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Determination of a source term and boundary heat flux in an inverse heat equation

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Abstract. In this paper, the determination of the heat source and heat flux at $x=0$ in one-dimensional inverse heat conduction problem (IHCP) is investigated. First with an suitable transformation, the problem is reduced, then the method of fundamental solutions (MFS) is used to solve the problem. Due to ill-posed the IHCP, the Tikhonov regularization method with Generalized cross validation (GCV) criterion are employed in numerical procedure. Finally, some numerical examples are presented to show the accuracy and effectiveness of the algorithm.

Keywords: IHCP, MFS, Heat source, Ill-posed, Tikhonov regularization method, GCV criterion.

1. Introduction

Boundary heat flux reconstruction and heat source identification are the most commonly encountered inverse problems in heat conduction. These problems have been studied over several decades due to their significance in a variety of scientific and engineering applications. In the process of transportation, diffusion and conduction of natural materials, the following heat equation is a suitable approximation [1]:

$$\frac{\partial U}{\partial t}-\frac{\partial^2 U}{\partial x^2}=f\left(x,t;U\right); \qquad \left(x,t\right)\in\left[0,1\right]\times\left[0,T\right],$$

where U represents the state variable, T is final time and f denotes physical law. Unfortunately, the characteristics of sources in actual problems are always unknown. This problem is an inverse problem [2]. Another example of the IHCP is the estimation of the heating history experienced by a shuttle or missile reentering the earth's atmosphere from space. The heat flux at the heated surface is needed [3].

IHCPs are mathematically ill-posed in the sense that the existence, uniqueness and stability of their solutions can not be assured. A number of numerical approaches have been developed toward the solution of these problems, the boundary element method [4], Ritz-Galerkin method [5] and iterative regularization method [6]. Recently, Y.C. Hon and T. Wei [7] successfully applied the method of fundamental solutions to approximate the solution of IHCP. A meshless and integration-free scheme for solving the problem. Following their works, many researchers applied this method to solve many inverse problems [8-12]. In this study we use the MFS with Tikhonov regularization method and GCV criterion to solve the inverse problem.

The organization of the paper is as follows: In section 2, the formulation of IHCP is presented. Section 3 is devoted to the numerical procedure, MFS. Several numerical examples are presented in section 4. Conclusion is finally discussed in section 5.

2. Mathematical formulation

In this work we consider the following inverse partial differential equation (PDE):

$$\frac{\partial U}{\partial t}\left(x,t\right)=\frac{\partial^2 U}{\partial x^2}\left(x,t\right)+f\left(x\right); \qquad 0<x<1,\ 0<t<T, \tag{1}$$

with initial condition:

$$U\left(x,0\right)=\varphi(x); \qquad 0\le x\le 1, \tag{2}$$

and boundary condition:

$$\beta \frac{\partial U}{\partial x}(1,t) + \gamma U(1,t) = g(t); \qquad 0 \le t \le T, \tag{3}$$

and overspecified conditions:

$$U(x^*,t) = h(t); \qquad 0 \le t \le T, \tag{4}$$

$$U(x,T) = \psi(x); \qquad 0 \le x \le 1, \tag{5}$$

where $x^* \in (0,1)$ and is known, T is the final time, β and γ are positive constants and φ, g, h and ψ are known continuous functions in their domain satisfying the compatibility conditions:

$$\varphi(x^*) = h(0), h(T) = \psi(x^*), \beta\varphi'(1) + \gamma\varphi(1) = g(0), \beta\psi'(1) + \gamma\psi(1) = g(T), \tag{6}$$

and heat source $f(x)$, heat flux $\frac{\partial U}{\partial x}(0,t) = q(t)$ and heat distribution $U(x,t)$ are unknowns to be determined.

If the triple $\left(U, \frac{\partial U}{\partial x}(0,t), f(x)\right)$ is known, then the direct initial boundary value problem (1)-(5) has a unique smooth solution $U(x,t)$ [13].

The IHCP is ill-posed, so we solve the inverse problem with numerical approach. To obtain a PDE containing only one unknown function using the following suitable transformation:

$$V(x,t) = U(x,t) + r(x), \tag{7}$$

$$r(x) = \int_0^x (x-\alpha) f(\alpha)\, d\alpha, \tag{8}$$

By considering (6), (7) and (8), the IHCP (1)-(5) is transformed into the following problem:

$$\frac{\partial V}{\partial t}(x,t) = \frac{\partial^2 V}{\partial x^2}(x,t); \qquad 0 < x < 1, \quad 0 < t < T, \tag{9}$$

