ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 7, No. 3, 2012, *pp*. 181-189

A Comparison of Numerical Solutions of Eleventh Order Two-point Boundary Value Problems

Amjad Hussain [1+], Syed Tauseef Mohyud-Din [2] and Ahmet Yildirim [3]

[1] Department of Mathematics, Zhejiang University, Hangzhou 310027, PR China
[2] Department of Mathematics, HITEC University, Taxila Cantt Pakistan
[3] 4146 SK No 16 Zeytinalani Mah. 35440 Urla-Izmir, Turkey
(*Received December 28, 2011, accepted March 10, 2012*)

Abstract. This study considers an application of differential transform method (DTM) to solving linear and nonlinear eleventh order two-point boundary value problems discussed in [S.S. Siddiqi, G. Akram and I. Zulfiqar. Solution of eleventh order boundary value problems using variational iteration technique. *European Journal of Scientific Research*. 2009, **30**: 505-525]. It is shown that the proposed method yields excellent approximations at minimum computational cost, whereas, variational iteration method is computationally expensive for such problems. Numerical results explicitly reveal the complete reliability and efficiency of the proposed algorithm.

Keywords: Eleventh order boundary value problems, Differential Transform Method, Comparison.

1. Introduction

Differential transform algorithm for obtaining approximate series solution to the differential equations is fairly well known. The technique was proposed by Zhou [1] in 1986, to solve linear and nonlinear problems arising in electrical circuits. Since then it has been extensively used by many others to solve a variety of problems, like for example Blasius boundary layer flow [2], Non-Newtonian fluid flow between two parallel plates [3], Nonlinear equations arising in heat transfer [4], vibration analysis of pipes covering fluids [5], free vibration analysis of a centrifugally stiffened beam [6], vibration of an elastic beam supported on elastic soil [7], Eigen-value problems [8-10], and so on. At the present time, DTM is considered sufficiently accurate for most scientific applications. Current research is aimed at evaluating this technique for eleventh order boundary value problems.

The boundary value problems of higher order have been investigated due to their mathematical importance and the potential for applications in numerous fields of science and engineering. Agarwal [11] presented the theorems stating the conditions for the existence and uniqueness of solutions of such BVPs, while no numerical methods are contained therein. Eleventh order boundary value problem have not taken much attention in the literature. Siddiqi et al. [12] applied variational iteration method (VIM) to obtain approximations for such problems there by converting the original problem into a system of integral equations. Although, VIM works well, but for technical reasons the calculations are more voluminous. To solve these problems at a minimum computational cost we implement differential transform method. It is found that the method is versatile and numerically programmable.

In this paper, we consider eleventh order boundary value problem in the form:

$$y^{(11)}(x) = f(x, y), \qquad x \in [a, b] \tag{1}$$

subject to the boundary conditions

[+] Corresponding author. Tel.: +86-571-87994844.
E-mail address: ahabbasi123@yahoo.com; ahabbasi123@hotmail.com.

$$\left.\begin{array}{ll} y(a)=A_1, & y(b)=B_1, \\ y^{(1)}(a)=A_2, & y^{(1)}(b)=B_2, \\ y^{(2)}(a)=A_3, & y^{(2)}(b)=B_3, \\ y^{(3)}(a)=A_4, & y^{(3)}(b)=B_4, \\ y^{(4)}(a)=A_5, & y^{(4)}(b)=B_5, \\ y^{(5)}(a)=C_1. & \end{array}\right\} \quad (2)$$

where $f(x,y)$ is assumed to be real and as many times differentiable as required for $x \in [a,b]$ and A_i, $B_i; i=1,\ldots,5$ and C_1 are real constants. In the next Section, we will tabulate some rules to implement differential transform method.

2. The Differential Transform Method

With reference to the articles [13-15], the basic definitions of differential transform are as follows

Definition 2.1 if $f(t)$ is analytic in the time domain T, then it can be differentiated continuously with respect to time t

$$\Phi(t,k) = \frac{d^k f(t)}{dt^k}, \qquad \text{for all } t \in T \tag{3}$$

for $t = t_i, \Phi(t,k) = \Phi(t_i,k)$ where k belongs to the set of non-negative integers, denoted as K-domain. Therefore (3) can be written as

$$F(k) = \Phi(t_i,k) = \left[\frac{d^k f(t)}{dt^k}\right]_{t=t_i}, \qquad \text{for all } k \in K \tag{4}$$

where $F(k)$ is called spectrum of $f(t)$ at $t = t_i$ in the K-domain.

