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Quartic B-Spline Collocation Method for Solving One-Dimensional Hyperbolic Telegraph Equation

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Abstract. In this paper, we use a numerical method based on B-spline function and collocation method to solve second-order linear hyperbolic telegraph equation. The scheme works in a similar fashion as finite difference methods. The results of numerical experiments are presented, and are compared with analytical solutions to confirm the good accuracy of the presented scheme.

Keywords: B-spline, collocation method, second-order hyperbolic telegraph equation, difference schemes.

1. Introduction

We consider the second-order linear hyperbolic telegraph equation in one-space dimension, given by

$$\frac{\partial^2 u}{\partial t^2} + 2\alpha \frac{\partial u}{\partial t} + \beta^2 u = \frac{\partial^2 u}{\partial x^2} + f(x,t), \; a \le x \le b, \;\; 0 \le t \le T, \tag{1}$$

subject to initial conditions

$$u(x,0) = f_0(x), \quad a \le x \le b, \tag{2}$$

$$\frac{\partial u(x,0)}{\partial t} = f_1(x), \quad a \le x \le b, \tag{3}$$

and Dirichlet boundary conditions

$$u(a,t) = g_0(t), \;\; u(b,t) = g_1(t), \;\; t \ge 0, \tag{4}$$

where α and β are known constant coefficients. We assume that $f_0(x), f_1(x)$ and their derivatives are continuous functions of x, and $g_i(t), i = 0,1,$ and their derivatives are continuous functions of t. Both the electric voltage and the current in a double conductor, satisfy the telegraph equation, where x is distance and t is time. For $\alpha > 0$ and $\beta = 0$, Eq. (1) represents a damped wave equation and for $\alpha > \beta > 0$, it is called telegraph equation.

The hyperbolic partial differential equations model the vibrations of structures (e.g. buildings, beams and machines) and are the basis for fundamental equations of atomic physics. Equations of the form Eq. (1) arise in the study of propagation of electrical signals in a cable of transmission line and wave phenomena. Interaction between convection and diffusion or reciprocal action of reaction and diffusion describes a number of nonlinear phenomena in physical, chemical and biological process [1]-[4]. In fact the telegraph equation is more suitable than ordinary diffusion equation in modeling reaction diffusion for such branches of sciences. For example biologists encounter these equations in the study of pulsate blood flow in arteries and in one- dimensional random motion of bugs along a hedge [5]. Also the propagation of acoustic waves in Darcy-type porous media [6], and parallel flows of viscous Maxwell fluids [7] are just some of the phenomena governed [8]-[9] by Eq. (1).

The theory of spline functions is very active field of approximation theory, boundary value problems and partial differential equations, when numerical aspects are considered. Among the various classes of splines,

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the polynomial spline has been received the greatest attention primarily because it admits a basis of B-splines [10]-[14] which can be accurately and efficiently computed. As the piecewise polynomial, B-splines have also become a fundamental tool for numerical methods to get the solution of the differential equations. In this paper, numerical solution of the hyperbolic telegraph equation by using the quartic B-spline collocation scheme is proposed. The collocation method together with B-spline approximations represents an economical alternative since it only requires the evaluation of the unknown parameters at the grid points. As is known, the success of the B-spline collocation method is dependent on the choice of B-spline basis. The quartic B-spline basis has been used to build up the approximation solutions for some differential equations. For instance see [15]-[20].

The layout of the article is as follows. In Section 2, we show that how we use the B-spline collocation method to approximate the solution of the hyperbolic telegraph equation. To demonstrate the efficiency of the proposed method, numerical experiments are carried out for several test problems and results are given in section 3. Finally the conclusion is given in the last Section. Finally some references are introduced at the end. Note that we have computed the numerical results by Matlab programming.

2. Quartic B-spline collocation method

Let be a uniform partition of an interval $[a,b]$ as follows $a = x_0 < x_1 < ... < x_N = b$ where $h = x_{j+1} - x_j$, $j = 0,1,..., N-1.$ The quartic B-splines are defined upon an increasing set of $N+1$ knots over the problem domain plus 8 additional knots outside the problem domain 8 additional knots are positioned as

$$x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_0 \text{ and } x_N < x_{N+1} < x_{N+2} < x_{N+3} < x_{N+4}.$$

The set of quartic B-spline $\{Q_{-2}, Q_{-1}, ..., Q_{N+1}\}$ form a basis over the problem domain $[a,b]$ [12].

