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Some Comparative Growth Rate of Composite Entire and Meromorphic Functions

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Abstract. In this paper we discuss some growth rates of composite entire and meromorphic functions improving some earlier results.

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1. Introduction, Definitions and Notations.

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a meromophic function and g be an entire function defined on $\mathbb{C}$. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [6] and [3].

In the sequel we use the following notation:

$$log^{[k]}x = log(log^{[k-1]}x) \text{ for } k = 1,2,3, \ldots \text{ and}$$
$$log^{[0]}x = x\,;$$

and

$$exp^{[k]}x = exp(exp^{[k-1]}x) \text{ for } k = 1,2,3, \ldots \text{ and}$$
$$exp^{[0]}x = x\,.$$

The following definitions are well known.

Definition 1 The order ρ_f and lower order λ_f of an entire function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \qquad and \qquad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

If f is meromorphic then

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \qquad and \qquad \lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

Juneja, Kapoor and Bajpai [4] defined the (p,q) th order and (p,q) th lower order of an entire function f respectively as follows:

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} r} \qquad and \qquad \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} r},$$

where p, q are positive integers and $p > q$.

When f is meromorphic, one can easily verify that

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$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r} \quad and \quad \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r},$$

where p, q are positive integers and $p > q$.

If $p = 2$ and $q = 1$ then we write $\rho_f(1,2) = \rho_f$ and $\lambda_f(1,2) = \lambda_f$.

The following definitions are also well known.

Definition 2 A meromorphic function $a \equiv a(z)$ is called small with respect to f if $T(r,a) = S(r,f)$.

Definition 3 Let $a_1, a_2, \ldots. a_k$ be linearly independent meromorphic functions and small with respect to f. We denote by $L(f) = W(a_1, a_2, \ldots. a_k, f)$ the Wronskian determinant of $a_1, a_2, \ldots., a_k, f$ i.e,

$$L(f) = \begin{vmatrix} a_1 & a_2 & \cdots & a_k & f \\ a_1' & a_2' & \cdots & a_k' & f' \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_1^k & a_2^k & \cdots & a_k^k & f^k \end{vmatrix}.$$

Definition 4 If $a \in \mathbb{C} \cup \{\infty\}$, the quantity

$$\delta(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} = \liminf_{r\to\infty} \frac{m(r,a;f)}{T(r,f)}$$

is called the Nevanlinma's deficiency of the value 'a'.

From the second fundamental theorem it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a;f) > 0$ is countable and $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) \le 2$ {cf. [3], p.43}. If in particular, $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$, we say that f has the maximum deficiency sum.

In the paper we establish some newly developed results based on the comparative growth properties of composite entire or meromorphic functions and wronskians generated by one of the factors on the basis of $(p;\ q)$ th order and $(p;\ q)$ th lower order where p, q are positive integers with $p > q$. We do not explain the standard notations and definitions in the theory of entire and meromorphic functions because those are available in [6] and [3].

2. Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 [1] Let f be meromorphic and g be entire then for all sufficiently large values of r,

$$T(r, f\circ g) \le \{1 + o(1)\} \frac{T(r,g)}{\log M(r,g)} T(M(r,g), f).$$

Lemma 2 [2] Let f be meromorphic and g be entire and suppose that $0 < \mu < \rho_g \le \infty$. Then for a sequence of values of r tending to infinity,

$$T(r, f\circ g) \ge T(\exp(r^{\mu}), f).$$

Lemma 3 [5] Let f be a transcendental meromorphic function having the maximum deficiency sum. Then

$$\lim_{r\to\infty} \frac{T(r, L(f))}{T(r,f)} = 1 + k - k\delta(\infty; f).$$

Lemma 4 If f be a transcendental meromorphic function with the maximum deficiency sum, then the (p,q) th order and (p,q) th lower order of $L(f)$ are same as those of f.

Proof. By Lemma 3,

$$\lim_{r\to\infty} \frac{\log^{[p-1]} T(r, L(f))}{\log^{[p-1]} T(r,f)}$$

where p is any positive integer > 1 exists and is equal to 1. So

$$\rho_{L(f)}(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, L(f))}{\log^{[q]} r}$$

$$= \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r} \cdot \lim_{r\to\infty} \frac{\log^{[p-1]} T(r,L(f))}{\log^{[p-1]} T(r,f)}$$
$$= \rho_f(p,q).1$$
$$= \rho_f(p,q).$$

Similarly it can be proved that

$$\lambda_{L(f)}(p,q) = \lambda_f(p,q).$$

This proves the lemma.

2. Theorems.

In this section we present the main results of the paper.