Definition 2.2 if $f(t)$ can be expressed by Taylor series, then $f(t)$ can be represented as

$$f(t) = \sum_{k=0}^{\infty}\left[\frac{(t-t_i)^k}{k!}\right]F(k). \tag{5}$$

Equation (5) is called inverse differential transform of $F(k)$.

Using the differential transform, a differential equation in the domain of interest can be transformed to an algebraic equation in the K-domain and then $f(t)$ can be obtained by finite-term Taylor's series plus a remainder, as

$$f(t) = \sum_{k=0}^{n}\left[\frac{(t-t_i)^k}{k!}\right]F(k) + R_{n+1}(t). \tag{6}$$

In order to speed up the convergence rate and the accuracy of calculation, the entire domain of t needs to be split into sub-domains [13, 17]. The fundamental operations performed by differential transform can readily be obtained and are listed in Table 1.

Table1

The fundamental operations of differential transform method.

Original function	Transformed function
$f(x)=\alpha u(x)+\beta v(x)$	$F(k)=\alpha U(k)+\beta V(k)$
$f(x)=u(x)v(x)$	$F(k)=\sum_{l=0}^{k}U(l)V(k-l)$
$f(x)=\frac{d^n u(x)}{dx^n}$	$F(k)=\frac{(k+n)!}{k!}U(k+n)$
$f(x)=x^m$	$F(k)=\delta(k-m)=\begin{cases}1 & k=m\\ 0 & k\neq m\end{cases}$
$f(x)=\exp(\lambda x)$	$F(k)=\frac{\lambda^k}{k!}$
$f(x)=\sin(\omega x+\theta)$	$F(k)=\frac{\omega^k}{k!}\sin(\frac{k\pi}{2}+\theta)$
$f(x)=\cos(\omega x+\theta)$	$F(k)=\frac{\omega^k}{k!}\cos(\frac{k\pi}{2}+\theta)$

3. Applications

In the last Section, we give basic definitions and tabulated some rule to implement DTM. In this Section, we will apply it for problems considered in [12].

Example3.1: For $x\in[0,1]$, we consider the following problem

$$y^{(11)}=-22(5+x)e^x+y, \tag{7}$$

subject to the boundary conditions

$$\left.\begin{array}{ll} y(0)=1, & y(1)=0,\\ y^{(1)}(0)=1, & y^{(1)}(1)=-2e,\\ y^{(2)}(0)=-1, & y^{(2)}(1)=-6e,\\ y^{(3)}(0)=-5, & y^{(3)}(1)=-12e,\\ y^{(4)}(0)=-11, & y^{(4)}(1)=-20e,\\ y^{(5)}(0)=-19. & \end{array}\right\} \tag{8}$$

By using basic definition of differential transform at $x=0$, equations (7) and (8) can be written in the K - domain as

$$Y(k+11)=\frac{k!}{(k+11)!}\left[Y(k)-\frac{110}{k!}-22\sum_{l=0}^{k}\frac{\delta(l-1)}{(k-l)!}\right], \tag{9}$$

and

$$\left.\begin{array}{ll} Y(0)=1,\ \ Y(1)=1,\ \ Y(2)=-\frac{1}{2}, & Y(3)=-\frac{5}{6},\ \ Y(4)=-\frac{11}{24},\ \ Y(5)=-\frac{19}{120}, \\ \sum_{k=0}^{n} Y(K)=0,\ \ \sum_{k=1}^{n} kY(k)=-2e, & \sum_{k=2}^{n} k(k-1)Y(k)=-6e, \\ \sum_{k=3}^{n} k(k-1)(k-2)Y(k)=-12e, & \sum_{k=4}^{n} k(k-1)(k-2)(k-3)Y(k)=-20e. \end{array}\right\} \quad (10)$$

$Y(k+11)$, for $k \geq 0$ can be calculated from (9) and (10). Finally, by using inverse differential transform, following series solution up to the order of $O(x^{16})$ is obtained.

$$y(x)=1+x-\frac{1}{2}x^2-\frac{5}{6}x^3-\frac{11}{24}x^4-\frac{19}{120}x^5+\alpha x^6+\beta x^7+\gamma x^8+\lambda x^9+\mu x^{10}-\frac{109}{39916800}x^{11}$$
$$-\frac{131}{479001600}x^{12}-\frac{31}{1245404160}x^{13}-\frac{181}{87178291200}x^{14}-\frac{109}{118879488000}x^{15}+O(x^{16}), \quad (11)$$

where the constants $\alpha, \beta, \gamma, \lambda,$ and μ are given by

$$\left.\begin{array}{lll} \alpha = y^{(6)}/6! = Y(6), & \beta = y^{(7)}/7! = Y(7), & \gamma = y^{(8)}/8! = Y(8), \\ \lambda = y^{(9)}/9! = Y(9), & \mu = y^{(10)}/10! = Y(10). & \end{array}\right\} \quad (12)$$