Let $Q_m(x), m = -2,-1,..., N+1,$

$$Q_m(x) = \frac{1}{h^4}\begin{cases} (x - x_{m-2})^4, & x \in [x_{m-2}, x_{m-1}], \\ (x - x_{m-2})^4 - 5(x - x_{m-1})^4, & x \in [x_{m-1}, x_m], \\ (x - x_{m-2})^4 - 5(x - x_{m-1})^4 + 10(x - x_m)^4, & x \in [x_m, x_{m+1}], \\ (x_{x+3} - x)^4 - 5(x_{x+2} - x)^4, & x \in [x_{m+1}, x_{m+2}], \\ (x_{x+3} - x)^4, & x \in [x_{m+2}, x_{m+3}], \\ 0 & \text{otherwise}, \end{cases} \tag{5}$$

be quartic B-splines, which vanish outside interval. Each quartic B-spline cover five elements so that an element is covered by five quartic B-splines.

Now the solution of the problem is considered as follows:

$$U_N(x,t) = \sum_{m=-2}^{N+1} \delta_m(t) Q_m(x), \tag{6}$$

where $\delta_m, m = -2,..., N+1$ are unknown time dependent quantities to be determined from boundary conditions and the initial conditions. The values of $Q_m(x)$ and its derivatives $Q_m^{'}(x)$, $Q_m^{''}(x)$ and $Q_m^{'''}(x)$ at the knots are given in Table 1.

Table 1: The values of $Q_m, Q_m^{'}, Q_m^{''}, Q_m^{'''}$.

x	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}
Q	0	1	11	11	1	0
$Q_i^{'}$	0	$-4/h$	$-12/h$	$12/h$	$4/h$	0
$Q_i^{''}$	0	$12/h^2$	$-12/h^2$	$-12/h^2$	$12/h^2$	0
$Q_i^{'''}$	0	$-24/h^3$	$72/h^3$	$-72/h^3$	$24/h^3$	0

For every x by using the Taylor expansion in the time direction, using the notation $u_i = u(x,t_i)$ where

$t_i = t_{i-1} + \Delta t$ we have the following difference schemes

$$\frac{\partial^2 u(x,t_i)}{\partial t^2} = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta t)^2} + O((\Delta t)^2), \tag{7}$$

$$\frac{\partial u(x,t_i)}{\partial t} = \frac{u_{i+1} - u_i}{\Delta t} + O(\Delta t), \tag{8}$$

$$u(x,t_i) = \frac{u_{i+1} + u_i}{2} + O(\Delta t), \tag{9}$$

$$\frac{\partial^2 u(x,t_i)}{\partial x^2} = \frac{u''_{i+1} + u''_i}{2} + O(\Delta t). \tag{10}$$

Now, let us discretize Eq. (1) according to schemes Eqs. (7)-(10) in the following form

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta t)^2} + 2\alpha \frac{u_{i+1} - u_i}{\Delta t} + \beta^2 \frac{u_{i+1} + u_i}{2} = \frac{u''_{i+1} + u''_i}{2} + f(x,t_i). \tag{11}$$

Rearranging Eq. (11) we obtain

$$(1 + 2\alpha\Delta t + \frac{\beta^2 (\Delta t)^2}{2})u_{i+1} - \frac{(\Delta t)^2}{2} u''_{i+1} =$$

$$(2 + 2\alpha\Delta t - \frac{\beta^2 (\Delta t)^2}{2})u_i - u_{i-1} + \frac{(\Delta t)^2}{2} u''_i + (\Delta t)^2 f(x,t_i), \tag{12}$$

