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Application of Homotopy Perturbation Transform Method for Solving Linear and Nonlinear Klein-Gordon Equations

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Abstract. In this paper, the homotopy perturbation transform method (HPTM) has been applied to obtain the solution of the linear and nonlinear Klein-Gordon equations. The homotopy perturbation transform method is a combined form of the Laplace transform method with the homotopy perturbation method. This scheme finds the solution without any discretization or restrictive assumptions and avoids the round-off errors. The fact that this technique solves nonlinear problems without using Adomian's polynomials can be considered as a clear advantage of this technique over the decomposition method. The results reveal that the proposed algorithm is very efficient, simple and can be applied to other nonlinear problems.

Mathematics Subject Classification: 35C05, 35C10.

Key words and Phrases: Laplace transform method, Homotopy perturbation method, Linear and Nonlinear Klein-Gordon equations, He's Polynomials.

1. Introduction

Nonlinear phenomena have important effects on applied mathematics, physics and related to engineering; many such physical phenomena are modeled in terms of nonlinear partial differential equations. For example, the Klein-Gordon equations which are of the form

$$u_{tt}(x,t) - u_{xx}(x,t) + au(x,t) = h(x,t), \tag{1}$$

with initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \tag{2}$$

appears in quantum field theory, relativistic physics, dispersive wave-phenomena, plasma physics, nonlinear optics and applied physical sciences. Several techniques including finite difference, collocation, finite element, inverse scattering, decomposition and variational iteration using Adomian's polynomials have been used to handle such equations [5, 6, 7, 48]. He [1-4] developed the homotopy perturbation technique for solving such physical problems. In recent years, many research workers have paid attention to study the solutions of nonlinear partial differential equations by using various methods. Among these are the Adomian decomposition method (ADM) [8], He's semi-inverse method [9], the tanh method, the homotopy perturbation method (HPM), the differential transform method and the variational iteration method (VIM) [10-17]. He [25-38] developed the homotopy perturbation method (HPM) by merging the standard homotopy and perturbation for solving various physical problems. The Laplace transform is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. Various ways have been proposed recently to deal with these nonlinearities such as the Adomian decomposition method

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[39] and the Laplace decomposition algorithm [40-44]. Furthermore, the homotopy perturbation method is also combined with the well-known Laplace transformation method [45] and the variational iteration method [46] to produce a highly effective technique for handling many nonlinear problems. In a recent paper Khan and Wu [23] proposed the homotopy perturbation transform method (HPTM) for solving the nonlinear equations. It is worth mentioning that this method is an elegant combination of the Laplace transformation, the homotopy perturbation method and He's polynomials and is mainly due to Ghorbani [20, 21]. The homotopy perturbation transform method provides the solution in a rapid convergent series which may lead to the solution in a closed form. The advantage of this method is its capability of combining two powerful methods for obtaining exact solutions for nonlinear equations. In this article, we apply the homotopy perturbation transform method (HPTM) for solving the linear and nonlinear Klein-Gordon equations to show the simplicity and straight forwardness of the method.

2. Homotopy perturbation transform method (HPTM)

To illustrate the basic idea of this method, we consider a general nonlinear partial differential equation with the initial conditions of the form:

$$Du(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \tag{3}$$

$$u(x,0) = h(x),\ u_t(x,0) = f(x),$$

where D is the second order linear differential operator $D = \partial^2/\partial t^2$, R is the linear differential operator of less order than D, N represents the general nonlinear differential operator and g(x, t) is the source term. Taking the Laplace transform (denoted in this paper by L) on both sides of eq. (3), we get

$$L[Du(x,t)] + L[Ru(x,t)] + L[Nu(x,t)] = L[g(x,t)]. \tag{4}$$

Using the differentiation property of the Laplace transform, we have

$$L[u(x,t)] = \frac{h(x)}{s} + \frac{f(x)}{s^2} + \frac{1}{s^2}L[g(x,t)] - \frac{1}{s^2}L[Ru(x,t)] - \frac{1}{s^2}L[Nu(x,t)]. \tag{5}$$

