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Inner-Outer Synchronization Analysis of Two Complex Networks with Delayed and Non-Delayed Coupling

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Abstract. In this paper, two kinds of synchronization between two complex networks with non-delayed and delayed coupling are discussed by the pinning control method, that is, inner synchronization and outer synchronization. Based on the Lyapunov stability theory and linear matrix inequality (LMI), some sufficient conditions for the synchronization are derived by adding linear feedback controllers to a part of nodes, and the linear feedback controllers are designed. Under suitable conditions, not only inner synchronization but also outer synchronization can be asymptotically achieved. Numerical simulations are presented to show the effectiveness of the proposed synchronization scheme.

Keywords: Complex network; Inner-outer synchronization; Pinning control; Non-delayed and delayed coupling;

1. Introduction

In the past few years, the control and synchronization problem of complex networks has been extensively investigated in various fields due to its many potential applications [1-3]. And various control schems including pinning control [4-5], adaptive control [6], impulsive control [7-9], etc., have been used to study the above problem. Generally speaking, network synchronization can be classified into inner synchronization and outer synchronization [10]. In brief, the synchronization in a network is called inner synchronization, i.e., the synchronization of all the nodes within a network, it has been investigated recently [3-9]. On the other hand, outer synchronization [10-14] occurs between two or more complex networks regardless of synchronization of the inner network. One important example is the infectious disease that spreads between different communities. Therefore, how to realize the synchronization between different networks is very interesting and challenging work. Li et al. [10] pioneered in studying the outer synchronization between two unidirectionally coupled complex networks and derived analytically a criterion for them having the identical topological structures. The adaptive-impulsive synchronization between two complex networks with non-delayed and delayed coupling was discussed in Ref. [14]. Another interesting work is how to realize the inner synchronization inside each network and the outer synchronization between two different networks simultaneously. Recently, Sun et al. [15] investigated the hybrid synchronization problem of two coupled complex networks by using the linear feedback and the adaptive feedback control methods, but the time delay was ignored. To simulate more realistic networks, time delay should be taken into account. Sun et al. [16] studied two kinds of synchronization between two discrete-time networks with time delays, including inner synchronization within each network and outer synchronization between two networks. In the above literatures [10-15], the networks are coupled by full states of nodes in the networks, which means all the states in the drive network must be transmitted to the response network. However, for the complexity of the network, it is difficult to realize the synchronization by adding controllers to all nodes. To reduce the number of the controllers, a natural approach is to control a complex network by pinning part of the nodes in the networks. As far as the authors know, there is few work on pinning synchronization between two coupled dynamical networks, although some pinning control schemes have been proposed for inner synchronization.

Motivated by the above discussions, this paper will focus on the synchronization problem of two coupled

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dynamical networks with both delayed and non-delayed coupling via pinning control method, including inner synchronization within each network and outer synchronization between two networks. Some criteria for the synchronization are derived. Analytical results show that two networks can realize the synchronization: the outer synchronization between the drive-response networks, and the inner synchronization in the drive network and the response newtwork, respectively.

2. Problem description and preliminaries

In this paper, we consider the two coupled complex dynamical network consisting of linearly coupled *N* identical dynamical nodes, with each node being an *n*-dimensional dynamic system respectively.

The drive coupled complex network is characterized by

$$\dot{x}_i(t) = f(x_i(t)) + c_1\sum_{j=1}^{N} a_{ij}\Gamma_1 x_j(t) + c_2\sum_{j=1}^{N} b_{ij}\Gamma_2 x_j(t-\tau),\ i = 1, 2, ..., N. \tag{1}$$

