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Exact Solutions of the ZK(M, N, K) Equation with Generalized Evolution and Time-Dependent Coefficients

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Abstract. In this paper, a ZK(m, n, k) equation with generalized evolution and time-dependent coefficients is investigated. Exp-function method combined with F-expansion method are used to determine eight families of exact solutions of exp-function type for this equation. When the parameters are taken as special values, every family of solution can be reduced to some solitary wave solutions and periodic wave solutions. The results presented in this paper improve the previous results.

Keywords: Exp-function method; F-expansion method; Variable-coefficient ZK(m, n, k) equation; Riccati equation; Exact solutions of exp-function type

MR(2000) Subject Classification: 35Q51; 35Q58; 36K65; 34A20.

1. Introduction

Nonlinear partial differential equations (NPDEs) are widely used to describe complex phenomena in various fields of science, especially in physics. Searching for exact soliton solutions of NPDEs plays an important and significant role in the study on the dynamics of those phenomena. Up to now, many effective ansatz methods have been presented, such as the tanh method [1], Jacobi elliptic function method [2], F-expansion method [3], the Exp-function method [4-7], auxiliary equation method [8,9], and so on. Here, it is worth to mention that the two methods, the Exp-function method and F-expansion method can be combined to form one method [10-13].

In this paper, by using Exp-function method combined with F-expansion method, we will study the ZK(m, n, k) equation with generalized evolution and time-dependent coefficients [14]

$$(u^l)_t + a(t)(u^m)_x + b(t)(u^n)_{xxx} + c(t)(u^k)_{yyx} = \alpha(t)u^l, \tag{1}$$

where $a(t)$, $b(t)$, $c(t)$ and $\alpha(t)$ are all time-dependent coefficients, while l, m, n and k are integers. Generally, Eq. (1) is not integrable. In [14], using a solitary wave ansatz in the form of $\sec h^p$ functions, Triki and Wazwaz obtained an exact one-soliton solution for Eq. (1).

In this work, we will explore more types of exact solutions for Eq. (1).

2. Description of the method

In this section, we review the combining the Exp-function method with F-expansion method [12,13] at first.

Given a nonlinear partial differential equation, for instance, in two variables, as follows:

$$p(u, u_x, u_t, u_{xx}, u_{xt}, \ldots) = 0, \tag{2}$$

where P is in general a nonlinear function of its variables.

We firstly use the Exp-function method to obtain new exact solutions of the following Riccati equation

$$\phi'(\xi) = \frac{d}{d\xi}\phi(\xi) = A + \gamma\phi^2(\xi), \tag{3}$$

where A and γ are arbitrary constants, then using the Riccati equation (3) as auxiliary equation and its exact solutions, we obtain exact solutions of the nonlinear partial differential equation(2).

Seeking for the exact solutions of Eq. (3), we introducing a complex variable η, defined by

$$\eta = \rho\xi + \xi_0, \tag{4}$$

where ρ is a constant to be determined later, ξ_0 is an arbitrary constant, Riccati equation (3) converts to

$$\rho\phi' - A - \gamma\phi^2 = 0, \tag{5}$$

where prime denotes the derivative with respect to η.

According to the Exp-function method, we assume that the solution of Eq. (5) can be expressed in the following form

$$\phi(\eta) = \frac{a_e \exp(e\eta) + \cdots + a_{-d}\exp(-d\eta)}{b_g \exp(g\eta) + \cdots + b_{-f}\exp(-f\eta)}, \tag{6}$$

where e, d, g and f are positive integers which are given by the homogeneous balance principle, a_e, $\cdots$, a_{-d}, $b_g, \cdots$, b_{-f} are unknown constants to be determined. To determine the values of e and g, we usually balance the linear term of the highest order in Eq. (5) with the highest order nonlinear term. Similarly, we can determine d and f by balancing the linear term of the lowest order in Eq. (5) with the lowest order nonlinear term, we obtain $e = g$, $d = f$. For simplicity, we set $e = g = 1$ and $d = f = 1$, then Eq. (6) becomes

$$\phi(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{b_1 \exp(\eta) + b_0 + b_{-1}\exp(-\eta)}, \tag{7}$$

Substituting Eq. (7) into Eq. (5), equating to zero the coefficients of all powers of $\exp(n\eta)$ $(n = -2, -1, 0, 1, 2)$ yields a set of algebraic equations for a_1, a_0, a_{-1}, b_1, b_0, b_{-1} and μ. Solving the system of algebraic equations by using Maple, we obtain the new exact solution of Eq. (3), which read

$$\phi_1 = \frac{-\sqrt{-\frac{A}{\gamma}}b_1 \exp(\gamma\sqrt{-\frac{A}{\gamma}}\xi + \xi_0) + a_{-1}\exp(-\gamma\sqrt{-\frac{A}{\gamma}}\xi - \xi_0)}{b_1 \exp(\gamma\sqrt{-\frac{A}{\gamma}}\xi + \xi_0) + \frac{a_{-1}}{\sqrt{-\frac{A}{\gamma}}}\exp(-\gamma\sqrt{-\frac{A}{\gamma}}\xi - \xi_0)}, \tag{8}$$

where a_{-1} and b_1 are free parameters ;

$$\phi_2 = \frac{\frac{(\gamma a_0^2 + Ab_0^2)}{4\gamma\sqrt{-\frac{A}{\gamma}}b_{-1}}\exp(2\gamma\sqrt{-\frac{A}{\gamma}}\xi + \xi_0) + a_0 + \sqrt{-\frac{A}{\gamma}}b_{-1}\exp(-2\gamma\sqrt{-\frac{A}{\gamma}}\xi - \xi_0)}{\frac{(\gamma a_0^2 + Ab_0^2)}{4Ab_{-1}}\exp(2\gamma\sqrt{-\frac{A}{\gamma}}\xi + \xi_0) + b_0 + b_{-1}\exp(-2\gamma\sqrt{-\frac{A}{\gamma}}\xi - \xi_0)}, \tag{9}$$

where a_0, b_0 and b_{-1} are free parameters.