Utilizing the transformed boundary conditions (10), we obtain following system of algebraic equations

$$\begin{aligned} \alpha+\beta+\gamma+\lambda+\mu &= -\frac{8172469307}{163459296000}, \\ 6\alpha+7\beta+8\gamma+9\lambda+10\mu &= \frac{34368590621}{6706022400}-2e, \\ 30\alpha+42\beta+56\gamma+72\lambda+90\mu &= \frac{15221960063}{1037836800}-6e, \\ 120\alpha+210\beta+336\gamma+504\lambda+720\mu &= \frac{12216031481}{479001600}-12e, \\ 360\alpha+840\beta+1680\gamma+3024\lambda+5040\mu &= \frac{5447801}{181440}-20e. \end{aligned} \quad (13)$$

Solving the system of equations (13), we calculate $\alpha, \beta, \gamma, \lambda,$ and μ as

$$\alpha=-0.0402778, \beta=-0.00813491, \gamma=-0.0013641, \lambda=-0.000195648, \mu=-0.0000245287.$$

Substituting values of these constants into (11), we get an approximation to the problem under consideration. Analytical solution for this problem is given by

$$y(x)=(1-x^2)e^x \quad (14)$$

Numerical results are reported in Table 2. This Table shows that our results are more accurate than those obtained in [12]. From this experiment it is evident that DTM is more efficient and numerically economical for eleventh order boundary value problems.

Example3.2: For $x \in [0,1]$, we consider the following problem

$$y^{(11)}=(1+22x-x^2)\cos x+(111-x^2)\sin x-y, \quad (15)$$

Table 2

Numerical comparison for example 3.1

		Absolute Errors for	
x	$y(x)$ Exact	DTM $(n=15)$	VIM $(n=15)$
0.0	1.0	0.0	0.0
0.1	1.09412	1.11022E−15	6.43929E−15
0.2	1.17255	3.79696E−14	2.40252E−13
0.3	1.22837	2.37144E−13	1.44063E−12
0.4	1.25313	6.58584E−13	3.84626E−12
0.5	1.23654	1.08158E−12	6.05849E−12
0.6	1.16616	1.13687E−12	6.09446E−12
0.7	1.02701	7.33413E−13	3.75366E−12
0.8	0.801195	2.32703E−13	1.14198E−12
0.9	0.467325	1.61537E−14	9.9476E−14
1.0	0.0	2.25731E−16	6.37165E−14

subject to the boundary conditions

$$\left.\begin{array}{ll} y(0)=1, & y(1)=0, \\ y^{(1)}(0)=0, & y^{(1)}(1)=-2\cos 1, \\ y^{(2)}(0)=-3, & y^{(2)}(1)=-2\cos 1+4\sin 1, \\ y^{(3)}(0)=0, & y^{(3)}(1)=6(\cos 1+\sin 1), \\ y^{(4)}(0)=13, & y^{(4)}(1)=12\cos 1-8\sin 1, \\ y^{(5)}(0)=0. & \end{array}\right\} \tag{16}$$

Taking the differential transform at $x=0$, equations (15) and (16) can be written in the K-domain as

$$Y(k+11)=\frac{k!}{(k+11)!}\left[\begin{array}{l} -Y(k)+\frac{1}{k!}\left(\cos(\frac{k\pi}{2})+111\sin(\frac{k\pi}{2})\right)+ \\ \sum_{l=0}^{k}\frac{1}{l!}\left(\left(22\delta(k-l-1)-\delta(k-l-2)\right)\cos(\frac{l\pi}{2})-\delta(k-l-2)\sin(\frac{l\pi}{2})\right) \end{array}\right], \tag{17}$$

and

$$\left.\begin{array}{l} Y(0)=1, \quad Y(1)=0, \quad Y(2)=-\frac{3}{2}, \quad Y(3)=0, \quad Y(4)=\frac{13}{24}, \quad Y(5)=0, \\ \sum_{k=0}^{n}Y(K)=0, \quad \sum_{k=1}^{n}kY(k)=-2\cos 1, \quad \sum_{k=2}^{n}k(k-1)Y(k)=-2\cos 1+4\sin 1, \\ \sum_{k=3}^{n}k(k-1)(k-2)Y(k)=6(\cos 1+\sin 1), \quad \sum_{k=4}^{n}k(k-1)(k-2)(k-3)Y(k)=12\cos 1-8\sin 1. \end{array}\right\} \tag{18}$$