Theorem 1 Let f be a transcendental meromorphic function with the maximum deficiency sum and g be an entire function such that $\rho_g(m,n) < \lambda_f(p,q) \le \rho_f(p,q) < \infty$ where p, q, m, n are positive integers with $p > q, m > n$. Then

$$\text{(i)} \quad \lim_{r\to\infty} \frac{\log^{[p-1]} T\left(\exp^{[n-1]} r, f \circ g\right)}{\log^{[p-2]} T(\exp^{[q-1]} r, L(f))} = 0 \ if \ q \ge m$$

and

$$\text{(ii)} \quad \lim_{r\to\infty} \frac{\log^{[p+m-q-2]} T\left(\exp^{[n-1]} r, f \circ g\right)}{\log^{[p-2]} T(\exp^{[q-1]} r, L(f))} = 0 \ if \ q < m.$$

Proof. Since $\rho_g(m,n) < \lambda_f(p,q)$ we can choose $\varepsilon(> 0)$ in such a way that

$$\rho_g(m,n) + \varepsilon < \lambda_f(p,q) - \varepsilon \tag{1}$$

As $T(r,g) \le log^+ M(r,g)$, we have from Lemma 1 for all sufficiently large values of r,

$$\log^{[p-1]} T\left(\exp^{[n-1]} r, f \circ g\right) \le \log^{[p-1]} T\left(M\left(\exp^{[n-1]} r, g\right), f\right) + O(1)$$
$$\text{i.e.,} \quad \log^{[p-1]} T\left(\exp^{[n-1]} r, f \circ g\right) \le (\rho_f(p,q) + \varepsilon) \log^{[q]} M\left(\exp^{[n-1]} r, g\right) + O(1). \tag{2}$$

Now the following two cases may arise.

Case I. Let $q \ge m$.

Then we have from (2) for all sufficiently large values of r,

$$\log^{[p-1]} T\left(\exp^{[n-1]} r, f \circ g\right) \le (\rho_f(p,q) + \varepsilon) \log^{[m-1]} M\left(\exp^{[n-1]} r, g\right) + O(1). \tag{3}$$

Again for all sufficiently large values of r,

$$\log^{[m]} M\left(\exp^{[n-1]} r, g\right) \le (\rho_g(m,n) + \varepsilon) \log^{[n]} \exp^{[n-1]} r$$
$$\text{i.e.,} \quad \log^{[m]} M\left(\exp^{[n-1]} r, g\right) \le \left(\rho_g(m,n) + \varepsilon\right) \log r$$
$$\text{i.e.,} \quad \log^{[m]} M\left(\exp^{[n-1]} r, g\right) \le \log r^{(\rho_g(m,n)+\varepsilon)}$$
$$\text{i.e.,} \quad \log^{[m-1]} M\left(\exp^{[n-1]} r, g\right) \le r^{(\rho_g(m,n)+\varepsilon)} \tag{4}$$

Now from (3) and (4) we have for all sufficiently large values of r,

$$\log^{[p-1]} T\left(\exp^{[n-1]} r, f \circ g\right) \le (\rho_f(p,q) + \varepsilon) r^{(\rho_g(m,n)+\varepsilon)} + O(1). \tag{5}$$

Case II. Let $q < m$.

Then for all sufficiently large values of r we get from (2) that

$$\log^{[p-1]} T\left(\exp^{[n-1]} r, f \circ g\right)$$
$$\le (\rho_f(p,q) + \varepsilon) exp^{[m-q]} \log^{[m]} M\left(\exp^{[n-1]} r, g\right) + O(1). \tag{6}$$

Again for all sufficiently large values of r,

$$\log^{[m]} M\left(\exp^{[n-1]} r, g\right) \le (\rho_g(m,n) + \varepsilon) \log^{[n]} \exp^{[n-1]} r$$
$$\text{i.e.,} \quad \log^{[m]} M\left(\exp^{[n-1]} r, g\right) \le \left(\rho_g(m,n) + \varepsilon\right) \log r$$
$$\text{i.e.,} \quad \log^{[m]} M\left(\exp^{[n-1]} r, g\right) \le \log r^{(\rho_g(m,n)+\varepsilon)}$$

$$\text{i. e.,}\quad \exp^{[m-q]}\log^{[m]} M\left(\exp^{[n-1]}r, g\right) \leq \exp^{[m-q]}\log r^{(\rho_g(m,n)+\varepsilon)}$$

$$\text{i. e.,}\quad \exp^{[m-q]}\log^{[m]} M\left(\exp^{[n-1]}r, g\right) \leq \exp^{[m-q-1]} r^{(\rho_g(m,n)+\varepsilon)}. \tag{7}$$