and the initial conditions are given in Eqs. (2) and (3) as follows

$$u(x,0) = f_0(t) = u_0, \tag{13}$$

$$u_t(x,0) = \frac{u_1 - u_0}{\Delta t} = f_1(x), \tag{14}$$

$$u_1 = u_0 + \Delta t f_1(x). \tag{15}$$

Substituting Eq. (15) into Eq. (12) then is obtained as follows

$$i = 1,\quad (1 + 2\alpha\Delta t + \frac{\beta^2 (\Delta t)^2}{2})u_2 - \frac{(\Delta t)^2}{2} u''_2 =$$

$$(2 + 2\alpha\Delta t - \frac{\beta^2 (\Delta t)^2}{2})u_1 - u_0 + \frac{(\Delta t)^2}{2} u''_1 + (\Delta t)^2 f(x,t_1), \tag{16}$$

$$i = 2,\quad (1 + 2\alpha\Delta t + \frac{\beta^2 (\Delta t)^2}{2})u_3 - \frac{(\Delta t)^2}{2} u''_3 =$$

$$(2 + 2\alpha\Delta t - \frac{\beta^2 (\Delta t)^2}{2})u_2 - u_1 + \frac{(\Delta t)^2}{2} u''_2 + (\Delta t)^2 f(x,t_2), \tag{17}$$

$$\vdots$$

$$i = n-1,\quad (1 + 2\alpha\Delta t + \frac{\beta^2 (\Delta t)^2}{2})u_n - \frac{(\Delta t)^2}{2} u''_n =$$

$$(2 + 2\alpha\Delta t - \frac{\beta^2 (\Delta t)^2}{2})u_{n-1} - u_{n-2} + \frac{(\Delta t)^2}{2} u''_{n-1} + (\Delta t)^2 f(x,t_{n-1}), \tag{18}$$

The approximate solution of Eqs. (16)-(18) are sought in the form of the B-spline functions $U_N(x,t)$, it follows that

$$i = 1,\quad (1 + 2\alpha\Delta t + \frac{\beta^2 (\Delta t)^2}{2})(U_N)_2 - \frac{(\Delta t)^2}{2} (U_N)''_2 =$$

$$(2 + 2\alpha\Delta t - \frac{\beta^2 (\Delta t)^2}{2})u_1 - u_0 + \frac{(\Delta t)^2}{2} u''_1 + (\Delta t)^2 f(x,t_1), \tag{19}$$

$$i=2,\quad (1+2\alpha\Delta t+\frac{\beta^2(\Delta t)^2}{2})(U_N)_3-\frac{(\Delta t)^2}{2}(U_N)_3^{''}=$$

$$(2+2\alpha\Delta t-\frac{\beta^2(\Delta t)^2}{2})u_2-u_1+\frac{(\Delta t)^2}{2}u_2^{''}+(\Delta t)^2 f(x,t_2), \tag{20}$$

$$\vdots$$

$$i=n-1,\quad (1+2\alpha\Delta t+\frac{\beta^2(\Delta t)^2}{2})(U_N)_n-\frac{(\Delta t)^2}{2}(U_N)_n^{''}=$$

$$(2+2\alpha\Delta t-\frac{\beta^2(\Delta t)^2}{2})u_{n-1}-u_{n-2}+\frac{(\Delta t)^2}{2}u_{n-1}^{''}+(\Delta t)^2 f(x,t_{n-1}), \tag{21}$$

and the boundary conditions (4) can be written as

$$\sum_{m=-2}^{N+2}\delta_m(t)Q_m(x_0)=g_0(t),\quad for\quad x=a,\quad 0\le t\le T, \tag{22}$$

$$\sum_{m=-2}^{N+2}\delta_m(t)Q_m(x_N)=g_1(t),\quad for\quad x=b,\quad 0\le t\le T. \tag{23}$$

