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The Prove of a Class of Variational Inequalities

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Abstract. If $f(x)$ is a differentiable convex function and its Heissen matrix is positive semi-definite,we can prove the inequality $(\tilde{x}-x^*)^T \nabla f(x) \geq -\frac{1}{2}(x-\tilde{x})^T(\nabla f(x)-\nabla f(\tilde{x}))$.
Meet the above inequality from the general convex function of the convergence card sequence of functions on the measurable set.

Keyword. Convex function, positive semidifinite, cauchy sequences, convergence

1. Introduction

let $f(x)$ be a differentiable convex function, and $\nabla f(x)$ be the gradient of the function. x^* is the only optimal value point, which makes $\nabla f(x^*)=0$.Then to any x and $\tilde{x}$, if H is positive semi-definite,

$$(\tilde{x}-x^*)^T \nabla f(x) \geq -\frac{1}{2}(x-\tilde{x})^T(\nabla f(x)-\nabla f(\tilde{x})) \qquad (1)$$

(see [11],[12],[13],[14],[15],[16],[17]), but if $f(x)$ only be a differentiable convex function we can get weaker inequality(see [18],[19]),can we get the inequality to any convex function?

2. Some properties

Conclusion 1: Let H be positive semi-definite.Then we have

$$a^T Hb \geq -\frac{1}{2}(a-b)^T H(a-b)$$

Proof: $(a-b)^T H(a-b)$

$=(a-b, H(a-b))$

$=(a, Ha)-(b, Ha)-(a, Hb)+(b, Hb)$

since H is positive semi-definite and $(a, Hb)=(b, Ha)$,

we have $(a, Hb) \geq -\frac{1}{2}(a-b, H(a-b))$。

Conclusion 2: assume that $f(x)$ is differentiable on R^n, then $f(x)$ convex if and only if

$$f(y)-f(x) \geq \nabla f(x)^T(y-x)\text{。}$$

Proof: if $f(x)$ is convex, then

$$f((1-\theta)x+\theta y) \leq (1-\theta)f(x)+\theta f(y), \quad \theta \in [0,1], x, y, \in R^n$$

So we have $f(y)-f(x) \geq \dfrac{f(x+\theta(y-x))-f(x)}{\theta}$

letting $\theta \to 0_+$, we get $f(y)-f(x) \geq (y-x)^T \nabla f(x)$.

Conversely if $f(x)=\frac{1}{2}x^T Hx+c^T x, H^T=H$ and H is semi-definite，the conclusion is obviously.

Theorem 1. Let $f(x)=\frac{1}{2}x^T Hx+c^T x, H^T=H$ and H be positive semi-ddefinite，Then we have

$$(\tilde{x}-x^*)^T \nabla f(x) \ge -\frac{1}{2}(x-\tilde{x})^T(\nabla f(x)-\nabla f(\tilde{x})) \qquad (2)$$

Proof: note that in this case $\nabla f(x)=Hx+c$ and thus the equivalent of（2）is

$$(\tilde{x}-x^*)^T(Hx+c) \ge -\frac{1}{2}(x-\tilde{x})^T H(x-\tilde{x}) \qquad (3)$$

By using $Hx^*+c=0$, we have $(\tilde{x}-x^*)^T\{(Hx+c)-H(x-x^*)\}=0$ and consequently $(\tilde{x}-x^*)^T(Hx+c)=(\tilde{x}-x^*)^T H(x-x^*)$

Since H is positive semi-definite ,we can get the conclusion .

For a general convex function， weaker than (1) the conclusion is clearly established.

Theorem 2.Let $f(x): R^n \to R$ be convex and differentiable. Then we have

$$(\tilde{x}-x^*)^T \nabla f(x) \ge -(x-\tilde{x})^T(\nabla f(x)-\nabla f(\tilde{x})). \text{(see [15],[16])} \qquad (4)$$

Proof: since $f(x)$ is convex and differentiable,we have

$$f(x^*) \ge f(x)+(x^*-x)^T \nabla f(x). \qquad (5)$$

Since $x*$ is the minimum point， $f(\tilde{x}) \ge f(x*)$. Therefore,it follows from (5)that

$$(x-x^*)^T \nabla f(x) \ge f(x)-f(\tilde{x}) \qquad (6)$$

Since f is convex， $f(x)-f(\tilde{x}) \ge \nabla f(\tilde{x})^T(x-\tilde{x})$ Using (5) (6) we have:

$$(x-\tilde{x})^T \nabla f(x) \ge (x-\tilde{x})^T \nabla f(\tilde{x}) \qquad (7)$$

Adding $(\tilde{x}-x)^T \nabla f(x)$ to the both sides of (7),we get

$$(\tilde{x}-x*)^T \nabla f(x) \ge (x-\tilde{x})^T(\nabla f(\tilde{x})-\nabla f(x)).$$

For general convex function ,can conclude that (1).