Operating with the Laplace inverse on both sides of eq. (5) gives

$$u(x,t) = G(x,t) - L^{-1}\left[\frac{1}{s^2}L[Ru(x,t) + Nu(x,t)]\right], \tag{6}$$

where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now we apply the homotopy perturbation method

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \tag{7}$$

and the nonlinear term can be decomposed as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u), \tag{8}$$

for some He's polynomials $H_n(u)$ (see [21, 46]) that are given by

$$H_n(u_0, u_1, ..., u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0}, \ n = 0, 1, 2, 3\ldots \tag{9}$$

Substituting eq. (7) and eq. (8) in eq. (6), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) - p\left(L^{-1}\left[\frac{1}{s^2}L\left[R\sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right), \tag{10}$$

which is the coupling of the Laplace transform and the homotopy perturbation method using He's polynomials. Comparing the coefficient of like powers of p, the following approximations are obtained.

$$
\begin{aligned}
p^0 &: u_0(x,t) = G(x,t), \\
p^1 &: u_1(x,t) = -L^{-1}\left[\frac{1}{s^2} L\left[R\,u_0(x,t) + H_0(u)\right]\right], \\
p^2 &: u_2(x,t) = -L^{-1}\left[\frac{1}{s^2} L\left[R\,u_1(x,t) + H_1(u)\right]\right], \\
p^2 &: u_2(x,t) = -L^{-1}\left[\frac{1}{s^2} L\left[R\,u_2(x,t) + H_2(u)\right]\right], \\
&\vdots
\end{aligned} \tag{11}
$$

3. Numerical Applications

In this section, we use homotopy perturbation transform method (HPTM) in solving the linear and nonlinear Klein-Gordon equations.

Example 3.1. Consider the following linear Klein-Gordon equation

$$u_{tt}(x,t) - u_{xx}(x,t) + u(x,t) = 0, \tag{12}$$

with the initial conditions

$$u(x,0) = 0,\ u_t(x,0) = x. \tag{13}$$

Applying the Laplace transform on both sides of eq. (12) subject to the initial conditions (13), we have

$$L[u(x,t)] = \frac{x}{s^2} + \frac{1}{s^2} L[u_{xx}(x,t) - u(x,t)]. \tag{14}$$

The inverse of Laplace transform implies that

$$u(x,t) = xt + L^{-1}\left[\frac{1}{s^2} L[u_{xx}(x,t) - u(x,t)]\right]. \tag{15}$$

Now, applying the homotopy perturbation method, we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = xt + p\left(L^{-1}\left[\frac{1}{s^2} L\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xx} - \sum_{n=0}^{\infty} p^n u_n(x,t)\right]\right]\right). \tag{16}$$

Comparing the coefficients of like powers of p, we have

$$
\begin{aligned}
p^0 &: u_0(x,t) = xt, \\
p^1 &: u_1(x,t) = L^{-1}\left[\frac{1}{s^2} L\left[(u_0)_{xx} - u_0\right]\right] = \frac{-xt^3}{3!}, \\
p^2 &: u_2(x,t) = -L^{-1}\left[\frac{1}{s^2} L\left[(u_1)_{xx} - u_1\right]\right] = \frac{xt^5}{5!}, \\
&\vdots
\end{aligned} \tag{17}
$$

Therefore the solution $u(x,t)$ is given by

$$u(x,t) = x\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots \right), \tag{18}$$

in series form, and

$$u(x,t) = x \sin t, \tag{19}$$

in closed form.

Example 3.2. Consider the following linear Klein-Gordon equation

$$u_{tt}(x,t) - u_{xx}(x,t) + u(x,t) = 2\sin x, \tag{20}$$

with the initial conditions

$$u(x,0) = \sin x,\ u_t(x,0) = 1. \tag{21}$$

Applying the Laplace transform on both sides of eq. (20) subject to the initial conditions (21), we have

$$L[u(x,t)] = \frac{2\sin x}{s^3} + \frac{\sin x}{s} + \frac{1}{s^2} + \frac{1}{s^2} L[u_{xx}(x,t) - u(x,t)]. \tag{22}$$