Now from (6) and (7) we have for all sufficiently large values of r,

$$\log^{[p-1]} T\left(\exp^{[n-1]}r, f\circ g\right) \leq (\rho_f(p,q)+\varepsilon)\exp^{[m-q-1]} r^{(\rho_g(m,n)+\varepsilon)} + O(1)$$

$$\text{i. e.,}\quad \log^{[p]} T\left(\exp^{[n-1]}r, f\circ g\right) \leq \exp^{[m-q-2]} r^{(\rho_g(m,n)+\varepsilon)} + O(1)$$

$$\text{i. e.,}\quad \log^{[p+m-q-2]} T\left(\exp^{[n-1]}r, f\circ g\right) \leq \log^{[m-q-2]}\exp^{[m-q-2]} r^{(\rho_g(m,n)+\varepsilon)} + O(1)$$

$$\text{i. e.,}\quad \log^{[p+m-q-2]} T\left(\exp^{[n-1]}r, f\circ g\right) \leq r^{(\rho_g(m,n)+\varepsilon)} + O(1). \tag{8}$$

Again for all sufficiently large values of r, we obtain in view of Lemma 4 that

$$\log^{[p-1]} T\left(\exp^{[q-1]}r, L(f)\right) \geq (\lambda_{L(f)}(p,q)-\varepsilon)\log^{[q]}\exp^{[q-1]}r$$

$$\text{i. e.,}\quad \log^{[p-1]} T\left(\exp^{[q-1]}r, L(f)\right) \geq (\lambda_f(p,q)-\varepsilon)\log r$$

$$\text{i. e.,}\quad \log^{[p-1]} T\left(\exp^{[q-1]}r, L(f)\right) \geq \log r^{(\lambda_f(p,q)-\varepsilon)}$$

$$\text{i. e.,}\quad \log^{[p-2]} T\left(\exp^{[q-1]}r, L(f)\right) \geq r^{(\lambda_f(p,q)-\varepsilon)}. \tag{9}$$

Now combining (5) of Case I and (9) we get for all sufficiently large values of r that

$$\frac{\log^{[p-1]} T(\exp^{[n-1]}r, f\circ g)}{\log^{[p-2]} T(\exp^{[q-1]}r, L(f))} \leq \frac{(\rho_f(p,q)+\varepsilon) r^{(\rho_g(m,n)+\varepsilon)} + O(1)}{r^{(\lambda_f(p,q)-\varepsilon)}}. \tag{10}$$

Now in view of (1) it follows from (10) that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} T(\exp^{[n-1]}r, f\circ g)}{\log^{[p-2]} T(\exp^{[q-1]}r, L(f))} = 0$$

$$\text{i. e.,}\quad \lim_{r\to\infty} \frac{\log^{[p-1]} T(\exp^{[n-1]}r, f\circ g)}{\log^{[p-2]} T(\exp^{[q-1]}r, L(f))} = 0.$$

This proves the first part of the theorem.

Again combining (8) of Case II and (9) we obtain for all sufficiently large values of r that

$$\frac{\log^{[p+m-q-2]} T(\exp^{[n-1]}r, f\circ g)}{\log^{[p-2]} T(\exp^{[q-1]}r, L(f))} \leq \frac{r^{(\rho_g(m,n)+\varepsilon)} + O(1)}{r^{(\lambda_f(p,q)-\varepsilon)}}. \tag{11}$$

Now in view of (1) it follows from (11) that

$$\limsup_{r\to\infty} \frac{\log^{[p+m-q-2]} T(\exp^{[n-1]}r, f\circ g)}{\log^{[p-2]} T(\exp^{[q-1]}r, L(f))} = 0$$

$$\text{i. e.,}\quad \lim_{r\to\infty} \frac{\log^{[p+m-q-2]} T(\exp^{[n-1]}r, f\circ g)}{\log^{[p-2]} T(\exp^{[q-1]}r, L(f))} = 0.$$

This establishes the second part of the theorem.

Remark 1 The condition $\rho_g(m,n) < \lambda_f(p,q)$ in Theorem 1 is essential as we see in the following example.

Example 1 Let $f = g = \exp z$ and $p = m = 2,\ q = n = 1$.

Then $\rho_g(m,n) = \lambda_f(p,q) = \rho_f(p,q) = 1$ and $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$.

