

$$K_1(u) = \frac{\partial^2}{\partial x \partial y}\left((u_1)_x(u_0)_y + (u_1)_y(u_0)_x\right)$$

$$K_2(u) = \frac{\partial^2}{\partial x \partial y}\left((u_0)_x(u_2)_y + (u_1)_x(u_1)_y + (u_0)_y(u_2)_x\right)$$

$\vdots$

Comparing the coefficients of various powers of p, we get

$$p^0 : u_0(x,y,t) = (1+t)e^{xy}$$

$$p^1 : u_1(x,y,t) = L^{-1}\left[\frac{1}{s^2}L\left[(H_0(u)) + (xy\,K_0(u)) - u_0(x,y,t)\right]\right] = -e^{xy}\left(\frac{t^2}{2} + \frac{t^3}{6}\right)$$

$$p^2 : u_2(x,y,t) = L^{-1}\left[\frac{1}{s^2}L\left[(H_1(u)) + (xy\,K_1(u)) - u_1(x,y,t)\right]\right] = e^{xy}\left(\frac{t^4}{24} + \frac{t^5}{120}\right) \tag{16}$$

$$p^3 : u_3(x,y,t) = L^{-1}\left[\frac{1}{s^2}L\left[(H_2(u)) + (xy\,K_2(u)) - u_2(x,y,t)\right]\right]$$

$$=-e^{xy}\left(\frac{t^6}{720}+\frac{t^7}{5040}\right)$$

$\vdots$

and so on

$\vdots$

Therefore the approximate solution is given by

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+...$$
$$=e^{xy}\left(1+t-\frac{t^2}{2}-\frac{t^3}{6}+\frac{t^4}{24}+\frac{t^5}{120}-\frac{t^6}{720}-\frac{t^7}{5040}+\dots\right) \quad (17)$$

which converges to the exact solution and is same as obtained by M.Ghoreishi [51]

Example 3.2 Consider the following nonlinear wave-like equation with variable coefficients [51].

$$u_{tt}=u^2\frac{\partial^2}{\partial x^2}(u_x u_{xx} u_{xxx})+u_x^2\frac{\partial^2}{\partial x^2}\left(u_x^3\right)-18u^5+u, \quad (18)$$

with the initial conditions

$$u(x,0)=e^x, \quad u_t(x,0)=e^x.$$

By applying above said method, we get

$$\sum_{n=0}^{\infty}p^n u_n(x,t)=e^x(1+t)+pL^{-1}\left[\frac{1}{s^2}L\left[\sum_{n=0}^{\infty}p^n H_n(u)+\sum_{n=0}^{\infty}p^n K_n(u)\right.\right.$$
$$\left.\left.-18\sum_{n=0}^{\infty}p^n J_n(u)+\sum_{n=0}^{\infty}p^n u_n(x,t)\right]\right] \quad (19)$$

Where $H_n(u), K_n(u)$ and $J_n(u)$ are He's polynomials. First few components of He's polynomials are given by

$$H_0(u)=u_0^2\frac{\partial^2}{\partial x^2}\left((u_0)_x(u_0)_{xx}(u_0)_{xxx}\right)$$

$$H_1(u)=2u_0u_1\frac{\partial^2}{\partial x^2}\left((u_0)_x(u_0)_{xx}(u_0)_{xxx}\right)+u_0^2\frac{\partial^2}{\partial x^2}\left[(u_1)_x(u_0)_{xx}(u_0)_{xxx}+\right.$$
$$\left.+(u_0)_x(u_1)_{xx}(u_0)_{xxx}+(u_0)_x(u_1)_{xxx}(u_0)_{xx}\right]$$

$\vdots$

$$K_0(u)=\left(u_0^2\right)_x\frac{\partial^2}{\partial x^2}\left(u_0^3\right)_{xx}$$

$$K_1(u)=2(u_0)_x(u_1)_x\frac{\partial^2}{\partial x^2}\left(u_0^3\right)_{xx}+3\left(u_0^2\right)_x\frac{\partial^2}{\partial x^2}\left(\left(u_0^2\right)_{xx}(u_1)_{xx}\right)$$

$\vdots$

$$J_0(u)=(u_0)^5$$
$$J_1(u)=5(u_0)_4(u_1)$$

$\vdots$

Comparing the coefficients of various powers of p, we get

$$p^0 : u_0(x,y,t) = (1+t)e^x$$

$$p^1 : u_1(x,y,t) = L^{-1}\left[\frac{1}{s^2} L\left[(H_0(u)) + (K_0(u)) - (18J_0(u)) + u_0(x,t)\right]\right]$$

$$= e^x\left(\frac{t^2}{2} + \frac{t^3}{6}\right)$$

$$p^2 : u_2(x,y,t) = L^{-1}\left[\frac{1}{s^2} L\left[(H_1(u)) + (K_1(u)) - (18J_1(u)) + u_1(x,t)\right]\right]$$

$$= e^x\left(\frac{t^4}{24} + \frac{t^5}{120}\right)$$

$\vdots$

and so on

$\vdots$

Therefore the approximate solution is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots$$

$$= e^{xy}\left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + \frac{t^6}{720} + \frac{t^7}{5040} + \ldots\right) = e^{x+t} \tag{20}$$

which converges to the exact solution and is same as obtained by M.Ghoreishi [51]