The inverse of Laplace transform implies that

$$u(x,t) = t^2 \sin x + \sin x + t + L^{-1}\left[\frac{1}{s^2} L[u_{xx}(x,t) - u(x,t)] \right]. \tag{23}$$

Now, applying the homotopy perturbation method, we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = t^2 \sin x + \sin x + t$$
$$+ p\left(L^{-1}\left[\frac{1}{s^2} L\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,t) \right)_{xx} - \sum_{n=0}^{\infty} p^n u_n(x,t) \right] \right] \right). \tag{24}$$

Comparing the coefficients of like powers of p, we have

$$p^0 : u_0(x,t) = t^2 \sin x + \sin x + t,$$

$$p^1 : u_1(x,t) = L^{-1}\left[\frac{1}{s^2} L\left[(u_0)_{xx} - u_0 \right] \right] = -t^2 \sin x - \frac{t^3}{3!} - \frac{t^4}{3!} \sin x, \tag{25}$$

$$p^2 : u_2(x,t) = L^{-1}\left[\frac{1}{s^2} L\left[(u_1)_{xx} - u_1 \right] \right] = \frac{t^4}{3!} \sin x + \frac{t^6}{90} \sin x + \frac{t^5}{5!},$$

$$\vdots$$

Therefore the solution $u(x,t)$ is given by

$$u(x,t) = \sin x + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots \right) = \sin x + \sin t. \tag{26}$$

Example 3.3. Consider the following nonlinear Klein-Gordon equation

$$u_{tt}(x,t) - u_{xx}(x,t) + u^2(x,t) = x^2 t^2, \tag{27}$$

with the initial conditions

$$u(x,0)=0,\ u_t(x,0)=x. \tag{28}$$

Applying the Laplace transform on both sides of eq. (27) subject to the initial conditions (28), we have

$$L[u(x,t)]=\frac{2x^2}{s^5}+\frac{x}{s^2}+\frac{1}{s^2}L[u_{xx}(x,t)-u^2(x,t)]. \tag{29}$$

The inverse of Laplace transform implies that

$$u(x,t)=\frac{x^2t^4}{12}+xt+L^{-1}\left[\frac{1}{s^2}L[u_{xx}(x,t)-u^2(x,t)]\right]. \tag{30}$$

Now, applying the homotopy perturbation method, we get

$$\sum_{n=0}^{\infty}p^n u_n(x,t)=\frac{x^2t^4}{12}+xt+p\left(L^{-1}\left[\frac{1}{s^2}L\left[\left(\sum_{n=0}^{\infty}p^n u_n(x,t)\right)_{xx}-\sum_{n=0}^{\infty}p^n H_n(u)\right]\right]\right), \tag{31}$$

where $H_n(u)$ are He's polynomial [21, 46] that represents the nonlinear terms. The first few components of He's polynomials, are given by

$$\begin{aligned}&H_0(u)=(u_0)^2,\\ &H_1(u)=2u_0u_1,\\ &\vdots\end{aligned} \tag{32}$$

Comparing the coefficients of like powers of p, we have

$$\begin{aligned}
p^0 &: u_0(x,t)=\frac{x^2t^4}{12}+xt,\\
p^1 &: u_1(x,t)=L^{-1}\left[\frac{1}{s^2}L\left[(u_0)_{xx}-H_0(u)\right]\right]\\
&\qquad=\frac{t^6}{180}-\frac{x^4t^{10}}{12960}-\frac{x^3t^7}{252}-\frac{x^2t^4}{12},\\
p^2 &: u_2(x,t)=L^{-1}\left[\frac{1}{s^2}L\left[(u_1)_{xx}-H_1(u)\right]\right]\\
&\qquad=-\frac{x^2t^{12}}{71280}-\frac{11xt^9}{22680}-\frac{t^6}{180}+\frac{x^6t^{16}}{18662400}+\frac{11x^4t^{10}}{45360}\\
&\qquad\quad+\frac{383x^5t^{13}}{15921360}+\frac{x^3t^7}{252},\\
&\vdots
\end{aligned} \tag{33}$$