Taking $a_1 = 1,\ a_2 = \ldots = a_k = 0$ and $k = 1$ in Definition 3 we get that

$$L(f) = \begin{vmatrix} 1 & \exp z \\ 0 & \exp z \end{vmatrix} = \exp z.$$

Now

$$T(r, f\circ g) = T\left(r, \exp^{[2]} z\right) \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}\ (r\to\infty)$$

and

$$T(r, L(f)) = T(r, \exp z) = \frac{r}{\pi}.$$

Then

$$\lim_{r\to\infty}\frac{\log^{[p+m-q-2]}T\left(\exp^{[n-1]}r,f\circ g\right)}{\log^{[p-2]}T\left(\exp^{[q-1]}r,L(f)\right)}$$

$$=\lim_{r\to\infty}\frac{\log T(r,f\circ g)}{T\left(r,L(f)\right)}$$

$$=\lim_{r\to\infty}\frac{r-\frac{1}{2}\log r+O(1)}{\frac{r}{\pi}}$$

$$=\pi\neq 0,\quad \text{which is contrary to Theorem 1.}$$

Theorem 2 Let f be a transcendental meromorphic function with the maximum deficiency sum and g be an entire function such that $\lambda_g(m,n)<\lambda_f(p,q)\leq\rho_f(p,q)<\infty$ where p,q,m,n are positive integers with $p>q, m>n$. Then

$$\text{(i)}\ \liminf_{r\to\infty}\frac{\log^{[p-1]}T\left(\exp^{[n-1]}r,f\circ g\right)}{\log^{[p-2]}T(\exp^{[q-1]}r,L(f))}=0\ if\ q\geq m$$

and

$$\text{(ii)}\ \liminf_{r\to\infty}\frac{\log^{[p+m-q-2]}T\left(\exp^{[n-1]}r,f\circ g\right)}{\log^{[p-2]}T(\exp^{[q-1]}r,L(f))}=0\ if\ q<m.$$

Proof. For a sequence of values of r tending to infinity that

$$\log^{[m]}M\left(\exp^{[n-1]}r,g\right)\leq(\lambda_g(m,n)+\varepsilon)\log^{[n]}\exp^{[n-1]}r$$
$$\text{i.e.,}\ \log^{[m]}M\left(\exp^{[n-1]}r,g\right)\leq\left(\lambda_g(m,n)+\varepsilon\right)\log r$$
$$\text{i.e.,}\ \log^{[m]}M\left(\exp^{[n-1]}r,g\right)\leq\log r^{(\lambda_g(m,n)+\varepsilon)}$$
$$\text{i.e.,}\ \log^{[m-1]}M\left(\exp^{[n-1]}r,g\right)\leq r^{(\lambda_g(m,n)+\varepsilon)} \tag{12}$$

Now from (3) and (12) we have for a sequence of values of r tending to infinity that

$$\log^{[p-1]}T\left(\exp^{[n-1]}r,f\circ g\right)\leq(\rho_f(p,q)+\varepsilon)r^{(\lambda_g(m,n)+\varepsilon)}+O(1). \tag{13}$$

Combining (9) and (13) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[p-1]}T\left(\exp^{[n-1]}r,f\circ g\right)}{\log^{[p-2]}T(\exp^{[q-1]}r,L(f))}\leq\frac{(\rho_f(p,q)+\varepsilon)r^{(\lambda_g(m,n)+\varepsilon)}+O(1)}{r^{(\lambda_f(p,q)-\varepsilon)}}. \tag{14}$$

Now in view of (1) it follows from (14) that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]}T\left(\exp^{[n-1]}r,f\circ g\right)}{\log^{[p-2]}T(\exp^{[q-1]}r,L(f))}=0.$$

This proves the first part of the theorem.

Again for a sequence of values of r tending to infinity that

$$\log^{[m]}M\left(\exp^{[n-1]}r,g\right)\leq(\lambda_g(m,n)+\varepsilon)\log^{[n]}\exp^{[n-1]}r$$
$$\text{i.e.,}\ \log^{[m]}M\left(\exp^{[n-1]}r,g\right)\leq\left(\lambda_g(m,n)+\varepsilon\right)\log r$$
$$\text{i.e.,}\ \log^{[m]}M\left(\exp^{[n-1]}r,g\right)\leq\log r^{(\lambda_g(m,n)+\varepsilon)}$$
$$\text{i.e.,}\ \exp^{[m-q]}\log^{[m]}M\left(\exp^{[n-1]}r,g\right)\leq\exp^{[m-q]}\log r^{(\lambda_g(m,n)+\varepsilon)}$$
$$\text{i.e.,}\ \exp^{[m-q]}\log^{[m]}M\left(\exp^{[n-1]}r,g\right)\leq\exp^{[m-q-1]}r^{(\lambda_g(m,n)+\varepsilon)}. \tag{15}$$