The spline solution of Eq. (19) with the boundary conditions (22) and (23) are obtained by solving to the following matrix equation. The value of spline functions at the knots $\{x_i\}_{i=0}^{N}$ are determined using Table 1. Then the B-spline method in matrix form can be written as follows:

$$AX=B, \tag{24}$$

where $X=[\delta_{-2},\delta_{-1},...,\delta_N,\delta_{N+1}]$, while $A\in R^{(N+3)\times(N+4)}$ and $B\in R^{(N+3)}$ are obtained from left and right hand sides of Eqs. (19) , (22) and (23), respectively as follows:

$$A=\begin{bmatrix} 1 & 11 & 11 & 1 & 0 & 0 & 0 & \cdots & 0\\ r_1 & r_2 & r_2 & r_1 & 0 & 0 & 0 & \cdots & 0\\ 0 & r_1 & r_2 & r_2 & r_1 & 0 & 0 & \cdots & 0\\ 0 & 0 & r_1 & r_2 & r_2 & r_1 & 0 & \cdots & 0\\ & & & & \vdots & & & & \\ 0 & 0 & \cdots & 0 & 0 & r_1 & r_2 & r_2 & r_1\\ 0 & 0 & \cdots & 0 & 0 & 1 & 11 & 11 & 1 \end{bmatrix},$$

and

$$B=\begin{bmatrix} g_0(t_1)\\ (\Delta t)^2 f(x_0,t_1)-u_0(x_0)+\frac{(\Delta t)^2}{2}u_1^{''}(x_0)+(2+2\alpha\Delta t-\frac{\beta^2(\Delta t)^2}{2})u_1(x_0)\\ (\Delta t)^2 f(x_1,t_1)-u_0(x_1)+\frac{(\Delta t)^2}{2}u_1^{''}(x_1)+(2+2\alpha\Delta t-\frac{\beta^2(\Delta t)^2}{2})u_1(x_1)\\ \vdots\\ (\Delta t)^2 f(x_N,t_1)-u_0(x_N)+\frac{(\Delta t)^2}{2}u_1^{''}(x_N)+(2+2\alpha\Delta t-\frac{\beta^2(\Delta t)^2}{2})u_1(x_N)\\ g_1(t_1) \end{bmatrix},$$

where $r_1=(1+2\alpha\Delta t+\frac{\beta^2(\Delta t)^2}{2})-\frac{6(\Delta t)^2}{h^2}$ and $r_2=11(1+2\alpha\Delta t+\frac{\beta^2(\Delta t)^2}{2})+\frac{6(\Delta t)^2}{h^2}$.

It is easy to see that, the same approximation can be applied the other Eqs. (20) and (21) together with the corresponding boundary conditions (22) and (23). We solve $n-1$ times the system (24) by means of a home-made program which is based on singular value de-composition (SVD) method [21] and in each step obtain $u(x_0,t_i),...,u(x_N,t_i)(i=1,..,n-1)$.

The condition number of A

$$k_s(A) = \|A\|_s \|A^{-1}\|_s, \qquad s = 1,2,\infty,$$

depends on α, β distance of collocation points and Δt. Therefore a small perturbation in initial data may produce a large amount of perturbation in the solution. Also the condition number grows with N for fixed values of α and β. Generally for a fixed number of collocation points N, smaller values of α and β produce better approximations, but the matrix A will be more ill-conditioned.

3. Numerical examples

In this section, the method discussed in Sections 2 is tested on the following problems. Pointwise error is measured by using the root mean square error L_2 and maximum error L_∞:

$$L_2 = |U - U_N|^2 = h\sum_{j=0}^{N} |U_j - (U_N)_j|^2, \qquad L_\infty = |U - U_N|_\infty = \max_j |U_j - (U_N)_j|.$$

Example 3.1: We consider the hyperbolic telegraph Eq. (1) with $\alpha = 10, \beta = 5$,

$f(x,t) = \alpha(1+ \tan^2(\frac{x+t}{2})) + \beta^2 \tan(\frac{x+t}{2})$ and $0 \le x \le 2$. The initial conditions are given by

$$\begin{cases} u(x,0) = \tan(\frac{x}{2}), \\ u_t(x,0) = \frac{1}{2}(1+\tan^2(\frac{x}{2})), \end{cases}$$

and the boundary conditions

$$\begin{cases} u(0,t) = \tan(\frac{t}{2}), \\ u(2,t) = \tan(\frac{2+t}{2}). \end{cases}$$

The analytical solution of this example [22] is $u(x,t) = \tan((x+t)/2)$. The root-mean- square error L_2 and maximum error L_∞ are presented in Table 2. The space-time graph of the estimated solution up to $t = 1$ is shown in Figure 1. The graph of analytical and estimated solutions for some different times and $x \in [0,2]$ is presented in Figure 2.