By choosing properly values of a_0, a_{-1}, b_0, b_{-1}, we find many kinds of hyperbolic function solutions and triangular periodic solutions of Eq. (3), which are listed as follows:

(ⅰ) When $\xi_0 = 0, b_1 = 1, a_{-1} = \pm\sqrt{-\frac{A}{\gamma}}, \frac{A}{\gamma} < 0$, the solution (8) becomes

$$\phi = -\sqrt{-\frac{A}{\gamma}}\tanh(\gamma\sqrt{-\frac{A}{\gamma}}\xi), \tag{10}$$

and

$$\phi=-\sqrt{-\frac{A}{\gamma}}\coth(\gamma\sqrt{-\frac{A}{\gamma}}\xi)\,. \tag{11}$$

(ii) When $\xi_0=0\,,b_1=i\,,a_{-1}=\mp\sqrt{\frac{A}{\gamma}}\,,\frac{A}{\gamma}>0$, the solution (8) becomes

$$\phi=\sqrt{\frac{A}{\gamma}}\tan(\gamma\sqrt{\frac{A}{\gamma}}\xi)\,, \tag{12}$$

and

$$\phi=-\sqrt{\frac{A}{\gamma}}\cot(\gamma\sqrt{\frac{A}{\gamma}}\xi)\,. \tag{13}$$

(iii) When $\xi_0=0\,,b_0=0\,,b_{-1}=1\,,\ a_0=\pm2\sqrt{-\frac{A}{\gamma}}\,,\frac{A}{\gamma}<0$, the solution (9) becomes

$$\phi=-\sqrt{-\frac{A}{\gamma}}[\coth(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)\pm\operatorname{csch}(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)]\,. \tag{14}$$

(iv) When $\xi_0=0\,,b_0=0\,,b_{-1}=i\,,\ a_0=\pm2\sqrt{-\frac{A}{\gamma}}\,,\frac{A}{\gamma}<0$, the solution (9) becomes

$$\phi=-\sqrt{-\frac{A}{\gamma}}[\tanh(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)\pm i\operatorname{sech}(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)]\,. \tag{15}$$

(v) When $\xi_0=0\,,b_0=0\,,b_{-1}=1\,,\ a_0=\pm2\sqrt{\frac{A}{\gamma}}\,,\frac{A}{\gamma}>0$, the solution (9) becomes

$$\phi=\sqrt{\frac{A}{\gamma}}[\tan(2\gamma\sqrt{\frac{A}{\gamma}}\xi)\pm\sec(2\gamma\sqrt{\frac{A}{\gamma}}\xi)]\,. \tag{16}$$

(vi) When $\xi_0=0\,,b_0=0\,,b_{-1}=i\,,\ a_0=\pm2\sqrt{\frac{A}{\gamma}}\,,\frac{A}{\gamma}>0$, the solution (9) becomes

$$\phi=-\sqrt{\frac{A}{\gamma}}[\cot(2\gamma\sqrt{\frac{A}{\gamma}}\xi)\mp\csc(2\gamma\sqrt{\frac{A}{\gamma}}\xi)]\,. \tag{17}$$

For simplicity, in the rest of the paper, we consider $\xi_0=0$.

3. Application to the variable-coefficient ZK(m, n, k) equation

3.1. Case I: $l=n=k$, $m\neq n$

Balancing the order of the nonlinear term $(u^m)_x$ with the term $(u^n)_{xxx}$ in (1), we obtain

$$mP+1=nP+3, \tag{18}$$

so that

$$P=\frac{2}{m-n}\,. \tag{19}$$

To get a closed form solution, it is natural to use the transformation

$$u=v^{\frac{1}{m-n}}, \tag{20}$$

and when $l=n=k$, Eq. (1) becomes

$$n(m-n)^2v^2v_t+a(t)m(m-n)^2v^3v_x+b(t)[n(2n-m)(3n-2m)(v_x)^3+$$

$$3n(2n-m)(m-n)vv_xv_{xx}+n(m-n)^2v^2v_{xxx}]+c(t)[n(2n-m)(3n-2m)v_x(v_y)^2+$$
$$2n(2n-m)(m-n)vv_yv_{yx}+n(2n-m)(m-n)vv_xv_{yy}+n(m-n)^2v^2v_{yyx}]-\alpha(t)(m-n)^3v^3=0 \tag{21}$$

This means that all the evolution terms that satisfy the condition $l=n=k$ contribute to the soliton formation.

In order to obtain new exact travelling wave solutions for Eq. (21), we use

$$v(x,y,t)=v(\xi)\,,\xi=B_1(t)x+B_2(t)y-\omega(t)t\,, \tag{22}$$

where $B_1(t)$, $B_2(t)$ and $\omega(t)$ are functions in t to be determined later, and substituting the (22) into Eq. (21), we obtain

$$n(m-n)^2v^2v_t+\alpha(t)m(m-n)^2B_1(t)v^3v'+n(2n-m)(3n-2m)(v')^3[b(t)B_1^{\ 3}(t)$$
$$+c(t)B_1(t)B_2^{\ 2}(t)]+3n(2n-m)(m-n)vv'v''[b(t)B_1^{\ 3}(t)+c(t)B_1(t)B_2^{\ 2}(t)]$$
$$+n(m-n)^2v^2v'''[b(t)B^3{}_1(t)+c(t)B_1(t)B_2^{\ 2}(t)]-\alpha(t)(m-n)^3v^3=0\,. \tag{23}$$

Now, we assume that the solution of Eq. (23) can be expressed in the following form

$$v=v(\xi)=\sum_{j=0}^{N}\alpha_j(t)\phi^j(\xi)+\sum_{j=0}^{N}\beta_j(t)\phi^{-j}(\xi)\,, \tag{24}$$

where N is positive integers which are given by the homogeneous balance principle, $\phi(\xi)$ is a solution of Eq. (3). Balancing v^2v''' term with v^3v' term in (23) gives $N=2$. Therefore, we obtain

$$v=\alpha_0(t)+\alpha_1(t)\phi(\xi)+\alpha_2(t)\phi^2(\xi)+\frac{\beta_1(t)}{\phi(\xi)}+\frac{\beta_2(t)}{\phi^2(\xi)}\,. \tag{25}$$

Substituting Eq. (25) into (23) and using the Riccati equation (3), collecting the coefficients of $\phi(\xi)$, we have

$$\frac{1}{D}[C_0(t)+C_1(t)\phi(\xi)+C_2(t)\phi^2(\xi)+\cdots+C_{17}(t)\phi^{17}(\xi)+C_{18}(t)\phi^{18}(\xi)]=0\,. \tag{26}$$