Now from (6) and (15) we have for a sequence of values of r tending to infinity that

$$\log^{[p-1]}T\left(\exp^{[n-1]}r,f\circ g\right)\leq(\rho_f(p,q)+\varepsilon)\exp^{[m-q-1]}r^{(\lambda_g(m,n)+\varepsilon)}+O(1)$$
$$\text{i.e.,}\ \log^{[p]}T\left(\exp^{[n-1]}r,f\circ g\right)\leq\exp^{[m-q-2]}r^{(\lambda_g(m,n)+\varepsilon)}+O(1)$$
$$\text{i.e.,}\ \log^{[p+m-q-2]}T\left(\exp^{[n-1]}r,f\circ g\right)\leq\ \log^{[m-q-2]}\exp^{[m-q-2]}r^{(\lambda_g(m,n)+\varepsilon)}+O(1)$$
$$\text{i.e.,}\ \log^{[p+m-q-2]}T\left(\exp^{[n-1]}r,f\circ g\right)\leq r^{(\lambda_g(m,n)+\varepsilon)}+O(1). \tag{16}$$

Combining (9) and (16) we obtain for a sequence of values of r tending to infinity that

$$\frac{\log^{[p+m-q-2]} T(\exp^{[n-1]} r, f\circ g)}{\log^{[p-2]} T(\exp^{[q-1]} r, L(f))} \le \frac{r^{(\lambda_g(m,n)+\varepsilon)} + O(1)}{r^{(\lambda_f(p,q)-\varepsilon)}}. \qquad (17)$$

Now in view of (1) it follows from (17) that

$$\liminf_{r\to\infty} \frac{\log^{[p+m-q-2]} T(\exp^{[n-1]} r, f\circ g)}{\log^{[p-2]} T(\exp^{[q-1]} r, L(f))} = 0.$$

This establishes the second part of the theorem.

Remark 2 The condition $\lambda_g(m,n) < \lambda_f(p,q)$ in Theorem 2 is necessary which is evident from the following example.

Example 2 Let $f = g = \exp z$ and $p = m = 2,\ q = n = 1.$

Then $\lambda_g(m,n) = \lambda_f(p,q) = \rho_f(p,q) = 1$ and $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2.$

Taking $a_1 = 1, a_2 = \ldots = a_k = 0$ and $k = 1$ in Definition 3 we obtain that

$$L(f) = \begin{vmatrix} 1 & expz \\ 0 & expz \end{vmatrix} = expz.$$

Now

$$T(r, f\circ g) = T(r, exp^{[2]} z) \sim \frac{expr}{(2\pi^3 r)^{\frac{1}{2}}}\ (r\to\infty)$$

and

$$T(r, L(f)) = T(r, expz) = \frac{r}{\pi}.$$

Therefore

$$\lim_{r\to\infty} \frac{\log^{[p+m-q-2]} T(\exp^{[n-1]} r, f\circ g)}{\log^{[p-2]} T\left(\exp^{[q-1]} r, L(f)\right)}$$

$$= \lim_{r\to\infty} \frac{\log T(r, f\circ g)}{T(r, L(f))}$$

$$= \lim_{r\to\infty} \frac{r - \frac{1}{2}\log r + O(1)}{\frac{r}{\pi}}$$

$$= \pi \neq 0, \quad \text{which is contrary to Theorem 2.}$$

Theorem 3 Let f be a transcendental meromorphic function such that $\rho_f(p,q) < \infty$ and $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$ where p, q are positive integers with $p > q > 1$. Also let g be entire. If $\lambda_{f\circ g}(p,q) = \infty$ then for every positive number α,

$$\lim_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[p-1]} T(r^\alpha, L(f))} = \infty.$$

Proof. Let us suppose that the conclusion of the theorem does not hold. Then we can find a constant $\beta > 0$ such that for a sequence of values of r tending to infinity

$$\log^{[p-1]} T(r, f\circ g) \le \beta \log^{[p-1]} T(r^\alpha, L(f)). \qquad (18)$$

Again for $q > 1$ from the definition of $\rho_{L(f)}(p,q)$ it follows that for all sufficiently large values of r and in view of Lemma 4

$$\log^{[p-1]} T(r^\alpha, L(f)) \le (\rho_{L(f)}(p,q) + \varepsilon)\log^{[q]}(r^\alpha)$$

$$\text{i.e., } \log^{[p-1]} T(r^\alpha, L(f)) \le (\rho_f(p,q) + \varepsilon)\log^{[q]} r + O(1) \qquad (19)$$

Thus from (18) and (19) we have for a sequence of values of r tending to infinity that

$$\log^{[p-1]} T(r, f\circ g) \le \beta(\rho_f(p,q) + \varepsilon)\log^{[q]} r + O(1)$$

$$\text{i. e.,}\quad \frac{\log^{[p-1]}T(r,f\circ g)}{\log^{[q]}r} \le \frac{\beta(\rho_f(p,q)+\varepsilon)\log^{[q]}r+O(1)}{\log^{[q]}r}$$

$$\text{i. e.,}\quad \liminf_{r\to\infty}\frac{\log^{[p-1]}T(r,f\circ g)}{\log^{[q]}r} = \lambda_{f\circ g}(p,q) < \infty.$$

This is a contradiction.