Therefore the solution $u(x,t)$ is given by

$$u(x,t)=xt. \tag{34}$$

Example 3.4. Consider the following nonlinear Klein-Gordon equation

$$u_{tt}(x,t)-u_{xx}(x,t)+u^2(x,t)=2x^2-2t^2+x^4t^4, \tag{35}$$

with the initial conditions

$$u(x,0)=0,\ u_t(x,0)=0. \tag{36}$$

Applying the Laplace transform on both sides of eq. (35) subject to the initial conditions (36), we have

$$L[u(x,t)]=\frac{2x^2}{s^3}-\frac{4}{s^5}+\frac{24x^4}{s^7}+\frac{1}{s^2}L[u_{xx}(x,t)-u^2(x,t)]. \tag{37}$$

The inverse of Laplace transform implies that

$$u(x,t)=x^2t^2-\frac{t^4}{6}+\frac{x^4t^6}{30}+L^{-1}\left[\frac{1}{s^2}L[u_{xx}(x,t)-u^2(x,t)]\right]. \tag{38}$$

Now, applying the homotopy perturbation method, we get

$$\sum_{n=0}^{\infty}p^n u_n(x,t)=x^2t^2-\frac{t^4}{6}+\frac{x^4t^6}{30}$$
$$+p\left(L^{-1}\left[\frac{1}{s^2}L\left[\left(\sum_{n=0}^{\infty}p^n u_n(x,t)\right)_{xx}-\sum_{n=0}^{\infty}p^n H_n(u)\right]\right]\right), \tag{39}$$

where $H_n(u)$ are He's polynomial [21, 46] that represents the nonlinear terms.

In a similar manner as above the solution $u(x,t)$ is given by

$$u(x,t)=x^2t^2. \tag{40}$$

Example 3.5. Consider the following nonlinear Klein-Gordon equation

$$u_{tt}(x,t)-u_{xx}(x,t)+u^2(x,t)=6xt(x^2-t^2)+x^6t^6, \tag{41}$$

with the initial conditions

$$u(x,0)=0,\ u_t(x,0)=0. \tag{42}$$

Applying the Laplace transform on both sides of eq. (41) subject to the initial conditions (42), we have

$$L[u(x,t)]=\frac{6x^3}{s^4}-\frac{36x}{s^6}+x^6\frac{8!}{s^{11}}+\frac{1}{s^2}L[u_{xx}(x,t)-u^2(x,t)]. \tag{43}$$

The inverse of Laplace transform implies that

$$u(x,t)=x^3t^3-\frac{3xt^5}{10}+\frac{x^6t^{10}}{90}+L^{-1}\left[\frac{1}{s^2}L[u_{xx}(x,t)-u^2(x,t)]\right]. \tag{44}$$

Now, applying the homotopy perturbation method, we get

$$\sum_{n=0}^{\infty}p^n u_n(x,t)=x^3t^3-\frac{3xt^5}{10}+\frac{x^6t^{10}}{90}$$
$$+p\left(L^{-1}\left[\frac{1}{s^2}L\left[\left(\sum_{n=0}^{\infty}p^n u_n(x,t)\right)_{xx}-\sum_{n=0}^{\infty}p^n H_n(u)\right]\right]\right), \tag{45}$$

where $H_n(u)$ are He's polynomial [21, 46] that represents the nonlinear terms.

In a similar manner as above the solution $u(x,t)$ is given by

$$u(x,t) = x^3 t^3. \tag{46}$$

4. Conclusions

In this paper, the homotopy perturbation transform method (HPTM) has been successfully applied to find the solution of the linear and nonlinear Klein-Gordon equations with initial conditions. The method is reliable and easy to use. The results show that the homotopy perturbation transform method is powerful and efficient technique in finding exact and approximate solutions for nonlinear differential equations. It is worth mentioning that the method is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result; the size reduction amounts to an improvement of the performance of the approach. The fact that the HPTM solves nonlinear problems without using Adomian's polynomials is a clear advantage of this technique over the decomposition method. In conclusion, the HPTM may be considered as a nice refinement in existing numerical techniques and might find the wide applications.

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