Table 2: Results at $\Delta t = 0.001$ and $\Delta x = 0.001$ in Example 3.1.

t	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$	$t = 1$
L_∞	2.774×10^{-4}	7.0782×10^{-4}	1.3848×10^{-3}	3.0930×10^{-3}	1.3424×10^{-2}
L_2	3.3189×10^{-8}	2.3067×10^{-7}	8.208×10^{-7}	3.237×10^{-6}	3.2782×10^{-5}

Example 3.2: Consider the hyperbolic telegraph Eq. (1) with $\alpha = 4, \beta = 2$,

$f(x,t) = (2 - 2\alpha + \beta^2)\exp(-t)\sin(x)$ and $0 \le x \le \pi$. The initial conditions are given by

$$\begin{cases} u(x,0) = \sin(x), \\ u_t(x,0) = -\sin(x), \end{cases} \tag{25}$$

and the boundary conditions

$$u(0,t) = u(\pi,t) = 0. \tag{26}$$

The analytical solution of this example [22] is $u(x,t) = \exp(-t)\sin(x)$. The space-time graph of the numerical solution up to $t = 2$ is presented in Figure 3. The graph of analytical and estimated solutions for some different times and $x \in [0,\pi]$ is presented in Figure 4. The accuracy of the B-spline method is measured by using the

L_2 and L_∞ errors. The errors are reported in Table 3.

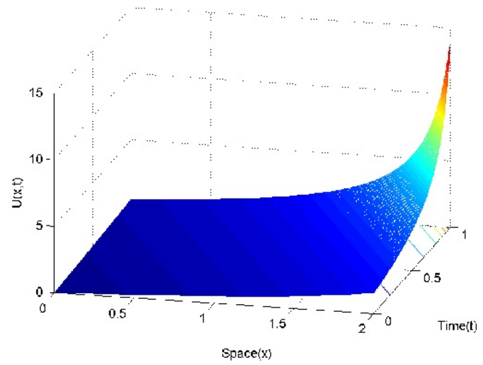

Fig. 1: Three-dimensional plot, with $\Delta t = 0.001$ and $\Delta x = 0.005$ in Example 3.1.

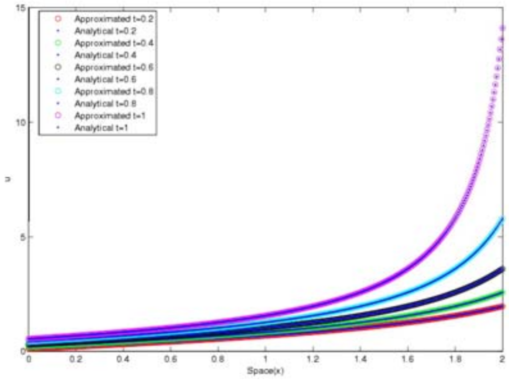

Fig. 2: Comparisons between numerical and analytical solutions of Eq. (1) in $t = 0.2s, t = 0.4s, t = 0.6s$, $t = 0.8s, t = 1s$ with $\Delta t = 0.001$ and $\Delta x = 0.005$ in Example 3.1.

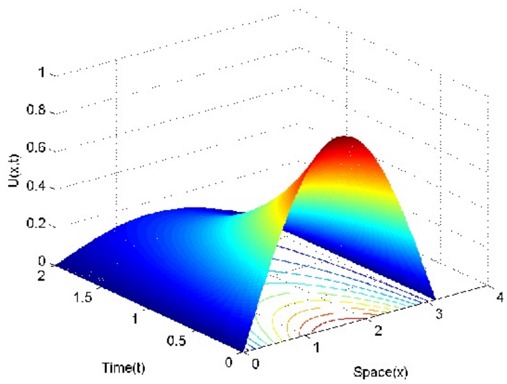

Fig. 3: Three-dimensional plot, with $\Delta t = 0.0001$ and $\Delta x = 0.02$ in Example 3.2.