3. The main result

Conclusion 3: $L^p(\mu)$ is a complete metric space，that is to say any cauchy sequence of the $L^p(\mu)$ converges to an element of the $L^p(\mu)$.

Theorem 3: If $1 \le p \le \infty$, and if $\{f_n\}$ is a Cauchy sequence in $L^p(\mu)$， with limit f ,then $\{f_n\}$ has a subsequence which converges pointwise almost everywhere to $f(x)$.

According to Theorem 3(see [20]),for a differentiable convex function $f(x)$， $x \in R^n$,can always find a Cauchy sequence $\{f_n\}$ of $L^p(\mu)$ .By the conclusion of Theorem 3 shows that a subsequence $\{f_{n_i}\}$ can always be found in the above Cauchy sequence ,which converge to $f(x)$. f_{n_i} is a differentiable convex function and meets $f_{n_i}(x)=\frac{1}{2}x^T H_{n_i}x+c^T x, H_{n_i}=H_{n_i}^T$, among which, H_{n_i} is positive semi-definite.

4. References

[1] J. Barzilai and J.M. Borwein. Two-point step size gradien methods. *IMA J. Numerical Analysis.* 1988, **8:** 141-148.

[2] E. Blum and W.Oettli. Mathematatische Optimierung. *Grundlagen und Verfahren. Okonometrie und Unternehmensforschung.* Berlin-Heidelberg-New York: Springer-Verlag, 1975.

[3] Y.H. Dai. Alternate step gradient method. *Optimization.* 2003, **52**: 395-415.

[4] Y.H. Dai and L.Z. Liao. R-linear convergence of the Barzilai and Borwein gradient method. *IMA J. Numerical Analysis.* 2002, **22**: 1-10.

[5] Y.H. Dai and Y. Yuan. Alternate minimization gradient method. *IMA J. Numerical Analysis.* 2003, **23**:377-393.

[6] B.S. He. A new method for a class of linear variational inequalities. *Math. Progr.* 1994, **66**: 137-144.

[7] B.S. He. Solving a class of linear projection equations. *Numerische Mathematik*. 1994, **68**: 71-80.

[8] B.S. He. A class of projection and contraction methods for monotone variational inequalities. *Applied Mathematics and optimization*. 1997, **bf 35**: 69-76.

[9] B.S. He. Inexact implicit methods for monotone general variational inequalities. *Math. Program.* 1999, **Series A, 86**: 199-217, 1999.

[10] B.S He and L.-Z Liao. Improvements of some projection methods for monotone nonlinear variational inequalities. 2002, JOTA, **112**: 111-128.

[11] X. Wang, B.S. He, and L.Z. Liao. Steplengths in the extragradient type methods. *J. of Comput. Appl. Math*. 2010, **233**: 2925-2939.

[12] B.S. He, X.Z. He, and Henry X. Liu. Solving a class of constrained 'black-box' inverse variational inequalities. *European J. Oper. Res*. 2010, **204**: 391-401.

[13] X.L. Fu, and B.S. He. Self-adaptive projection-based prediction correction method for constrained variational inequalities. *Front. Math. China*. 2010, **5** (1): 3-21.

[14] H. Yang, W. Xu, B.S. He, and Q. Meng. Road pricing for congestion control with unknown demand and cost functions. *Trans. Res.* 2010, **Part C. 18**: 157-175.

[15] B.S. He, X. Wang, and J.F. Yang. A comparison of different contraction methods for monotone variational inequalities. *J. Comput. Math*. 2009, **27** (4): 459-473.

[16] B.S. He, X.L. Fu, and Z.K. Jiang. Proximal-point algorithm using a linear proximal term. *J. Optim. Theory Appl*. 2009, **141** (2): 299-319.

[17] B.S. He, X.Z. He, Henry X. Liu, and T. Wu. Self-adaptive projection method for co-coercive variational inequalities. *European J. Oper. Res*. 2009, **196** (1): 43-48.

[18] B.S. He. Parallel splitting augmented Lagrangian methods for monotone structured variational inequalities. *Comput. Optim. Appl*. 2009, **42** (2): 195-212.

[19] B. Martinet. Regularisation, d'in ´equations variationelles par approximations succesives. *Rev. Francaised'Inform. Recherche Oper*. 1970, **4**: 154-159.

[20] W. Rudin. *Real And Complex Analysis*(Third Edition[M]). New York: McGraw-Hill Companies, Inc. 1987, pp. 65-69.