This proves the theorem.

Remark 3 Theorem 3 is also valid with "limit superior" instead of "limit" if $\lambda_{f\circ g}(p,q)=\infty$ is replaced by $\rho_{f\circ g}(p,q)=\infty$ and the other conditions remaining the same.

Corollary 1 Under the assumptions of Remark 3,

$$\limsup_{r\to\infty}\frac{\log^{[p-2]}T(r,f\circ g)}{\log^{[p-2]}T(r^{\alpha},L(f))} = \infty .$$

Proof. From Remark 3 we obtain for all sufficiently large values of r and for $K > 1$,

$$\log^{[p-1]}T(r,f\circ g) > K\log^{[p-1]}T(r^{\alpha},L(f))$$

$$\text{i. e.,}\quad \log^{[p-2]}T(r,f\circ g) > \{\log^{[p-2]}T(r^{\alpha},L(f))\}^{K},$$

from which the corollary follows.

Corollary 2 Under the same conditions of Theorem 3 if $q = 1$

$$\lim_{r\to\infty}\frac{\log^{[p-1]}T(r,f\circ g)}{\log^{[p-1]}T(r^{\alpha},L(f))} = \infty .$$

Corollary 3 Under the same conditions of Remark 3 if $q = 1$

$$\lim_{r\to\infty} sup\frac{\log^{[p-1]}T(r,f\circ g)}{\log^{[p-1]}T(r^{\alpha},L(f))} = \infty .$$

Remark 4 The condition $\lambda_{f\circ g}(p,1)=\infty$ in Corollary 2 is necessary as we see in the following example.

Example 3 Let $f = exp\, z,\ g = z$ and $p = 2,\ q = 1,\ \alpha = 1.$

Then $\rho_f(p,1) = \lambda_{f\circ g}(p,1) = 1.$

Also $\sum_{a\neq\infty}\delta(a;f) + \delta(\infty;f) = 2.$

Taking $a_1 = 1, a_2 = \ldots = a_k = 0$ and $k = 1$ in Definition 3 we obtain that

$$L(f) = \begin{vmatrix} 1 & expz \\ 0 & expz \end{vmatrix} = expz.$$

Now

$$T(r,f\circ g) = T(r,expz) = \frac{r}{\pi}.$$

Then

$$\lim_{r\to\infty}\frac{\log^{[p-1]}T(r,f\circ g)}{\log^{[p-1]}T(r^{\alpha},L(f))}$$

$$= \lim_{r\to\infty}\frac{\log T(r,f\circ g)}{\log T(r,f)}$$

$$= \lim_{r\to\infty}\frac{\log\frac{r}{\pi}}{\log\frac{r}{\pi}}$$

$$= 1 \neq \infty, \qquad \text{which is contrary to Corollary 2.}$$

Remark 5 Considering $f = exp\, z,\ g = z$ and $p = 2, q = 1,\ \alpha = 1$ one can easily verify that the condition $\rho_{f\circ g}(p,1)=\infty$ in Corollary 3 is essential.

Theorem 4 Let f be a transcendental meromorphic function with the maximum deficiency sum such that $0 < \lambda_f(p,q) \le \rho_f(p,q) < \infty$ .Also let g be an entire function . Then

$$\text{(i)} \ \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[p-1]} T(\exp(r^{\mu}), L(f))} = \infty \quad if\ q = 1$$

$$\text{(ii)} \ \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[p-1]} T(\exp(r^{\mu}), L(f))} \geq \frac{\mu' \lambda_f(p,q)}{\mu \rho_f(p,q)} \ if\ q = 2$$

and

$$\text{(iii)} \ \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[p-1]} T(\exp(r^{\mu}), L(f))} = \frac{\lambda_f(p,q)}{\rho_f(p,q)} \quad if\ q > 1$$

where $0 < \mu < \mu' < \rho_g$ and p, q are positive integers with $p > q$.