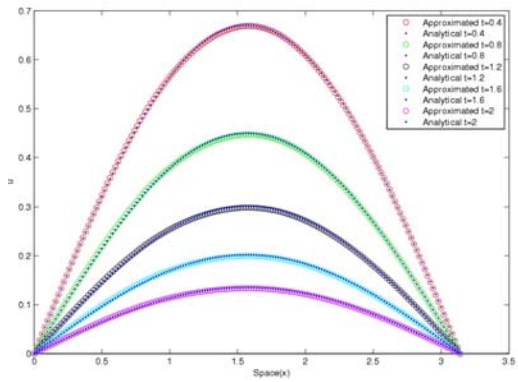

Fig. 4: Comparisons between numerical and analytical solutions of Eq. (1) in $t = 0.4s, t = 0.8s, t = 1.2s$, $t = 1.6s, t = 2s$ with $\Delta t = 0.0001$ and $\Delta x = 0.02$ in Example 3.2.

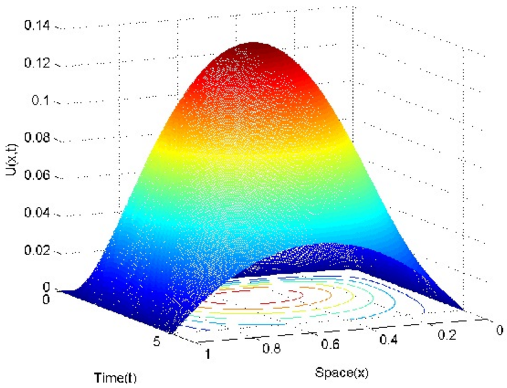

Fig. 5: Three-dimensional plot, with $\Delta t = 0.01$ and $\Delta x = 0.005$ in Example 3.3.

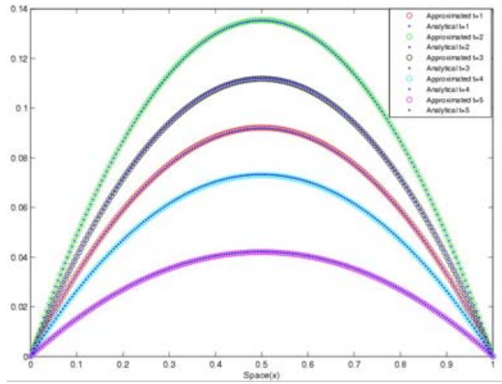

Fig. 6: Comparisons between numerical and analytical solutions of Eq. (1) in $t = 1s, t = 2s, t = 3s,$ $t = 4s, t = 5s$ with $\Delta t = 0.01$ and $\Delta x = 0.005$ in Example 3.3.

Example 3.3: In this example, we consider the hyperbolic telegraph Eq.(1) with $\alpha = \frac{1}{2}, \beta = 1,$ $f(x,t) = (2 - 2t + t^2)(x - x^2)\exp(-t) + 2t^2 \exp(-t)$ and $0 \le x \le 1$. The initial conditions are given by

$$u(x,0) = u_t(x,0) = 0, \tag{27}$$

and the boundary conditions

$$u(0,t) = u(1,t) = 0. \tag{28}$$

Table 3: Results at $\Delta t = 0.0001$ and $\Delta x = 0.02$ in Example 3.2.

t	$t=0.4$	$t=0.8$	$t=1.2$	$t=1.6$	$t=2$
L_∞	2.9×10^{-3}	3.2×10^{-3}	2.8×10^{-3}	2.3×10^{-3}	1.8×10^{-3}
L_2	5.8690×10^{-6}	1.0192×10^{-5}	9.3591×10^{-6}	6.9011×10^{-6}	4.5782×10^{-6}

Table 4: Results at $\Delta t = 0.01$ and $\Delta x = 0.005$ in Example 3.3.

t	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
L_∞	1.9175×10^{-4}	1.1387×10^{-4}	1.7053×10^{-4}	2.0271×10^{-4}	9.8405×10^{-5}
L_2	2.1120×10^{-8}	6.5830×10^{-9}	1.5660×10^{-8}	2.1734×10^{-8}	5.2713×10^{-9}