Because the expresses to these coefficients D, $C_0(t)=0$, $C_1(t)=0$, $C_2(t)=0$, $C_3(t)=0$, $\cdots$, $C_{17}(t)=0$, $C_{18}(t)=0$ of $\phi(\xi)$ in Eq. (26) are too lengthiness, so we omit them. But we can directly use the command "solve" in mathematical software Maple to solve the following set of algebraic equations

$$C_0(t)=0\,,\ C_1(t)=0\,,\ C_2(t)=0\,,\ C_3(t)=0\,,\ \cdots,\ C_{17}(t)=0\,,\ C_{18}(t)=0\,. \tag{27}$$

Solved the above algebraic equations, we obtain the following two sets of solutions

Case 1

$$B_1(t)=B_1,\ \ B_2(t)=B_2\,,$$
$$\alpha_0(t)=\lambda_0e^{\frac{m-n}{n}\int\alpha(t)dt},\ \alpha_1(t)=0\,,\ \alpha_2(t)=\lambda_2e^{\frac{m-n}{n}\int\alpha(t)dt},\ \beta_1(t)=0\,,\ \beta_2(t)=0\,,$$
$$\omega(t)=\frac{1}{t}\left\{-\frac{4n^2\gamma^2\lambda_0B_1}{\lambda_2(m-n)^2}\int\left[B_1^{\ 2}b(t)+B_2^{\ 2}c(t)\right]dt+C\right\},$$
$$\alpha(t)=-\frac{2n(m+n)\gamma^2[B_1^{\ 2}b(t)+B_2^{\ 2}c(t)]}{\lambda_2(m-n)^2e^{\frac{m-n}{n}\int\alpha(t)dt}},\ \ A=\frac{\lambda_0}{\lambda_2}\gamma\,, \tag{28}$$

where B_1, B_2, λ_0, λ_2 and C are arbitrary constants.

Case 2

$$B_1(t)=B_1,\ \ B_2(t)=B_2\,,$$
$$\alpha_0(t)=\mu_0e^{\frac{m-n}{n}\int\alpha(t)dt},\ \alpha_1(t)=0\,,\ \alpha_2(t)=0\,,\ \beta_1(t)=0\,,\ \beta_2(t)=\mu_2e^{\frac{m-n}{n}\int\alpha(t)dt},$$

$$\omega(t)=\frac{1}{t}\left\{-\frac{4n^2\gamma^2\mu_2 B_1}{\mu_0(m-n)^2}\int\left[B_1^{\,2}b(t)+B_2^{\,2}c(t)\right]dt+C\right\},$$

$$\alpha(t)=-\frac{2n(m+n)\gamma^2\mu_2[B_1^{\,2}b(t)+B_2^{\,2}c(t)]}{\mu_0^{\,2}(m-n)^2 e^{\frac{m-n}{n}\int\alpha(t)dt}},\quad A=\frac{\mu_2}{\mu_0}\gamma, \tag{29}$$

where B_1, B_2, μ_0, μ_2 and C are arbitrary constants.

Thus from Eqs. (25) and (28), (29) we obtain families of exact solutions to Eq. (23) as follows.

$$v=\lambda_0 e^{\frac{m-n}{n}\int\alpha(t)dt}+\lambda_2 e^{\frac{m-n}{n}\int\alpha(t)dt}\phi^2(\xi), \tag{30}$$

$$v=\mu_0 e^{\frac{m-n}{n}\int\alpha(t)dt}+\mu_2 e^{\frac{m-n}{n}\int\alpha(t)dt}\frac{1}{\phi^2(\xi)}, \tag{31}$$

where $\phi(\xi)$ is a solution of Eq. (3).

Substituting new solutions (8) and (9) of Riccati equation into solutions (30) and (31), using the transformation (20), we have the following several families of solutions to Eq. (1).

Family 1

$$u_1(x,y,t)=\left\{\lambda_0 e^{\frac{m-n}{n}\int\alpha(t)dt}+\lambda_2 e^{\frac{m-n}{n}\int\alpha(t)dt}\times\left[\frac{-\sqrt{-\frac{\lambda_0}{\lambda_2}}b_1\exp(\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)+a_{-1}\exp(-\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)}{b_1\exp(\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)+\frac{a_{-1}}{\sqrt{-\frac{\lambda_0}{\lambda_2}}}\exp(-\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)}\right]^2\right\}^{\frac{1}{m-n}}, \tag{32}$$

where $\xi=B_1x+B_2y+\dfrac{4n^2\gamma^2\lambda_0 B_1}{\lambda_2(m-n)^2}\int\left[B_1^{\,2}b(t)+B_2^{\,2}c(t)\right]dt-C$,

$$\alpha(t)=-\frac{2n(m+n)\gamma^2[B_1^{\,2}b(t)+B_2^{\,2}c(t)]}{\lambda_2(m-n)^2 e^{\frac{m-n}{n}\int\alpha(t)dt}},\quad l=n=k.$$

If we set $b_1=1$, $a_{-1}=\pm\sqrt{-\dfrac{A}{\gamma}}=\pm\sqrt{-\dfrac{\lambda_0}{\lambda_2}}$, and $\dfrac{A}{\gamma}=\dfrac{\lambda_0}{\lambda_2}<0$ in Eq. (32), we obtain

$$u_{1(1)}(x,y,t)=\left(\lambda_0 e^{\frac{m-n}{n}\int\alpha(t)dt}-\lambda_0 e^{\frac{m-n}{n}\int\alpha(t)dt}\tanh^2(\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)\right)^{\frac{1}{m-n}} \tag{33}$$

$$=\left(\lambda_0 e^{\frac{m-n}{n}\int\alpha(t)dt}\sec h^2(\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)\right)^{\frac{1}{m-n}}, \tag{34}$$

and

$$u_{1(2)}(x,y,t)=\left(\lambda_0 e^{\frac{m-n}{n}\int\alpha(t)dt}-\lambda_0 e^{\frac{m-n}{n}\int\alpha(t)dt}\coth^2(\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)\right)^{\frac{1}{m-n}} \tag{35}$$

$$=\left(-\lambda_0 e^{\frac{m-n}{n}\int\alpha(t)dt}\operatorname{csc}h^2(\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)\right)^{\frac{1}{m-n}}. \tag{36}$$