$$V(x^*,t) - V(x^*,0) = h(t) - \varphi(x^*), \tag{10}$$

$$\beta\left(\frac{\partial V}{\partial x}(1,t) - \frac{\partial V}{\partial x}(1,0)\right) + \gamma\left(V(1,t) - V(1,0)\right) = g(t) - \beta\varphi'(1) - \gamma\varphi(1), \tag{11}$$

$$V(x,T) - V(x,0) = \psi(x) - \varphi(x), \tag{12}$$

In equations (10)-(12) we have $0 \le x \le 1$ and $0 \le t \le T$. By solving the backward direct problem (9)-(12), the approximated solution $V(x,t)$ is obtained and with (2) and (7) we have:

$$r(x) = V(x,0) - \varphi(x), \tag{13}$$

so, for approximating $f(x)$, we differentiate from (8) as:

$$f(x) = r''(x) = \frac{\partial^2 V}{\partial x^2}(x,0) - \varphi''(x). \tag{14}$$

Now from (7) and (13) we conclude:

$$U(x,t) = V(x,t) - r(x). \tag{15}$$

3. The method of fundamental solutions

Discretizing of the initial-boundary conditions (10)-(12) may be considered as:

$$V\left(x^*,t_i\right)-V\left(x^*,0\right)=h\left(t_i\right)-\varphi\left(x^*\right);\quad i=1,\ldots,n, \tag{16}$$

$$\beta\left(\frac{\partial V}{\partial x}\left(1,t_{i-n}\right)-\frac{\partial V}{\partial x}\left(1,0\right)\right)+\gamma\left(V\left(1,t_{i-n}\right)-V\left(1,0\right)\right)=g\left(t_{i-n}\right)-\beta\varphi'(1)-\gamma\varphi(1);\quad i=n+1,\ldots,n+m, \tag{17}$$

$$V\left(x_{i-n-m},T\right)-V\left(x_{i-n-m},0\right)=\psi\left(x_{i-n-m}\right)-\varphi\left(x_{i-n-m}\right);\quad i=n+m+1,\ldots,n+m+l. \tag{18}$$

The fundamental solution of Eq. (9) is given as:

$$K\left(x,t\right)=\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}H\left(t\right),$$

where $H\left(t\right)$ is the Heaviside step function. Assume $\tau>T$ is a constant. Then the following time shift function:

$$\phi\left(x,t\right)=K\left(x,t+\tau\right), \tag{19}$$

is also a solution of Eq.(9) [7]. The approximation to the solution of the problem (9)-(12) can be expressed as the following:

$$V^*\left(x,t\right)=\sum_{j=1}^{n+m+l}\lambda_j\phi\left(x-x_j,t-t_j\right),$$

where ϕ is given by (19) and λ_j are unknown constants. Using conditions (16)-(18), the values of the λ_j can be obtained by solving the following matrix equation:

$$A\lambda=b, \tag{20}$$

where

$$A=\begin{pmatrix}\phi\left(x^*-x_j,t_i-t_j\right)-\phi\left(x^*-x_j,0-t_j\right)\\ L\\ \phi\left(x_k-x_j,T-t_j\right)-\phi\left(x_k-x_j,0-t_j\right)\end{pmatrix}_{(n+m+l)\times(n+m+l)},$$

$$L=\beta\left(\frac{\partial\phi}{\partial x}\left(1-x_j,t_s-t_j\right)-\frac{\partial\phi}{\partial x}\left(1-x_j,0-t_j\right)\right)+\gamma\left(\phi\left(1-x_j,t_s-t_j\right)-\phi\left(1-x_j,0-t_j\right)\right),$$

$$\lambda=\left(\lambda_1,\lambda_2,\ldots,\lambda_{n+m+l}\right)^t,$$

$$b=\left(h_i-\varphi\left(x^*\right),g_s-\beta\varphi'(1)-\gamma\varphi(1),\psi_k-\varphi_k\right)^t,$$

and $i=1,2,\ldots,n, s=n+1,n+2,\ldots,n+m, k=n+m+1,n+m+2,\ldots,n+m+l$ and $j=1,2,\ldots,n+m+l$.

Since the IHCP is ill-posed, the matrix A in Eq.(20) is ill-conditioned. We use the Tikhonov regularization method with the GCV criterion to solve Eq.(20). The Tikhonov regularized solution λ_α for Eq.(20) is defined to be the solution to the following least square problem:

$$\min_\lambda\left\{\left\|A\lambda-b\right\|^2+\alpha^2\left\|\lambda\right\|^2\right\},$$

where $\|.\|$ denotes the usual Euclidean norm and α is called the regularization parameter. We use the GCV method to determine a suitable value of α [1].