Proof. Since $0 < \mu < \mu' < \rho_g$ then from Lemma 2 we get for a sequence of values of r tending to infinity that

$$\log^{[p-1]} T(r, f\circ g) \geq \log^{[p-1]} T(exp(r^{\mu'}), f)$$
$$\text{i. e.,} \ \log^{[p-1]} T(r, f\circ g) \geq (\lambda_f(p,q) - \varepsilon)\log^{[q]} exp(r^{\mu'})$$
$$\text{i. e.,} \ \log^{[p-1]} T(r, f\circ g) \geq (\lambda_f(p,q) - \varepsilon)\log^{[q-1]}(r^{\mu'}). \quad (20)$$

Again from the definition of $\rho_{L(f)}(p,q)$ and in view of Lemma 4 it follows for all sufficiently large values of r that

$$\log^{[p-1]} T(\exp(r^{\mu}), L(f)) \leq (\rho_{L(f)}(p,q) + \varepsilon)\log^{[q]} exp(r^{\mu})$$
$$\text{i.e.,} \ \log^{[p-1]} T(\exp(r^{\mu}), L(f)) \leq (\rho_f(p,q) + \varepsilon)\log^{[q-1]} r^{\mu}. \quad (21)$$

Thus from (20) and (21) we have for a sequence of values of r tending to infinity that

$$\frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[p-1]} T(\exp(r^{\mu}), L(f))} \geq \frac{(\lambda_f(p,q) - \varepsilon)\log^{[q-1]}(r^{\mu'})}{(\rho_f(p,q) + \varepsilon)\log^{[q-1]} r^{\mu}}. \quad (22)$$

Since $\mu < \mu'$, the theorem follows from (22).

Remark 6 The condition $\mu < \rho_g$ in Theorem 4 is essential as we see in the following example.

Example 4 Let $f = g = exp\ z$ and $p = m = 2,\ q = n = 1.$ Also let $\mu = 1$.

Then $\lambda_f(p,q) = \rho_f(p,q) = 1$ and $\rho_g = 1$. Also $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$.

Taking $a_1 = 1, a_2 = \ldots = a_k = 0$ and $k = 1$ in Definition 3 we obtain that

$$L(f) = \begin{vmatrix} 1 & expz \\ 0 & expz \end{vmatrix} = expz.$$

Now

$$T(r, f\circ g) = T(r, exp^{[2]} z) \sim \frac{expr}{(2\pi^3 r)^{\frac{1}{2}}} \ (r\to\infty)$$

and

$$T(\exp(r^{\mu}), L(f)) = T(expr, expz) = \frac{expr}{\pi}.$$

Therefore

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[p-1]} T(\exp(r^{\mu}), L(f))}$$
$$= \limsup_{r\to\infty} \frac{\log T(r, exp^{[2]} z)}{T(expr, expz)}$$
$$= \limsup_{r\to\infty} \frac{r - \frac{1}{2}\log r + O(1)}{r + O(1)}$$
$$= 1 \neq 0, \quad \text{which is contrary to Theorem 4.}$$

Theorem 5 Let f be a transcendental meromorphic function with the maximum deficiency sum. Also let g be entire such that $0 < \lambda_f(p,q) \leq \rho_f(p,q) < \infty$ and $\rho_g(m,n) < \infty$ where p, q, m, n are positive integers with $p > q, m > n$. Then

$$\text{(i)} \ \limsup_{r\to\infty} \frac{\log^{[p]} T\left(\exp^{[n-1]} r, f\circ g\right)}{\log^{[p-1]} T(\exp^{[q-1]} r, L(f))} \le \frac{\rho_g(m,n)}{\lambda_f(p,q)} \quad if\ q \ge m$$

and

$$\text{(ii)} \ \limsup_{r\to\infty} \frac{\log^{[p+m-q-1]} T\left(\exp^{[n-1]} r, f\circ g\right)}{\log^{[p-1]} T(\exp^{[q-1]} r, L(f))} \le \frac{\rho_g(m,n)}{\lambda_f(p,q)} \quad if\ q < m.$$

Proof. In view of Lemma 4 we have for all sufficiently large values of r

$$\log^{[p-1]} T\left(\exp^{[q-1]} r, L(f)\right) \ge \left(\lambda_{L(f)}(p,q) - \varepsilon\right)\log^{[q]}\exp^{[q-1]} r$$

$$\text{i. e.,} \ \log^{[p-1]} T\left(\exp^{[q-1]} r, L(f)\right) \ge \left(\lambda_f(p,q) - \varepsilon\right)\log r. \tag{23}$$