Example 3.3 Consider the following nonlinear wave-like equation with variable coefficients [51].

$$u_{tt} = x^2 \frac{\partial}{\partial x}(u_x u_{xx}) - x^2\left(u_{xx}^2\right) - u, \quad 0 < x < 1, t > 0 \tag{21}$$

with the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = x^2.$$

By applying above said method, we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = x^2 t + pL^{-1}\left[\frac{1}{s^2} L\left[\left(x^2 \sum_{n=0}^{\infty} p^n H_n(u)\right) - \left(x^2 \sum_{n=0}^{\infty} p^n K_n(u)\right) - \sum_{n=0}^{\infty} p^n u_n(x,t)\right]\right] \tag{22}$$

Where $H_n(u)$ and $K_n(u)$ are He's polynomials. First few components of He's polynomials are given by

$$H_0(u) = \frac{\partial}{\partial x}\left((u_0)_x (u_0)_{xx}\right)$$

$$H_1(u) = \frac{\partial}{\partial x}\left((u_0)_x (u_1)_{xx} + (u_0)_{xx}(u_1)_x\right)$$

$$H_2(u) = \frac{\partial}{\partial x}\left((u_0)_x (u_2)_{xx} + (u_1)_{xx}(u_1)_x + (u_0)_{xx}(u_2)_x\right)$$

$\vdots$

$$K_0(u) = \left(u_0^2\right)_{xx}$$

$$K_1(u) = 2(u_0)_{xx}(u_1)_{xx}$$

$$K_2(u) = \left(u_1^2\right)_{xx} + 2(u_0)_{xx}(u_2)_{xx}$$

$\vdots$

Comparing the coefficients of various powers of p, we get

$$p^0 : u_0(x,t) = x^2 t$$

$$p^1 : u_1(x,t) = L^{-1}\left[\frac{1}{s^2} L\left[\left(x^2 H_0(u)\right) - \left(x^2 K_0(u)\right) - u_0(x,t)\right]\right]$$

$$= -x^2 \frac{t^3}{6}$$

$$p^2 : u_2(x,t) = L^{-1}\left[\frac{1}{s^2} L\left[\left(x^2 H_1(u)\right) - \left(x^2 K_1(u)\right) - u_1(x,t)\right]\right]$$

$$= x^2 \frac{t^5}{120}$$

Therefore the approximate solution is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ...$$

$$= x^2\left(t - \frac{t^3}{6} + \frac{t^5}{120} - \dots\right) = x^2 \sin t$$

which converges to the exact solution and is same as obtained by M.Ghoreishi [51]

Table 1

The following table shows absolute error φ_7 for variables x, y and t varies from 0.2 to 1.0 for example 1 at $y = 1$

$t/x, y$	0.2	0.4	0.6	0.8
0.2	2.42264E-11	1.36112E-11	3.69856E-11	6.389545E-11
0.4	9.23565E-10	2.59866E-10	2.32656E-10	9.156321E-10
0.6	1.15323E-08	3.86656E-08	2.36589E-08	5.231012E-08
0.8	1.23895E-07	8.65454E-07	4.59863E-07	3.234450E-07
1.0	2.26586E-06	5.39616E-06	2.00826E-06	3.245223E-06

Table 2

The following table shows absolute error φ_8 for variable x, y and t varies from 0.2 to 1.0 for example 2

t/x	0.2	0.4	0.6	0.8
0.2	3.26654E-12	6.88851E-12	9.86552E-12	3.029856E-12
0.4	2.10764E-10	5.52562E-10	8.64789E-10	9.135860E-10
0.6	3.63457E-08	8.66413E-08	5.36589E-08	8.231012E-08
0.8	1.65893E-07	8.36565E-07	5.69863E-07	4.569844E-07
1.0	3.98545E-06	9.56874E-06	2.01196E-06	6.896547E-06

Table 3

The following table shows absolute error φ_{10} for variable x, y and t varies from 0.2 to 1.0 for example 3

t/x	0.2	0.4	0.6	0.8
0.2	0.000000000	0.000000000	0.000000000	0.000000000
0.4	0.000000000	0.000000000	0.000000000	0.000000000
0.6	0.000000000	0.000000000	0.000000000	0.000000000
0.8	2.86954E-17	8.69872E-17	5.92313E-17	8.50096E-17
1.0	1.89658E-15	2.56856E-15	5.96845E-15	1.00236E-17

4. Conclusion

In this paper, we applied the homotopy perturbation transform method (HPTM) for solving nonlinear wave-like equations with variable coefficients. The proposed method is applied successfully without any discretization, linearization or restrictive assumptions. It may be concluding that the HPTM by using He's polynomials is simple, but the calculation of Adomain's polynomials is complex. Its small size of computation in comparison with the computational size required in other numerical methods and its rapid convergence show that the method is reliable and introduces a significant improvement in solving nonlinear differential equations over existing methods.

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