Denote the regularized solution of Eq.(20) by λ^{α^*}. The approximated solution V_α^* for problems (9)-(12) may be given as:

$$V_{\alpha}^{*}(x,t)=\sum_{j=1}^{n+m+l}\lambda_j^{\alpha^*}\phi\left(x-x_j,t-t_j\right). \qquad (21)$$

From (13) and (21), we have:

$$r^{*}(x)=\sum_{j=1}^{n+m+l}\lambda_j^{\alpha^*}\phi\left(x-x_j,0-t_j\right)-\varphi(x). \qquad (22)$$

The solution of problem (1)-(5) with considering relations (15), (21) and (22) is:

$$U_{\alpha}^{*}(x,t)=\sum_{j=1}^{n+m+l}\lambda_j^{\alpha^*}\left[\phi\left(x-x_j,t-t_j\right)-\phi\left(x-x_j,0-t_j\right)\right]+\varphi(x),$$

and

$$f^{*}(x)=\sum_{j=1}^{n+m+l}\lambda_j^{\alpha^*}\frac{\partial^2\phi}{\partial x^2}\left(x-x_j,0-t_j\right)-\varphi''(x),$$

So we have

$$\frac{\partial U_{\alpha}^{*}}{\partial x}(0,t)=\sum_{j=1}^{n+m+l}\lambda_j^{\alpha^*}\left[\frac{\partial\phi}{\partial x}\left(0-x_j,t-t_j\right)-\frac{\partial\phi}{\partial x}\left(0-x_j,0-t_j\right)\right]+\varphi'(0).$$

The numerical results in section 4 indicate that the proposed scheme is stable and efficient.

4. Numerical examples

For simplicity, we set $T=1$ in all following examples.

Example 1. Consider the following IHCP:

$$\frac{\partial U}{\partial t}(x,t)=\frac{\partial^2 U}{\partial x^2}(x,t)+f(x); \qquad 0<x<1,\ 0<t<1,$$

$$U(x,0)=x^2+\sin(2\pi x); \qquad 0\le x\le 1,$$

$$U(1,t)=1+2t; \qquad 0\le t\le 1,$$

With the overspecified conditions:

$$U(x^*,t)=x^{*^2}+2x^*t+\sin(2\pi x^*); \qquad x^*\in(0,1),\quad 0\le t\le 1,$$
$$U(x,1)=x^2+2x+\sin(2\pi x); \qquad 0\le x\le 1.$$

The exact solution of this problem is:

$$U(x,t)=x^2+2xt+\sin(2\pi x); \qquad 0\le x\le 1,\quad 0\le t\le 1,$$
$$f(x)=2x-2+4\pi^2\sin(2\pi x); \qquad 0\le x\le 1,$$

and

$$\frac{\partial U}{\partial x}(0,t)=2t+2\pi; \qquad 0\le t\le 1.$$

We solve the above problem with noiseless data for various values of x^* and choice the values $x,t\in[0,1]$ for discretizing the initial-boundary conditions (16)-(18) with two methods. One is random and another is the roots of Chebyshev polynomial that can be obtained from following formula. If T_n is a Chebyshev polynomial of degree $n\ge 1$ then it has n roots in interval $[-1,1]$ as:

$$z_j=\cos\left(\frac{2j-1}{2n}\pi\right); \qquad j=1,2,\ldots,n,$$

and since $x,t\in[0,1]$, we put the discretizing points as:

$$x_j = \frac{z_j + 1}{2}; \qquad j = 1,2,...,n.$$

The numerical results are shown in table 1. To test the accuracy of the approximated solution, we use the root mean square error (RMS) defined as:

$$RMS\left(f(x)\right) = \sqrt{\frac{1}{n_k}\sum_{i=1}^{n_k}\left(f_i^{Exa.} - f_i^{Num.}\right)^2},$$

$$RMS\left(q(t)\right) = \sqrt{\frac{1}{n_t}\sum_{i=1}^{n_t}\left(q_i^{Exa.} - q_i^{Num.}\right)^2},$$