Setting $b_1 = i$, $a_{-1} = \mp\sqrt{\frac{A}{\gamma}} = \mp\sqrt{\frac{\lambda_0}{\lambda_2}}$, $\frac{A}{\gamma} = \frac{\lambda_0}{\lambda_2} > 0$ in Eq. (32), we get

$$u_{1(3)}(x,y,t) = \left(\lambda_0 e^{\frac{m-n}{n}\int\alpha(t)dt} + \lambda_0 e^{\frac{m-n}{n}\int\alpha(t)dt}\tan^2(\gamma\sqrt{\frac{\lambda_0}{\lambda_2}}\xi)\right)^{\frac{1}{m-n}} \tag{37}$$

$$=\left(\lambda_0 e^{\frac{m-n}{n}\int\alpha(t)dt}\sec^2(\gamma\sqrt{\frac{\lambda_0}{\lambda_2}}\xi)\right)^{\frac{1}{m-n}}, \tag{38}$$

and

$$u_{1(4)}(x,y,t) = \left(\lambda_0 e^{\frac{m-n}{n}\int\alpha(t)dt} + \lambda_0 e^{\frac{m-n}{n}\int\alpha(t)dt}\cot^2(\gamma\sqrt{\frac{\lambda_0}{\lambda_2}}\xi)\right)^{\frac{1}{m-n}} \tag{39}$$

$$=\left(\lambda_0 e^{\frac{m-n}{n}\int\alpha(t)dt}\csc^2(\gamma\sqrt{\frac{\lambda_0}{\lambda_2}}\xi)\right)^{\frac{1}{m-n}}. \tag{40}$$

Family 2

$$u_2(x,y,t) = \left\{\mu_0 e^{\frac{m-n}{n}\int\alpha(t)dt} + \mu_2 e^{\frac{m-n}{n}\int\alpha(t)dt}\times\right.$$

$$\left.\left[\frac{b_1\exp(\gamma\sqrt{-\frac{\mu_2}{\mu_0}}\xi) + \frac{a_{-1}}{\sqrt{-\frac{\mu_2}{\mu_0}}}\exp(-\gamma\sqrt{-\frac{\mu_2}{\mu_0}}\xi)}{-\sqrt{-\frac{\mu_2}{\mu_0}}b_1\exp(\gamma\sqrt{-\frac{\mu_2}{\mu_0}}\xi) + a_{-1}\exp(-\gamma\sqrt{-\frac{\mu_2}{\mu_0}}\xi)}\right]^2\right\}^{\frac{1}{m-n}}, \tag{41}$$

where $\xi = B_1 x + B_2 y + \left\{\frac{4n^2\gamma^2\mu_2 B_1}{\mu_0(m-n)^2}\int\left[B_1{}^2 b(t) + B_2{}^2 c(t)\right]dt - C\right\}$,

$$\alpha(t) = -\frac{2n(m+n)\gamma^2\mu_2[B_1{}^2 b(t) + B_2{}^2 c(t)]}{\mu_0{}^2(m-n)^2 e^{\frac{m-n}{n}\int\alpha(t)dt}}, \quad l = n = k.$$

Family 3

$$u_3(x,y,t) = \left\{\lambda_0 e^{\frac{m-n}{n}\int\alpha(t)dt} + \lambda_2 e^{\frac{m-n}{n}\int\alpha(t)dt}\times\right.$$

$$\left\{\left[\frac{\frac{(\gamma a_0{}^2+Ab_0{}^2)}{4\gamma\sqrt{-\frac{A}{\gamma}}b_{-1}}\exp(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)+a_0+\sqrt{-\frac{A}{\gamma}}b_{-1}\exp(-2\gamma\sqrt{-\frac{A}{\gamma}}\xi)}{\frac{(\gamma a_0{}^2+Ab_0{}^2)}{4Ab_{-1}}\exp(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)+b_0+b_{-1}\exp(-2\gamma\sqrt{-\frac{A}{\gamma}}\xi)}\right]^2\right\}^{\frac{1}{m-n}}, \tag{42}$$

where $\xi = B_1x+B_2y+\frac{4n^2\gamma^2\lambda_0 B_1}{\lambda_2(m-n)^2}\int\left[B_1{}^2b(t)+B_2{}^2c(t)\right]dt-C$,

$$\alpha(t)=-\frac{2n(m+n)\gamma^2[B_1{}^2b(t)+B_2{}^2c(t)]}{\lambda_2(m-n)^2e^{\frac{m-n}{n}\int\alpha(t)dt}},\quad A=\frac{\lambda_0}{\lambda_2}\gamma,\quad l=n=k.$$

If we set $b_0=0, b_{-1}=1$, $a_0=\pm2\sqrt{-\frac{A}{\gamma}}=\pm2\sqrt{-\frac{\lambda_0}{\lambda_2}}$, and $\frac{A}{\gamma}=\frac{\lambda_0}{\lambda_2}<0$ in Eq. (42), we obtain

$$u_{3(1)}(x,y,t)=\left\{\lambda_0e^{\frac{m-n}{n}\int\alpha(t)dt}-\lambda_0e^{\frac{m-n}{n}\int\alpha(t)dt}\left[\coth(2\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)\pm\operatorname{csch}(2\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)\right]^2\right\}^{\frac{1}{m-n}}$$

$$=\left(\frac{2\lambda_0e^{\frac{m-n}{n}\int\alpha(t)dt}}{1\mp\cosh(2\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)}\right)^{\frac{1}{m-n}}, \tag{43}$$

Setting $b_0=0, b_{-1}=i$, $a_0=\pm2\sqrt{-\frac{A}{\gamma}}=\pm2\sqrt{-\frac{\lambda_0}{\lambda_2}}$, $\frac{A}{\gamma}=\frac{\lambda_0}{\lambda_2}<0$ in Eq. (42), we get

$$u_{3(2)}(x,y,t)=\left\{\lambda_0e^{\frac{m-n}{n}\int\alpha(t)dt}-\lambda_0e^{\frac{m-n}{n}\int\alpha(t)dt}\left[\tanh(2\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)\pm i\operatorname{sech}(2\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)\right]^2\right\}^{\frac{1}{m-n}}. \tag{44}$$

Setting $b_0=0, b_{-1}=1$, $a_0=\pm2\sqrt{\frac{A}{\gamma}}=\pm2\sqrt{\frac{\lambda_0}{\lambda_2}}$, and $\frac{A}{\gamma}=\frac{\lambda_0}{\lambda_2}>0$ in Eq. (42), we have

$$u_{3(3)}(x,y,t)=\left\{\lambda_0e^{\frac{m-n}{n}\int\alpha(t)dt}+\lambda_0e^{\frac{m-n}{n}\int\alpha(t)dt}\left[\tan(2\gamma\sqrt{\frac{\lambda_0}{\lambda_2}}\xi)\pm\sec(2\gamma\sqrt{\frac{\lambda_0}{\lambda_2}}\xi)\right]^2\right\}^{\frac{1}{m-n}}. \tag{45}$$