Table 5: Results at $\Delta t = 0.001$ and $\Delta x = 0.005$ in Example 3.4.

t	$t=0.2$	$t=0.4$	$t=0.6$	$t=0.8$	$t=1$
L_∞	2.4279×10^{-5}	7.9315×10^{-5}	1.2097×10^{-4}	1.4883×10^{-4}	1.6462×10^{-4}
L_2	1.6998×10^{-10}	2.6707×10^{-9}	6.7849×10^{-9}	1.0726×10^{-8}	1.3438×10^{-8}

The analytical solution of this example [22] is $u(x,t) = (x - x^2)t^2 \exp(-t)$. The accuracy of the scheme is measured by using the L_2 and L_∞ errors. The errors are reported in Table 4.

The space-time graph of the numerical solution up to $t = 5$ is presented in Figure 5. The graph of analytical and estimated solutions for several different times and $x \in [0,1]$ is presented in Figure 6.

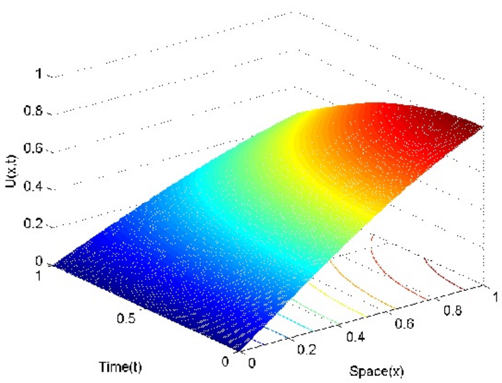

Fig. 7: Three-dimensional plot, with $\Delta t = 0.001$ and $\Delta x = 0.005$ in Example 3.4.

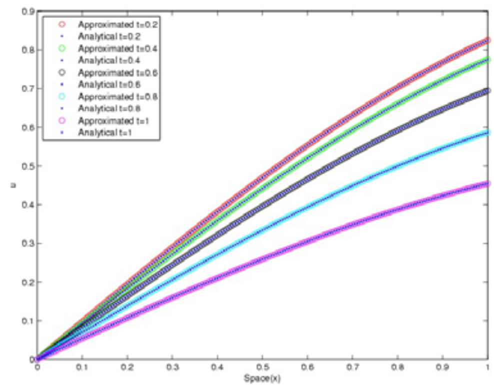

Fig. 8: Comparisons between numerical and analytical solutions of Eq. (1) in $t = 0.2s, t = 0.4s, t = 0.6s$, $t = 0.8s, t = 1s$ with $\Delta t = 0.001$ and $\Delta x = 0.005$ in Example 3.4.

Example 3.4: Consider Eq. (1) with $\alpha = 6, \beta = 2, f(x,t) = (2 - 2t + t^2)(x - x^2)\exp(-t) + 2t^2 \exp(-t)$ and $0 \le x \le 1$ and the following conditions:

$$\begin{cases} f_0(x) = \sin(x), \\ f_1(x) = g_0(t) = 0, \\ g_1(t) = \cos(t)\sin(1), \\ f(x,t) = -2\alpha \sin(t)\sin(x) + \beta^2 \cos(t)\sin(x). \end{cases}$$

The analytical solution of this example [22] is $u(x,t)=\cos(t)\sin(x)$. The root-mean- square error and and maximum error are presented in Table 5, also the space-time graph of the estimated solution up to $t=1$ is presented in Figure 7. The graph of analytical and estimated solutions for some different times and $x\in[0,1]$ is presented in Figure 8.

4. Conclusion

In this paper, we have been discussed on second-order hyperbolic telegraph equation. A numerical treatment for the second- order hyperbolic telegraph equation is proposed using a collocation method with the quartic B-spline functions. The numerical solutions are compared with the exact solution by finding L_2 and L_∞ errors. Most importantly, quartic B-spline methods are especially advisable for obtaining numerical solutions of differential equations when higher continuity of the solutions exist.

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