where n_t and n_k are total number of testing points in the domain $[0,1]\times[0,1]$, $q_i^{Exa.}$ and $q_i^{Num.}$ are the exact and approximated values at this point, respectively, and also is hold for $f(x)$. Tables 1 and 2 show the values of $RMS(q(t))$ and $RMS(f(x))$, respectively, with noiseless data for the roots of Chebyshev polynomial and random choosing of $x,t \in [0,1]$. From tables 1 and 2 we conclude that the numerical results are more accurate when we choose discretization points as roots of Chebyshev polynomial. Table 3 shows the values of regularization parameter, $RMS(q(t))$ and $RMS(f(x))$ with three methods of choosing regularization parameter, GCV criterion, quasi optimality method [14] and the L-curve scheme [14]. From table 3 we conclude that the result of GCV method is the most accurate. We put $n = m = l = 11$ and $\tau = 1.2$ in tables 1, 2 and 3 and $x^* = 0.01$ in table3. Table 4 shows the values of $q(t)$ and $f(x)$ with discrete noisy data, $\tilde{g}_i = g_i + \sigma.rand(1)$, $\tilde{h}_i = h_i + \sigma.rand(1)$ and $\tilde{\psi}_i = \psi_i + \sigma.rand(1)$ where g_i, h_i and ψ_i are the exact data and $rand(1)$ is a random number between $(-1,1)$ and the magnitude σ indicates the error level. We put $x^* = 0.01, n = m = l = 20, \sigma = 1\%$ and $\tau = 5$ in table 4. Figure 1 and figure 2 show the values of $f(x)$ and $q(t)$ with noisy and noiseless data, respectively.

x^*	$RMS(q(t))$:random	$RMS(q(t))$:Chebyshev
0.01	1.2051602×10^{-5}	7.9688800×10^{-7}
0.1	2.1281652×10^{-5}	1.9156389×10^{-6}
0.2	1.3410111×10^{-5}	3.6316214×10^{-6}
0.9	8.2734550×10^{-5}	5.7529302×10^{-5}

Table 1. The values of $RMS(q(t))$ for various values of x^* and random choice of x and $t \in [0,1]$ and choosing the roots of Chebyshev polynomial for x and t with noiseless data when $n = m = l = 11$ and $\tau = 1.2$.

x^*	$RMS(f(x))$:random	$RMS(f(x))$:Chebyshev
0.01	1.2887351×10^{-4}	8.3219848×10^{-6}
0.1	1.6510818×10^{-4}	2.6536207×10^{-5}
0.2	1.3640647×10^{-4}	3.7394973×10^{-5}
0.9	8.9909328×10^{-4}	1.5510584×10^{-4}

Table 2. The values of $RMS(f(x))$ for various values of x^* and random choice of x and $t \in [0,1]$ and choosing the roots of Chebyshev polynomial for x and t with noiseless data when $n = m = l = 11$ and $\tau = 1.2$.

	GCV	$Quasi\ optimality$	$L-Curve$
$RP.$	1.6087857×10^{-14}	0.9469968	1.5225606×10^{-13}
$RMS(q(t))$	7.9688800×10^{-7}	2.8604627	5.2256551×10^{-6}
$RMS(f(x))$	8.3219848×10^{-6}	0.2730006	5.6483463×10^{-5}

Table 3. The values of $RMS(q(t))$ and $RMS(f(x))$ for various values of choosing regularization parameter (RP.) when $n = m = l = 11$, $\tau = 1.2$ and $x^* = 0.01$ with noiseless data.

t	$q(t)_{Exa.}$	$q(t)_{Num.}$		x	$f(x)_{Exa.}$	$f(x)_{Num.}$
0.0	6.2831855	6.2831855		0.0	-2.0000000	-1.9729112
0.2	6.6831851	6.6805987		0.2	35.9462051	35.9454727
0.4	7.0831852	7.0892920		0.4	22.0048313	21.9941330
0.6	7.4831853	7.4870520		0.6	-24.0048313	-24.0013065
0.8	7.8831854	7.8820233		0.8	-37.9462051	-37.9098396
1.0	8.2831850	8.2898026		1.0	-0.0000000	0.0716840

Table 4. The values of exact and numeric $q(t)$ and $f(x)$ with discrete noisy data, $\tilde{g}_i = g_i + \sigma.rand(1), \tilde{h}_i = h_i + \sigma.rand(1), \tilde{\psi}_i = \psi_i + \sigma.rand(1)$ and $\sigma = 1\%$ when $n = m = l = 20, \tau = 5$ and $x^* = 0.01$.

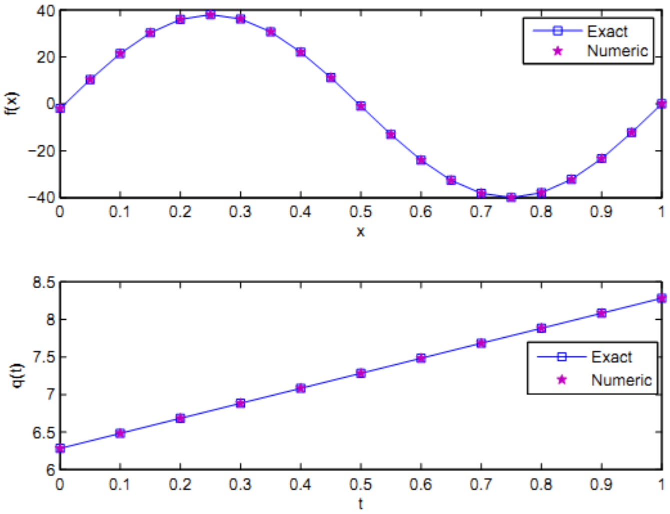

Figure 1. The values of $f(x)$ and $q(t)$ with noiseless data when $n = m = l = 11$, $\tau = 1.2$ and $x^* = 0.01$.