Setting $b_0=0, b_{-1}=i$, $a_0=\pm2\sqrt{\frac{A}{\gamma}}=\pm2\sqrt{\frac{\lambda_0}{\lambda_2}}$, and $\frac{A}{\gamma}=\frac{\lambda_0}{\lambda_2}>0$ in Eq. (42), we have

$$u_{3(4)}(x,y,t)=\left\{\lambda_0e^{\frac{m-n}{n}\int\alpha(t)dt}+\lambda_0e^{\frac{m-n}{n}\int\alpha(t)dt}\left[\cot(2\gamma\sqrt{\frac{\lambda_0}{\lambda_2}}\xi)\mp\sec(2\gamma\sqrt{\frac{\lambda_0}{\lambda_2}}\xi)\right]^2\right\}^{\frac{1}{m-n}}$$

$$=\left(\frac{2\lambda_0 e^{\frac{m-n}{n}\int\alpha(t)dt}}{1\pm\cos(2\gamma\sqrt{\frac{\lambda_0}{\lambda_2}}\xi)}\right)^{\frac{1}{m-n}}. \tag{46}$$

Family 4

$$u_4(x,y,t)=\left\{\mu_0 e^{\frac{m-n}{n}\int\alpha(t)dt}+\mu_2 e^{\frac{m-n}{n}\int\alpha(t)dt}\times\right.$$

$$\left.\left[\frac{\frac{(\gamma a_0{}^2+Ab_0{}^2)}{4Ab_{-1}}\exp(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)+b_0+b_{-1}\exp(-2\gamma\sqrt{-\frac{A}{\gamma}}\xi)}{\frac{(\gamma a_0{}^2+Ab_0{}^2)}{4\gamma\sqrt{-\frac{A}{\gamma}}b_{-1}}\exp(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)+a_0+\sqrt{-\frac{A}{\gamma}}b_{-1}\exp(-2\gamma\sqrt{-\frac{A}{\gamma}}\xi)}\right]^2\right\}^{\frac{1}{m-n}}, \tag{47}$$

where $\xi=B_1x+B_2y+\left\{\frac{4n^2\gamma^2\mu_2B_1}{\mu_0(m-n)^2}\int\left[B_1{}^2b(t)+B_2{}^2c(t)\right]dt-C\right\}$,

$$\alpha(t)=-\frac{2n(m+n)\gamma^2\mu_2[B_1{}^2b(t)+B_2{}^2c(t)]}{\mu_0{}^2(m-n)^2e^{\frac{m-n}{n}\int\alpha(t)dt}},\quad A=\frac{\mu_2}{\mu_0}\gamma,\quad l=n=k.$$

If we set $b_0=0, b_{-1}=i$, $a_0=\pm2\sqrt{-\frac{A}{\gamma}}=\pm2\sqrt{-\frac{\mu_2}{\mu_0}}$, $\frac{A}{\gamma}=\frac{\mu_2}{\mu_0}<0$ in Eq. (47), we obtain

$$u_{4(1)}(x,y,t)=\left\{\mu_0 e^{\frac{m-n}{n}\int\alpha(t)dt}-\mu_0 e^{\frac{m-n}{n}\int\alpha(t)dt}\frac{1}{\left[\tanh(2\gamma\sqrt{-\frac{\mu_2}{\mu_0}}\xi)\pm i\,\mathrm{sech}(2\gamma\sqrt{-\frac{\mu_2}{\mu_0}}\xi)\right]^2}\right\}^{\frac{1}{m-n}}. \tag{48}$$

Setting $b_0=0, b_{-1}=1$, $a_0=\pm2\sqrt{\frac{A}{\gamma}}=\pm2\sqrt{\frac{\mu_2}{\mu_0}}$, and $\frac{A}{\gamma}=\frac{\mu_2}{\mu_0}>0$ in Eq. (47), we get

$$u_{4(2)}(x,y,t)=\left\{\mu_0 e^{\frac{m-n}{n}\int\alpha(t)dt}+\mu_2 e^{\frac{m-n}{n}\int\alpha(t)dt}\frac{1}{\left[\tan(2\gamma\sqrt{\frac{\mu_2}{\mu_0}}\xi)\pm\sec(2\gamma\sqrt{\frac{\mu_2}{\mu_0}}\xi)\right]^2}\right\}^{\frac{1}{m-n}}. \tag{49}$$

3.2. Case II: $m=n=k$, $l\neq n$

Balancing the order of the nonlinear term $(u^l)_t$ with the term $(u^n)_{xxx}$ in (1), we obtain

$$lP+1=nP+3, \tag{50}$$

so that

$$P=\frac{2}{l-n}. \tag{51}$$

To get a closed form solution, it is natural to use the transformation

$$u = v^{\frac{1}{l-n}}, \tag{52}$$

and when $m = n = k$, Eq. (1) becomes

$$l(l-n)^2 v^3 v_t + a(t)n(l-n)^2 v^2 v_x + b(t)[n(2n-l)(3n-2l)(v_x)^3 +$$
$$3n(2n-l)(l-n)vv_x v_{xx} + n(l-n)^2 v^2 v_{xxx}] + c(t)[n(2n-l)(3n-2l)v_x(v_y)^2 +$$
$$2n(2n-l)(l-n)vv_y v_{yx} + n(2n-l)(l-n)vv_x v_{yy} + n(l-n)^2 v^2 v_{yyx}] - \alpha(t)(l-n)^3 v^4 = 0. \tag{53}$$

This means that all the evolution terms that satisfy the condition $m = n = k$ contribute to the soliton formation.