Table 3

Numerical comparison for example 3.2

x	$y(x)$ Exact	Absolute Errors for DTM $(n=14)$	Absolute Errors for VIM $(n=15)$
0.0	1.0	0.0	0.0
0.1	0.985054	7.77156E−16	3.88578E−15
0.2	0.940864	3.37508E−14	1.46216E−13
0.3	0.869356	2.10276E−13	8.80518E−13
0.4	0.773691	5.82978E−13	2.35822E−12
0.5	0.658187	9.56013E−13	3.8014E−12
0.6	0.528215	1.00375E−12	5.14766E−12
0.7	0.39007	6.46427E−13	1.56224E−11
0.8	0.250814	2.04559E−13	8.99409E−11
0.9	0.118106	1.39472E−14	4.70031E−10
1.0	0.0	3.85976E−17	2.06386E−9

Using (17) and (18) $Y(k+11)$ for $k \geq 0$, can be calculated conveniently. Following series solution up to the order of $O(x^{15})$ can be obtained by using inverse differential transform

$$y(x) = 1 - \frac{3}{2}x^2 + \frac{13}{24}x^4 + \alpha x^6 + \beta x^7 + \gamma x^8 + \lambda x^9 + \mu x^{10} + \frac{19}{68428800}x^{12} - \frac{61}{29059430400}x^{14} + O(x^{15}), \tag{19}$$

where the constants $\alpha, \beta, \gamma, \lambda,$ and μ are given by (12). By virtue of the transformed boundary conditions (18), these constants are calculated numerically as

$$\alpha = -0.0430556, \beta = -6.38182\times10^{-9}, \gamma = 0.0014137, \lambda = -8.20952\times10^{-9}, \mu = -0.0000250748.$$

With these values of $\alpha, \beta, \gamma, \lambda,$ and μ equation (19) yields an excellent approximation for example 3.2. The exact solution for this problem is given by

$$y(x) = (1-x^2)\cos x. \tag{20}$$

The comparison of numerical results obtained by using DTM and those obtained in [12] by using VIM, for example 3.2, is presented in Table 3. Again, it is seen that DTM produces excellent approximations as compared to VIM [12] while reducing the volume of numerical calculations to the minimum level.

Example3.3: For $x \in [0,1]$, we consider the following nonlinear problem

$$y^{(11)} = 11(\cos x - \sin x) + x^2 - y^2 - x(\cos x + \sin x) - x^2 \sin 2x, \tag{21}$$

subject to the boundary conditions

Table 4

Numerical comparison for example 3.3

x	$y(x)$ Exact	Absolute Errors for DTM $(n=15)$	DTM $(n=22)$	VIM $(n=15)$
0.0	0.0	0.0	0.0	0.0
0.1	−0.0895171	1.50782E−13	1.38778E−17	1.02696E−15
0.2	−0.156279	5.23079E−12	0.0	5.34017E−14
0.3	−0.197945	3.06515E−11	0.0	5.59025E−13
0.4	−0.212657	9.7418E−11	0.0	3.32057E−12
0.5	−0.199079	3.68908E−10	0.0	1.47861E−11
0.6	−0.156416	1.99656E−9	8.32667E−17	5.37756E−11
0.7	−0.0844372	1.02592E−8	2.77556E−17	1.65525E−10
0.8	0.0165195	4.38587E−8	4.16334E−17	4.41529E−10
0.9	0.145545	1.58466E−7	8.32667E−17	1.03969E−9
1.0	0.3011	4.99498E−7	5.55112E−17	2.19082E−9

$$\left.\begin{array}{l} y(0)=0,\quad y^{(1)}(0)=-1,\quad y^{(2)}(0)=2,\quad y^{(3)}(0)=3,\quad y^{(4)}(0)=-4,\quad y^{(5)}(0)=-5,\\ y(1)=\sin 1-\cos 1,\quad y^{(1)}(1)=2\sin 1,\quad y^{(2)}(1)=3\cos 1+\sin 1,\\ y^{(3)}(1)=2(\cos 1-2\sin 1),\quad y^{(4)}(1)=-5\cos 1-3\sin 1, \end{array}\right\} \tag{22}$$