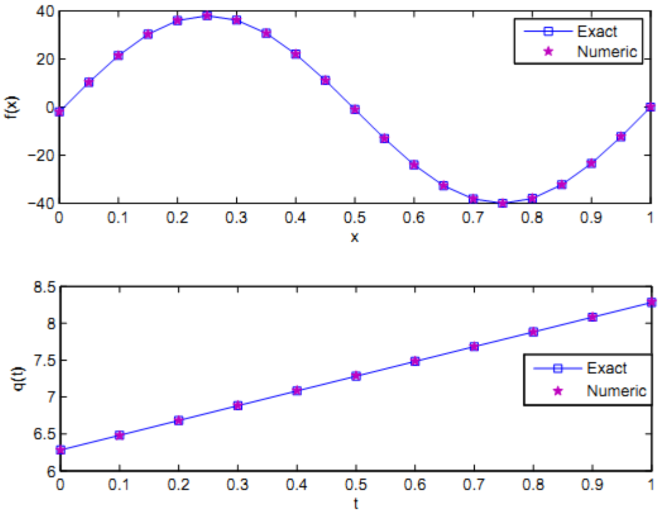

Figure 2. The values of $f(x)$ and $q(t)$ with noisy data when $n = m = l = 20$, $\tau = 5$ and $x^* = 0.01$ $(\sigma = 1\%)$.

Example 2. let us consider the following problem:

$$\frac{\partial U}{\partial t}(x,t) = \frac{\partial^2 U}{\partial x^2}(x,t) + f(x); \qquad 0 < x < 1, \ 0 < t < 1,$$

$$U(x,0)=0; \qquad 0 \le x \le 1,$$

$$\frac{\partial U}{\partial x}(1,t)+3U(1,t)=\pi\left(e^{-\pi^2 t}-1\right); \qquad 0 \le t \le 1,$$

With the overspecified conditions:

$$U(x^*,t)=\left(1-e^{-\pi^2 t}\right)\sin\left(\pi x^*\right); \qquad x^* \in (0,1), \quad 0 \le t \le 1,$$
$$U(x,1)=0.9999\sin\left(\pi x\right); \qquad 0 \le x \le 1.$$

The exact solution of this problem is:

$$U(x,t)=\left(1-e^{-\pi^2 t}\right)\sin\left(\pi x\right); \qquad 0 \le x \le 1, \quad 0 \le t \le 1,$$
$$f(x)=\pi^2\sin\left(\pi x\right); \qquad 0 \le x \le 1,$$

and

$$\frac{\partial U}{\partial x}(0,t)=\pi\left(1-e^{-\pi^2 t}\right); \qquad 0 \le t \le 1.$$

Tables 5 and 6 show the values of $RMS\left(q(t)\right)$ and $RMS\left(f(x)\right)$, respectively, with noiseless data for the roots of Chebyshev polynomial and random choosing of $x,t \in [0,1]$. From table 5 and 6 we conclude that the numerical results are more accurate when we choose discretization points as roots of Chebyshev polynomial. Table 7 shows the values of regularization parameter, $RMS\left(q(t)\right)$ and $RMS\left(f(x)\right)$ with three methods of choosing regularization parameter. From table 7 we conclude that the result of GCV method is the most accurate. We put $n=m=l=11$ and $\tau=1.2$ in table 5, 6 and 7 and $x^*=0.01$ in table 7. Table 8 shows the values of $q(t)$ and $f(x)$ with discrete noisy data as in example1 when $\sigma=0.1\%$. We put $x^*=0.01, n=m=l=20$ and $\tau=5$ in table 4. Figure 3 and figure 4 show the values of $f(x)$ and $q(t)$ with noisy and noiseless data, respectively.

x^*	$RMS\left(q(t)\right)$: *random*	$RMS\left(q(t)\right)$: *Chebyshev*
0.01	3.2516816×10^{-6}	5.0948250×10^{-7}
0.1	2.6673742×10^{-5}	9.4214829×10^{-7}
0.2	2.6992220×10^{-5}	1.4025782×10^{-6}
0.9	2.7580350×10^{-5}	1.4095266×10^{-5}