In order to obtain new exact travelling wave solutions for Eq. (53), we use

$$v(x, y, t) = v(\xi), \quad \xi = B_1(t)x + B_2(t)y - \omega(t)t. \tag{54}$$

where $B_1(t)$, $B_2(t)$ and $\omega(t)$ are functions in t to be determined later, and substituting the (54) into Eq. (53), we obtain

$$l(l-n)^2 v^3 v_t + \alpha(t)n(l-n)^2 B_1(t)v^2 v' + n(2n-l)(3n-2l)(v')^3[b(t)B_1^3(t)$$
$$+c(t)B_1(t)B_2^2(t)] + 3n(2n-l)(l-n)vv'v''[b(t)B_1^3(t) + c(t)B_1(t)B_2^2(t)]$$
$$+n(l-n)^2 v^2 v'''[b(t)B^3{}_1(t) + c(t)B_1(t)B_2^2(t)] - \alpha(t)(l-n)^3 v^4 = 0. \tag{55}$$

Now, we assume that the solution of Eq. (55) can be expressed in the following form

$$v = v(\xi) = \sum_{j=0}^{N} \alpha_j(t)\phi^j(\xi) + \sum_{j=0}^{N} \beta_j(t)\phi^{-j}(\xi), \tag{56}$$

where N is positive integers which are given by the homogeneous balance principle, $\phi(\xi)$ is a solution of Eq. (3). Balancing $v^2 v'''$ term with $v^3 v'$ term in (55) gives $N = 2$. Therefore, we obtain

$$v = \alpha_0(t) + \alpha_1(t)\phi(\xi) + \alpha_2(t)\phi^2(\xi) + \frac{\beta_1(t)}{\phi(\xi)} + \frac{\beta_2(t)}{\phi^2(\xi)}. \tag{57}$$

Substituting Eq. (57) into (55) and using the Riccati equation (3), collecting the coefficients of $\phi(\xi)$, we have

$$\frac{1}{D}[C_0(t) + C_1(t)\phi(\xi) + C_2(t)\phi^2(\xi) + \cdots + C_{17}(t)\phi^{17}(\xi) + C_{18}(t)\phi^{18}(\xi)] = 0. \tag{58}$$

Because the expresses to these coefficients D, $C_0(t) = 0$, $C_1(t) = 0$, $C_2(t) = 0$, $C_3(t) = 0$, $\cdots$, $C_{17}(t) = 0$, $C_{18}(t) = 0$ of $\phi(\xi)$ in Eq. (58) are too lengthiness, so we omit them. But we can directly use the command "solve" in mathematical software Maple to solve the following set of algebraic equations

$$C_0(t) = 0,\ C_1(t) = 0,\ C_2(t) = 0,\ C_3(t) = 0,\ \cdots,\ C_{17}(t) = 0,\ C_{18}(t) = 0. \tag{59}$$

Solved the above algebraic equations, we obtain the following two sets of solutions

Case 1

$$B_1(t) = B_1, \quad B_2(t) = B_2,$$
$$\alpha_0(t) = \lambda_0 e^{\frac{l-n}{l}\int \alpha(t)dt},\ \alpha_1(t) = 0,\ \alpha_2(t) = \lambda_2 e^{\frac{l-n}{l}\int \alpha(t)dt},\ \beta_1(t) = 0,\ \beta_2(t) = 0,$$
$$\omega(t) = \frac{1}{t}\left\{ \frac{2n(l+n)\gamma^2 B_1}{\lambda_2(l-n)^2} \int \left[B_1^2 b(t) + B_2^2 c(t) \right] e^{\frac{n-l}{l}\int \alpha(t)dt} dt + C \right\},$$
$$\alpha(t) = \frac{4\lambda_0 n^2 \gamma^2 [B_1^2 b(t) + B_2^2 c(t)]}{\lambda_2 (l-n)^2}, \quad A = \frac{\lambda_0}{\lambda_2}\gamma, \tag{60}$$

where B_1, B_2, λ_0, λ_2 and C are arbitrary constants.

Case 2

$B_1(t)=B_1,\ \ B_2(t)=B_2,$

$\alpha_0(t)=\mu_0 e^{\frac{l-n}{l}\int\alpha(t)dt},\ \alpha_1(t)=0,\ \alpha_2(t)=0,\ \beta_1(t)=0,\ \beta_2(t)=\mu_2 e^{\frac{l-n}{l}\int\alpha(t)dt},$

$\omega(t)=\frac{1}{t}\left\{\frac{2n(l+n)\mu_2\gamma^2 B_1}{\mu_0{}^2(l-n)^2}\int\left[B_1{}^2 b(t)+B_2{}^2 c(t)\right]e^{\frac{n-l}{l}\int\alpha(t)dt}dt+C\right\},$

$$\alpha(t)=\frac{4\mu_2 n^2\gamma^2[B_1{}^2 b(t)+B_2{}^2 c(t)]}{\mu_0(l-n)^2},\ \ A=\frac{\mu_2}{\mu_0}\gamma, \tag{61}$$

where B_1, B_2, μ_0, μ_2 and C are arbitrary constants.

Thus from Eqs. (57) and (60), (61) we obtain families of exact solutions to Eq. (55) as follows.

$$v=\lambda_0 e^{\frac{l-n}{l}\int\alpha(t)dt}+\lambda_2 e^{\frac{l-n}{l}\int\alpha(t)dt}\phi^2(\xi), \tag{62}$$

$$v=\mu_0 e^{\frac{l-n}{l}\int\alpha(t)dt}+\mu_2 e^{\frac{l-n}{l}\int\alpha(t)dt}\frac{1}{\phi^2(\xi)}, \tag{63}$$

where $\phi(\xi)$ is a solution of Eq. (3).

Substituting new solutions (8) and (9) of Riccati equation into solutions (62) and (63), using the transformation (52), we have the following several families of solutions to Eq. (1).

Family 5

$$u_5(x,y,t)=\left\{\lambda_0 e^{\frac{l-n}{l}\int\alpha(t)dt}+\lambda_2 e^{\frac{l-n}{l}\int\alpha(t)dt}\times\left[\frac{-\sqrt{-\frac{\lambda_0}{\lambda_2}}b_1\exp(\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)+a_{-1}\exp(-\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)}{b_1\exp(\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)+\frac{a_{-1}}{\sqrt{-\frac{\lambda_0}{\lambda_2}}}\exp(-\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)}\right]^2\right\}^{\frac{1}{l-n}}, \tag{64}$$

where $\xi=B_1x+B_2y-\frac{2n(l+n)\gamma^2 B_1}{\lambda_2(l-n)^2}\int\left[B_1{}^2 b(t)+B_2{}^2 c(t)\right]e^{\frac{n-l}{l}\int\alpha(t)dt}dt-C,$

$$\alpha(t)=\frac{4\lambda_0 n^2\gamma^2[B_1{}^2 b(t)+B_2{}^2 c(t)]}{\lambda_2(l-n)^2},\quad m=n=k.$$