Case I. If $q \ge m$ then from (5) and (23) we get for all sufficiently large values of r that

$$\frac{\log^{[p]} T\left(\exp^{[n-1]} r, f\circ g\right)}{\log^{[p-1]} T(\exp^{[q-1]} r, L(f))} \le \frac{\left(\rho_g(m,n)+\varepsilon\right)\log r + O(1)}{\left(\lambda_f(p,q) - \varepsilon\right)\log r}$$

$$\text{i. e.,} \ \limsup_{r\to\infty} \frac{\log^{[p]} T\left(\exp^{[n-1]} r, f\circ g\right)}{\log^{[p-1]} T(\exp^{[q-1]} r, L(f))} \le \frac{\rho_g(m,n)+\varepsilon}{\lambda_f(p,q) - \varepsilon}.$$

Since $\varepsilon\ (> 0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[p]} T\left(\exp^{[n-1]} r, f\circ g\right)}{\log^{[p-1]} T(\exp^{[q-1]} r, L(f))} \le \frac{\rho_g(m,n)}{\lambda_f(p,q)}.$$

This proves the first part of the theorem.

Case II. If $q < m$ then from (8) and (23) we obtain for all sufficiently large values of r that

$$\frac{\log^{[p+m-q-1]} T\left(\exp^{[n-1]} r, f\circ g\right)}{\log^{[p-1]} T(\exp^{[q-1]} r, L(f))} \le \frac{\left(\rho_g(m,n)+\varepsilon\right)\log r + O(1)}{\left(\lambda_f(p,q) - \varepsilon\right)\log r}$$

$$\text{i. e.,} \ \limsup_{r\to\infty} \frac{\log^{[p+m-q-1]} T\left(\exp^{[n-1]} r, f\circ g\right)}{\log^{[p-1]} T(\exp^{[q-1]} r, L(f))} \le \frac{\rho_g(m,n)+\varepsilon}{\lambda_f(p,q) - \varepsilon}.$$

As $\varepsilon\ (> 0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[p+m-q-1]} T\left(\exp^{[n-1]} r, f\circ g\right)}{\log^{[p-1]} T(\exp^{[q-1]} r, L(f))} \le \frac{\rho_g(m,n)}{\lambda_f(p,q)}.$$

Remark 7 The condition $\rho_g(m,n) < \infty$ in Theorem 5 is necessary which is evident from the following example.

Example 5 Let $f = exp\, z,\ g = exp^{[2]} z$ and $p = m = 2,\ q = n = 1$.

Then $\lambda_f(p,q) = \rho_f(p,q) = 1$ and $\rho_g(m,n) = \infty$. Also $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$.

Taking $a_1 = 1,\ a_2 = \ldots = a_k = 0$ and $k = 1$ in Definition 3 we obtain that

$$L(f) = \begin{vmatrix} 1 & expz \\ 0 & expz \end{vmatrix} = expz.$$

Now

$$T(r, f\circ g) \ge \frac{1}{3}\log M\left(\frac{r}{2}, f\circ g\right)$$

$$\text{i. e.,} \ \log^{[2]} T(r, f\circ g) \ge \log^{[3]} M\left(\frac{r}{2}, f\circ g\right) + O(1)$$

$$\text{i. e.,} \ \log^{[2]} T(r, f\circ g) \ge \log^{[3]}\exp^{[3]}\left(\frac{r}{2}\right) + O(1)$$

$$\text{i. e.,} \ \log^{[2]} T(r, f\circ g) \ge \left(\frac{r}{2}\right) + O(1)$$

and

$$\log T(\mathrm{r}, L(f)) = \log T(r, expz) = \log\left(\frac{r}{\pi}\right) = \log\mathrm{r} + \mathrm{O}(1).$$

Therefore

$$\limsup_{r\to\infty}\frac{\log^{[p+m-q-1]}T\left(\exp^{[n-1]}r, f\circ g\right)}{\log^{[p-1]}T(\exp^{[q-1]}r, L(f))} \geq \limsup_{r\to\infty}\frac{\log^{[2]}T(r, f\circ g)}{\log T(\mathrm{r}, L(f))}$$

$$\text{i.e.,}\ \limsup_{r\to\infty}\frac{\log^{[p+m-q-1]}T\left(\exp^{[n-1]}r, f\circ g\right)}{\log^{[p-1]}T(\exp^{[q-1]}r, L(f))} \geq \limsup_{r\to\infty}\frac{\frac{r}{2} + O(1)}{\log\mathrm{r} + \mathrm{O}(1)}$$

$$\text{i.e.,}\ \limsup_{r\to\infty}\frac{\log^{[p+m-q-1]}T\left(\exp^{[n-1]}r, f\circ g\right)}{\log^{[p-1]}T(\exp^{[q-1]}r, L(f))} = \infty,$$

which is contrary to Theorem 5.

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