Table 5. The values of $RMS\left(q(t)\right)$ for various values of x^* and random choice of x and $t \in [0,1]$ and choosing the roots of Chebyshev polynomial for x and t with noiseless data when $n=m=l=11$ and $\tau=1.2$

x^*	$RMS\left(f(x)\right)$: *random*	$RMS\left(f(x)\right)$: *Chebyshev*
0.01	1.4873355×10^{-5}	4.0674022×10^{-6}
0.1	1.3452856×10^{-4}	2.6702419×10^{-6}
0.2	1.3899122×10^{-4}	2.5899801×10^{-6}
0.9	2.6992612×10^{-4}	2.4934692×10^{-5}

Table 6. The values of $RMS\left(f(x)\right)$ for various values of x^* and random choice of x and $t \in [0,1]$ and choosing the roots of Chebyshev polynomial for x and t with noiseless data when $n=m=l=11$ and $\tau=1.2$.

	GCV	Quasi optimality	$L-Curve$
$RP.$	1.6846206×10^{-14}	2.0245921	3.5406765×10^{-14}
$RMS(q(t))$	2.6702419×10^{-6}	5.7369061	3.1985928×10^{-6}
$RMS(f(x))$	9.4214829×10^{-7}	2.5436726	9.4762476×10^{-7}

Table 7. The values of $RMS(q(t))$ and $RMS(f(x))$ for various values of choosing regularization parameter (RP.) when $n=m=l=11$, $\tau=1.2$ and $x^*=0.01$ with noiseless data.

t	$q(t)_{Exa.}$	$q(t)_{Num.}$		x	$f(x)_{Exa.}$	$f(x)_{Num.}$
0.0	0.0000000	0.0000000		0.0	0.0000000	0.0148435
0.2	2.7051904	2.7060506		0.2	5.8012080	5.8022895
0.4	3.0809715	3.0846586		0.4	9.3865519	9.3881464
0.6	3.1331718	3.1355097		0.6	9.3865519	9.3915215
0.8	3.1404228	3.1421163		0.8	5.8012080	5.8042960
1.0	3.1414301	3.1440303		1.0	0.0000000	-0.0011152

Table 8. The values of exact and numeric $q(t)$ and $f(x)$ with discrete noisy data, $\tilde{g}_i = g_i + \sigma.rand(1), \tilde{h}_i = h_i + \sigma.rand(1), \tilde{\psi}_i = \psi_i + \sigma.rand(1)$ and $\sigma = 0.1\%$ when $n=m=l=20, \tau=5$ and $x^*=0.01$.

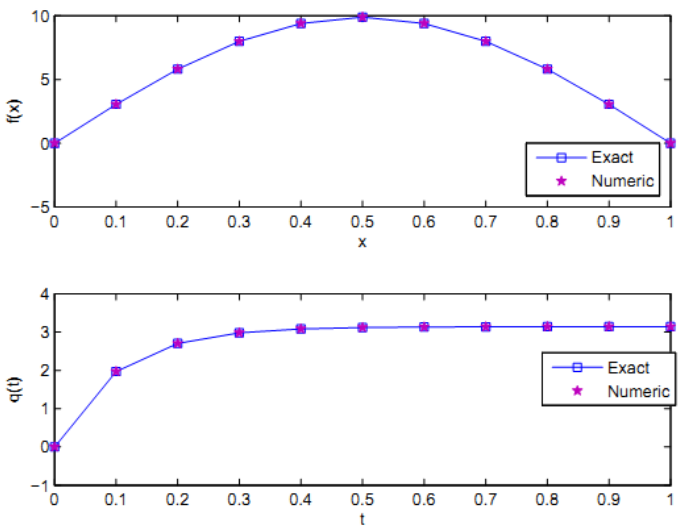

Figure 3. The values of $f(x)$ and $q(t)$ with noiseless data when $n=m=l=11$, $\tau=1.2$ and $x^*=0.01$.

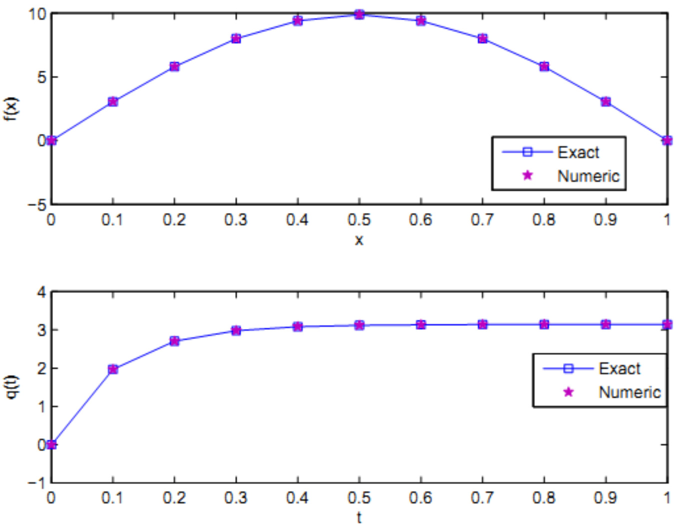

Figure 4. The values of $f(x)$ and $q(t)$ with noisy data when $n = m = l = 20,\ \tau = 5$ and $x^* = 0.01$ $(\sigma = 0.1\%)$.