If we set $b_1=1$, $a_{-1}=\pm\sqrt{-\frac{A}{\gamma}}=\pm\sqrt{-\frac{\lambda_0}{\lambda_2}}$, and $\frac{A}{\gamma}=\frac{\lambda_0}{\lambda_2}<0$ in Eq. (64), we obtain

$$u_{5(1)}(x,y,t)=\left(\lambda_0 e^{\frac{l-n}{l}\int\alpha(t)dt}-\lambda_0 e^{\frac{l-n}{l}\int\alpha(t)dt}\tanh^2(\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)\right)^{\frac{1}{l-n}} \tag{65}$$

$$=\left(\lambda_0 e^{\frac{l-n}{l}\int\alpha(t)dt}\sec h^2(\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)\right)^{\frac{1}{l-n}}, \tag{66}$$

and

$$u_{5(2)}(x,y,t)=\left(\lambda_0 e^{\frac{l-n}{l}\int\alpha(t)dt}-\lambda_0 e^{\frac{l-n}{l}\int\alpha(t)dt}\coth^2(\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)\right)^{\frac{1}{l-n}} \tag{67}$$

$$=\left(-\lambda_0 e^{\frac{l-n}{l}\int\alpha(t)dt}\operatorname{csc}h^2(\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)\right)^{\frac{1}{l-n}}. \tag{68}$$

Setting $b_1=i$, $a_{-1}=\mp\sqrt{\frac{A}{\gamma}}=\mp\sqrt{\frac{\lambda_0}{\lambda_2}}$, and $\frac{A}{\gamma}=\frac{\lambda_0}{\lambda_2}>0$ in Eq. (64), we get

$$u_{5(3)}(x,y,t)=\left(\lambda_0 e^{\frac{l-n}{l}\int\alpha(t)dt}+\lambda_0 e^{\frac{l-n}{l}\int\alpha(t)dt}\tan^2(\gamma\sqrt{\frac{\lambda_0}{\lambda_2}}\xi)\right)^{\frac{1}{l-n}} \tag{69}$$

$$=\left(\lambda_0 e^{\frac{l-n}{l}\int\alpha(t)dt}\sec^2(\gamma\sqrt{\frac{\lambda_0}{\lambda_2}}\xi)\right)^{\frac{1}{l-n}}, \tag{70}$$

and

$$u_{5(4)}(x,y,t)=\left(\lambda_0 e^{\frac{l-n}{l}\int\alpha(t)dt}+\lambda_0 e^{\frac{l-n}{l}\int\alpha(t)dt}\cot^2(\gamma\sqrt{\frac{\lambda_0}{\lambda_2}}\xi)\right)^{\frac{1}{l-n}} \tag{71}$$

$$=\left(\lambda_0 e^{\frac{l-n}{l}\int\alpha(t)dt}\csc^2(\gamma\sqrt{\frac{\lambda_0}{\lambda_2}}\xi)\right)^{\frac{1}{l-n}}. \tag{72}$$

Family 6

$$u_6(x,y,t)=\left\{\mu_0 e^{\frac{l-n}{l}\int\alpha(t)dt}+\mu_2 e^{\frac{l-n}{l}\int\alpha(t)dt}\times\right.$$

$$\left.\left[\frac{b_1\exp(\gamma\sqrt{-\frac{\mu_2}{\mu_0}}\xi)+\frac{a_{-1}}{\sqrt{-\frac{\mu_2}{\mu_0}}}\exp(-\gamma\sqrt{-\frac{\mu_2}{\mu_0}}\xi)}{-\sqrt{-\frac{\mu_2}{\mu_0}}b_1\exp(\gamma\sqrt{-\frac{\mu_2}{\mu_0}}\xi)+a_{-1}\exp(-\gamma\sqrt{-\frac{\mu_2}{\mu_0}}\xi)}\right]^2\right\}^{\frac{1}{l-n}}, \tag{73}$$

where $\xi=B_1x+B_2y-\frac{2n(l+n)\mu_2\gamma^2B_1}{\mu_0{}^2(l-n)^2}\int\left[B_1{}^2b(t)+B_2{}^2c(t)\right]e^{\frac{n-l}{l}\int\alpha(t)dt}dt-C$,

$$\alpha(t)=\frac{4\mu_2n^2\gamma^2[B_1{}^2b(t)+B_2{}^2c(t)]}{\mu_0(l-n)^2},\quad m=n=k.$$

Family 7

$$u_7(x,y,t)=\left\{\lambda_0 e^{\frac{l-n}{l}\int\alpha(t)dt}+\lambda_2 e^{\frac{l-n}{l}\int\alpha(t)dt}\times\right.$$

$$\left.\left[\frac{\dfrac{(\gamma a_0{}^2+Ab_0{}^2)}{4\gamma\sqrt{-\frac{A}{\gamma}}b_{-1}}\exp(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)+a_0+\sqrt{-\frac{A}{\gamma}}b_{-1}\exp(-2\gamma\sqrt{-\frac{A}{\gamma}}\xi)}{\dfrac{(\gamma a_0{}^2+Ab_0{}^2)}{4Ab_{-1}}\exp(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)+b_0+b_{-1}\exp(-2\gamma\sqrt{-\frac{A}{\gamma}}\xi)}\right]^2\right\}^{\frac{1}{m-n}}, \tag{74}$$

where $\xi=B_1x+B_2y-\dfrac{2n(l+n)\gamma^2B_1}{\lambda_2(l-n)^2}\int\left[B_1{}^2b(t)+B_2{}^2c(t)\right]e^{\frac{n-l}{l}\int\alpha(t)dt}dt-C$,

$$\alpha(t)=\frac{4\lambda_0n^2\gamma^2[B_1{}^2b(t)+B_2{}^2c(t)]}{\lambda_2(l-n)^2},\ \ A=\frac{\lambda_0}{\lambda_2}\gamma,\quad m=n=k.$$

If we set $b_0=0, b_{-1}=1$, $a_0=\pm2\sqrt{-\frac{A}{\gamma}}=\pm2\sqrt{-\frac{\lambda_0}{\lambda_2}}, \frac{A}{\gamma}=\frac{\lambda_0}{\lambda_2}<0$ in Eq. (74), we obtain

$$u_{7(1)}(x,y,t)=\left\{\lambda_0e^{\frac{l-n}{l}\int\alpha(t)dt}-\lambda_0e^{\frac{l-n}{l}\int\alpha(t)dt}\left[\coth(2\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)\pm\text{csch}(2\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)\right]^2\right\}^{\frac{1}{l-n}}$$

$$=\left(\frac{2\lambda_0e^{\frac{l-n}{l}\int\alpha(t)dt}}{1\mp\cosh(2\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)}\right)^{\frac{1}{l-n}}. \tag{75}$$