Taking the differential transform at $x=0$, the transformed version of (21) can be written as

$$Y(k+11)=\frac{k!}{(k+11)!}\left[\begin{array}{l}\frac{11}{k!}\left(\cos(\frac{k\pi}{2})-\sin(\frac{k\pi}{2})\right)+\delta(k-2)\\ \sum_{l=0}^{k}\left(-Y(l)Y(k-l)-\frac{1}{l!}\left(\begin{array}{l}\left(\cos(\frac{l\pi}{2})+\sin(\frac{l\pi}{2})\right)\delta(k-l-1)\\ +2^{l}\,\delta(k-l-2)\sin(\frac{l\pi}{2})\end{array}\right)\right)\end{array}\right], \tag{23}$$

And that of (22) can be written as

$$\left.\begin{array}{l} Y(0)=0,\quad Y(1)=-1,\quad Y(2)=1,\quad Y(3)=\frac{1}{2},\quad Y(4)=-\frac{1}{6},\quad Y(5)=-\frac{1}{24},\\ \sum_{k=0}^{n}Y(K)=\sin 1-\cos 1,\ \sum_{k=1}^{n}kY(k)=2\sin 1,\quad \sum_{k=2}^{n}k(k-1)Y(k)=3\cos 1+\sin 1,\\ \sum_{k=3}^{n}k(k-1)(k-2)Y(k)=2(\cos 1-2\sin 1),\quad \sum_{k=4}^{n}k(k-1)(k-2)(k-3)Y(k)=-5\cos 1-3\sin 1. \end{array}\right\} \tag{24}$$

In view of equations (23) and (24), series solution up to the order of $O(x^{16})$ can be obtained by using inverse differential transform

$$\begin{aligned} y(x)=&-x+x^2+\frac{1}{2}x^3-\frac{1}{6}x^4-\frac{1}{24}x^5+\alpha x^6+\beta x^7+\gamma x^8+\lambda x^9+\mu x^{10}+\frac{1}{3628800}x^{11}\\ &-\frac{1}{39916800}x^{12}-\frac{1}{479001600}x^{13}+\frac{1}{6227020800}x^{14}+\frac{1}{87178291200}x^{15}+O(x^{16}), \end{aligned} \tag{25}$$

where the constants $\alpha, \beta, \gamma, \lambda,$ and μ were defined in (12). By virtue of the transformed boundary conditions (24), these constants are calculated numerically as

$$\alpha = 0.00833333, \beta = 0.00138889, \gamma = -0.000198413, \lambda = -0.000024801, \mu = 2.75556\times 10^{-6}.$$

With these values of $\alpha, \beta, \gamma, \lambda,$ and μ equation (25) gives us an approximate solution to the problem under consideration. The problem can be solved analytically and the exact solution is

$$y(x) = x(\sin x - \cos x). \tag{26}$$

The comparison of numerical results obtained by using DTM and those obtained in [12] by using VIM, for example 3.3, is reported in Table 4. Although, the use of DTM do away one with voluminous calculations presented in [12], but the solution is slightly less accurate in this case. To fix the problem we proceed as below.

Repeating the same procedure for 22-term approximate solution, the following series solution is obtained:

$$\begin{aligned} y(x) = {} & -x + x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^4 - \frac{1}{24}x^5 + 0.833333x^6 + 0.00138889x^7 - 0.000198413x^8 \\ & - 0.0000248016x^9 + 2.75573\times 10^{-6}x^{10} + \frac{1}{3628800}x^{11} - \frac{1}{39916800}x^{12} \\ & - \frac{1}{479001600}x^{13} + \frac{1}{6227020800}x^{14} + \frac{1}{87178291200}x^{15} - \frac{1}{1307674368000}x^{16} \\ & - \frac{1}{20922789888000}x^{17} + 2.81146\times 10^{-15}x^{18} + 1.56192\times 10^{-16}x^{19} \\ & - 8.22064\times 10^{-18}x^{20} - 4.11032\times 10^{-19}x^{21} + 1.95729\times 10^{-20}x^{22} + O(x^{23}), \end{aligned} \tag{27}$$

Using (27), we calculate $y(x)$ at different values of the variable x, and the results so obtained are shown in the Table 4. From this Table, It is clear that by using 22 terms the series solution approaches very close to the exact solution and the absolute error is indeed very small.

4. Concluding remarks

In this work, we presented an analysis of differential transform method for evaluating eleventh order boundary value problems with two-point boundary conditions. By comparing our results with those obtained by using variational iteration method it is concluded that differential transform method yields excellent approximations at minimum computational cost without first evaluating the Lagrange multipliers as in the case of VIM. Moreover, it is seen that the efficiency of this approach can be dramatically enhanced by computing further components of the approximating series solution.

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