Example 3. In this example let us consider the following problem:

$$\frac{\partial U}{\partial t}(x,t) = \frac{\partial^2 U}{\partial x^2}(x,t) + f(x); \qquad 0 < x < 1,\ 0 < t < 1,$$

$$U(x,0) = 2(\sin 2x + \cos 2x) + 0.25x^4; \qquad 0 \le x \le 1,$$

$$\frac{\partial U}{\partial x}(1,t) = -5.3018e^{-4t} + 6t + 1; \qquad 0 \le t \le 1,$$

With the overspecified conditions:

$$U(x^*,t) = 2e^{-4t}(\sin 2x^* + \cos 2x^*) + 3\left(t^2 + tx^* + 0.0833x^{*^4}\right); \qquad x^* \in (0,1), \quad 0 \le t \le 1,$$

$$U(x,1) = 0.0366(\sin 2x + \cos 2x) + 3\left(1 + x + 0.0833x^4\right); \qquad 0 \le x \le 1.$$

The exact solution of this problem is:

$$U(x,t) = 2e^{-4t}(\sin 2x + \cos 2x) + 3\left(t^2 + tx + 0.0833x^4\right); \qquad 0 \le x \le 1, \quad 0 \le t \le 1,$$

$$f(x) = 0; \qquad 0 \le x \le 1,$$

and

$$\frac{\partial U}{\partial x}(0,t) = 4e^{-4t} + 3t; \qquad 0 \le t \le 1.$$

Tables 9 and 10 show the values of $RMS(q(t))$ and $RMS(f(x))$, respectively, with noiseless data for the roots of Chebyshev polynomial and random choosing of $x,t \in [0,1]$. From table 9 and 10 we conclude that the numerical results are more accurate when we choose discretization points as roots of Chebyshev polynomial. Table 11 shows the values of regularization parameter, $RMS(q(t))$ and $RMS(f(x))$ with three methods of choosing regularization parameter. From table 11 we conclude that the result of GCV method is the most accurate. We put $n = m = l = 11$ and $\tau = 5$ in tables 9, 10 and 11 and $x^* = 0.2$ in table 11. Table 12 shows the values of $q(t)$ and $f(x)$ with discrete noisy data as in example1 where $\sigma = 1\%$. We put $x^* = 0.2, n = m = l = 11$ and $\tau = 5$ in table 12. Figures 5 and 6 show the values of $f(x)$ and $q(t)$ with noisy and noiseless data, respectively.

x^*	$RMS(q(t))$:random	$RMS(q(t))$:Chebyshev
0.01	1.9100628×10^{-4}	1.6098575×10^{-4}
0.1	2.0902304×10^{-4}	1.5256544×10^{-4}
0.2	1.8386828×10^{-4}	1.3861558×10^{-4}
0.9	0.0013397	0.0010482

Table 9. The values of $RMS(q(t))$ for various values of x^* and random choice of x and $t \in [0,1]$ and choosing the roots of Chebyshev polynomial for x and t with noiseless data when $n = m = l = 11$ and $\tau = 5$.

x^*	$RMS(f(x))$:random	$RMS(f(x))$:Chebyshev
0.01	5.3691405×10^{-4}	4.1352982×10^{-4}
0.1	4.4680462×10^{-4}	3.2773466×10^{-4}
0.2	3.7652593×10^{-4}	2.6523585×10^{-4}
0.9	0.0040282	0.0026973

Table 10. The values of $RMS(f(x))$ for various values of x^* and random choice of x and $t \in [0,1]$ and choosing the roots of Chebyshev polynomial for x and t with noiseless data when $n = m = l = 11$ and $\tau = 5$.