Setting $b_0=0, b_{-1}=i$, $a_0=\pm2\sqrt{-\frac{A}{\gamma}}=\pm2\sqrt{-\frac{\lambda_0}{\lambda_2}}$, $\frac{A}{\gamma}=\frac{\lambda_0}{\lambda_2}<0$ in Eq. (74), we get

$$u_{7(2)}(x,y,t)=\left\{\lambda_0e^{\frac{l-n}{l}\int\alpha(t)dt}-\lambda_0e^{\frac{l-n}{l}\int\alpha(t)dt}\left[\tanh(2\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)\pm i\,\text{sech}(2\gamma\sqrt{-\frac{\lambda_0}{\lambda_2}}\xi)\right]^2\right\}^{\frac{1}{l-n}} \tag{76}$$

Setting $b_0=0, b_{-1}=1$, $a_0=\pm2\sqrt{\frac{A}{\gamma}}=\pm2\sqrt{\frac{\lambda_0}{\lambda_2}}$, and $\frac{A}{\gamma}=\frac{\lambda_0}{\lambda_2}>0$ in Eq. (74), we have

$$u_{7(3)}(x,y,t)=\left\{\lambda_0e^{\frac{l-n}{l}\int\alpha(t)dt}+\lambda_0e^{\frac{l-n}{l}\int\alpha(t)dt}\left[\tan(2\gamma\sqrt{\frac{\lambda_0}{\lambda_2}}\xi)\pm\sec(2\gamma\sqrt{\frac{\lambda_0}{\lambda_2}}\xi)\right]^2\right\}^{\frac{1}{l-n}}. \tag{77}$$

Setting $b_0=0, b_{-1}=i$, $a_0=\pm2\sqrt{\frac{A}{\gamma}}=\pm2\sqrt{\frac{\lambda_0}{\lambda_2}}$, and $\frac{A}{\gamma}=\frac{\lambda_0}{\lambda_2}>0$ in Eq. (74), we have

$$u_{7(4)}(x,y,t)=\left\{\lambda_0e^{\frac{l-n}{l}\int\alpha(t)dt}+\lambda_0e^{\frac{l-n}{l}\int\alpha(t)dt}\left[\cot(2\gamma\sqrt{\frac{\lambda_0}{\lambda_2}}\xi)\mp\sec(2\gamma\sqrt{\frac{\lambda_0}{\lambda_2}}\xi)\right]^2\right\}^{\frac{1}{l-n}}$$

$$=\left(\frac{2\lambda_0 e^{\frac{l-n}{l}\int\alpha(t)dt}}{1\pm\cos(2\gamma\sqrt{\frac{\lambda_0}{\lambda_2}}\xi)}\right)^{\frac{1}{l-n}}. \quad (78)$$

Family 8

$$u_8(x,y,t)=\left\{\mu_0 e^{\frac{l-n}{l}\int\alpha(t)dt}+\mu_2 e^{\frac{l-n}{l}\int\alpha(t)dt}\times\right.$$

$$\left.\left[\frac{\frac{(\gamma a_0^{\ 2}+Ab_0^{\ 2})}{4Ab_{-1}}\exp(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)+b_0+b_{-1}\exp(-2\gamma\sqrt{-\frac{A}{\gamma}}\xi)}{\frac{(\gamma a_0^{\ 2}+Ab_0^{\ 2})}{4\gamma\sqrt{-\frac{A}{\gamma}}b_{-1}}\exp(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)+a_0+\sqrt{-\frac{A}{\gamma}}b_{-1}\exp(-2\gamma\sqrt{-\frac{A}{\gamma}}\xi)}\right]^2\right\}^{\frac{1}{l-n}}, \quad (79)$$

where $\xi=B_1x+B_2y-\frac{2n(l+n)\mu_2\gamma^2B_1}{\mu_0^{\ 2}(l-n)^2}\int\left[B_1^{\ 2}b(t)+B_2^{\ 2}c(t)\right]e^{\frac{n-l}{l}\int\alpha(t)dt}dt-C$,

$$\alpha(t)=\frac{2\mu_2n^2\gamma^2[B_1^{\ 2}b(t)+B_2^{\ 2}c(t)]}{\mu_0(l-n)^2},\quad A=\frac{\mu_2}{\mu_0}\gamma,\quad m=n=k.$$

If we set $b_0=0, b_{-1}=i$, $a_0=\pm2\sqrt{-\frac{A}{\gamma}}=\pm2\sqrt{-\frac{\mu_2}{\mu_0}}, \frac{A}{\gamma}=\frac{\mu_2}{\mu_0}<0$ in Eq. (79), we obtain

$$u_{8(1)}(x,y,t)=\left\{\mu_0 e^{\frac{l-n}{l}\int\alpha(t)dt}-\mu_2 e^{\frac{l-n}{l}\int\alpha(t)dt}\frac{1}{\left[\tanh(2\gamma\sqrt{-\frac{\mu_2}{\mu_0}}\xi)\pm i\,\mathrm{sech}(2\gamma\sqrt{-\frac{\mu_2}{\mu_0}}\xi)\right]^2}\right\}^{\frac{1}{l-n}}. \quad (80)$$

Setting $b_0=0, b_{-1}=1$, $a_0=\pm2\sqrt{\frac{A}{\gamma}}=\pm2\sqrt{\frac{\mu_2}{\mu_0}}, \frac{A}{\gamma}=\frac{\mu_2}{\mu_0}>0$ in Eq. (79), we get

$$u_{8(2)}(x,y,t)=\left\{\mu_0 e^{\frac{l-n}{l}\int\alpha(t)dt}+\mu_2 e^{\frac{l-n}{l}\int\alpha(t)dt}\frac{1}{\left[\tan(2\gamma\sqrt{\frac{\mu_2}{\mu_0}}\xi)\pm\sec(2\gamma\sqrt{\frac{\mu_2}{\mu_0}}\xi)\right]^2}\right\}^{\frac{1}{l-n}}. \quad (81)$$

4. Conclusions

In this paper, by using Exp-function method combined with F-expansion method, we studied the $ZK(m,n,k)$ equation with generalized evolution and time-dependent coefficients, eight families of exact solutions of exp-function type are obtained. When the parameters are taken as special values, every family of solution can be reduced to some solitary wave solutions and periodic wave solutions, the majority of these results are very different to those in Ref. [14]. This shows that some solutions with different wave forms can be expressed by the same one solution of exp-function type. From these abundant results, it is easy to know that the Exp-function method combined F-expansion method is useful to many nonlinear partial equations.

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