	GCV	Quasi optimality	$L-$Curve
$RP.$	1.1154973×10^{-15}	0.2935133	6.0636541×10^{-12}
$RMS(q(t))$	1.3861558×10^{-4}	3.2266636	0.0050862
$RMS(f(x))$	2.6523585×10^{-4}	8.9432737	0.0241315

Table 11. The values of $RMS(q(t))$ and $RMS(f(x))$ for various values of choosing regularization parameter (RP.) when $n = m = l = 11$, $\tau = 5$ and $x^* = 0.2$ with noiseless data.

t	$q(t)_{Exa.}$	$q(t)_{Num.}$		x	$f(x)_{Exa.}$	$f(x)_{Num.}$
0.0	4.0000000	4.0000005		0.0	0.0	0.0553125
0.2	1.7973158	1.8208135		0.2	0.0	0.0644821
0.4	0.8075861	0.8245509		0.4	0.0	0.0350606
0.6	0.3628718	0.3829993		0.6	0.0	-0.0033705
0.8	0.1630488	0.1978113		0.8	0.0	-0.0219372
1.0	0.0732626	0.0710324		1.0	0.0	-0.0057789

Table 12. The values of exact and numeric $q(t)$ and $f(x)$ with discrete noisy data, $\tilde{g}_i = g_i + \sigma.rand(1), \tilde{h}_i = h_i + \sigma.rand(1), \tilde{\psi}_i = \psi_i + \sigma.rand(1)$ and $\sigma = 1\%$ when $n = m = l = 11, \tau = 5$ and $x^* = 0.2$.

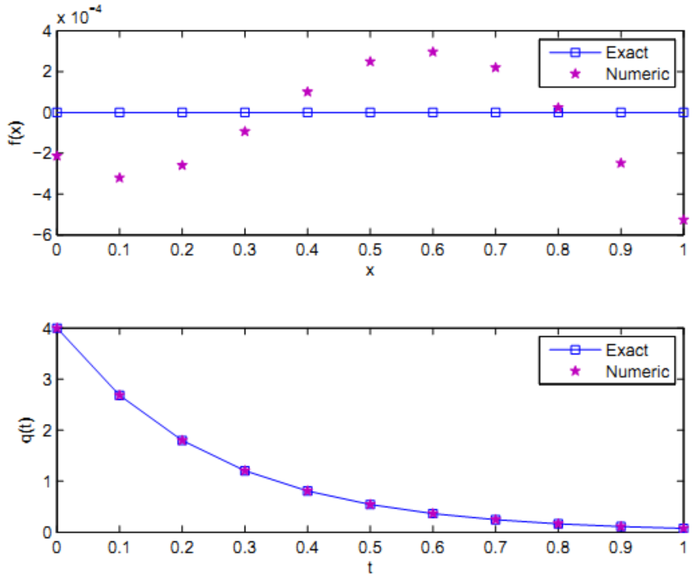

Figure 5. The values of $f(x)$ and $q(t)$ with noiseless data when $n=m=l=11,\ \tau=5$ and $x^*=0.2$.

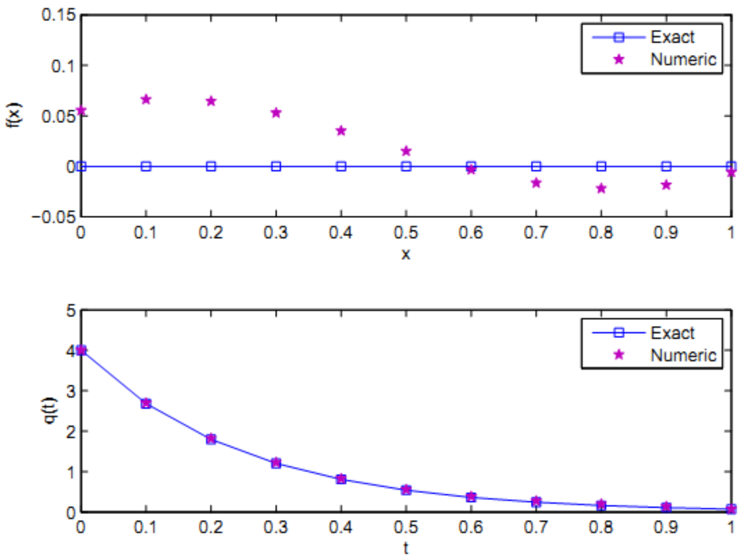

Figure 6. The values of $f(x)$ and $q(t)$ with noisy data when $n=m=l=11,\ \tau=5$ and $x^*=0.2$ $(\sigma=1\%)$.

5. CONCLUSION

In this paper the method of fundamental solution with the Tikhonov regularization technique has been developed for obtaining stable space-wise dependent heat source and heat flux at x=0. Numerical results were presented for three inverse problems. We take two approaches for choosing the discretizing points, random and the roots of Chebyshev polynomial. The obtained results show that:

1. Using roots of Chebyshev polynomial is more accurate than random choice.
2. When data contaminated by noise, the numerical solution are stable.
3. The method is accurate and reliable.

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