Consider the response coupled complex dynamical network as follows:

$$\dot{y}_i(t) = f(y_i(t)) + c_1\sum_{j=1}^{N} a_{ij}\Gamma_1 y_j(t) + c_2\sum_{j=1}^{N} b_{ij}\Gamma_2 y_j(t-\tau) + u_i,\ i = 1, 2, ..., N. \tag{2}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), ..., x_{in}(t))^T \in R^n$ is the drive state vector of the *i*th node, $y_i(t) = (y_{i1}(t), y_{i2}(t), ..., y_{in}(t))^T \in R^n$ is the response state vector of the i th node, $f: R^n \to R^n$ is a smooth function, the constant $c_1 > 0$ and $c_2 > 0$ denote the nondelayed and delayed coupling strength respectively, $\tau > 0$ is the time delay. u_i are linear controllers to be designed. $\Gamma_1 = diag(\gamma_1^1, \gamma_1^2, \cdots, \gamma_1^n)$ and $\Gamma_2 = diag(\gamma_2^1, \gamma_2^2, \cdots, \gamma_2^n)$ are positive definite diagonal inner coupling matrices of the networks. $A = (a_{ij})_{N\times N} \in R^{N\times N}$ and $B = (b_{ij})_{N\times N} \in R^{N\times N}$ are the nondelayed and delayed weight configuration matrices respectively, where a_{ij} and b_{ij} are defined as follows: If there is a connection from node i to node j $(j \neq i)$, then the coupling $a_{ij} \neq 0$; otherwise, $a_{ij} = 0$ $(j \neq i)$, and the diagonal elements of matrix *A* are defined as $a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}$, $i = 1, 2, \ldots N$. B has the same meaning as that of matrix A.

Suppose $C([t_0-\tau, t_0], R^n)$ be the Banach space of continuous vector-valued functions mapping the interval $[t_0-\tau, t_0]$ into R^n with the norm $\|\phi\| = \sup_{t_0-\tau\le s\le t_0}\|\phi(s)\|$. For the functional differential equation (1), its initial conditions are given by $x_i(t) = \phi_i(t) \in C([t_0-\tau, t_0], R^n)$. It is assumed that (1) has an unique solution with respect to these initial conditions. For the functional differential equation (2), its initial conditions are given by $y_i(t) = \varphi_i(t) \in C([t_0-\tau, t_0], R^n)$. And, at least, there exists a constant i $(i = 1, 2, \cdots, N)$ such that $\varphi_i(t) \neq \phi_i(t)$ for $t \in [t_0-\tau, t_0]$.

In order to derive our main results, some necessary definitions and lemmas are needed.

Definition 1 The coupled network (1) and the coupled network (2) are said to attain inner and outer synchronization simultaneously if

$$\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0,\ \lim_{t\to\infty}(y_i(t) - y_j(t)) = 0,\ \lim_{t\to\infty}(y_i(t) - x_i(t)) = 0, i, j = 1, 2, ...N.$$

Assumption 1 (A1) (see[17]) Assuming that there is a positive-definite diagonal matrix $P = diag(p_1, p_2, ..., p_n)$ and a diagonal matrix $D = diag(d_1, d_2, ..., d_n)$ such that *f* satisfies the following inequality:

$$(x-y)^T P(f(x,t) - f(y,t) - \Delta(x-y)) \le -\eta(x-y)^T(x-y),$$

for some $\eta > 0$, all $x, y \in R^n$ and $t > 0$.

Lemma 1 (see [17]) Assuming $A \in R^{N\times N}$ satisfies the following conditions:

(1) $a_{ij} \geq 0 (j \neq i), a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}, i = 1, 2, ...N;$

(2) A is irreducible, then, one has

(i) Real parts of the eigenvalues of A are all negative except an eigenvalue 0 with the multiplicity 1.

(ii) A has the right eigenvector $(1,1,\cdots,1)^T$ corresponding to the eigenvalue 0.

(iii) Let $\xi = (\xi_1, \xi_2, \cdots, \xi_N)^T$ be the left eigenvector of A corresponding to the eigenvalue 0 satisfying $\sum_{i=1}^{N} \xi_i = 1$, then , we can let $\xi_i > 0$ hold for all $i = 1, 2, ...N.$

Lemma 2 (see [6]) If $G = (g_{ij})_{N\times N} \in R^{N\times N}$ is an irreducible matrix and satisfies $g_{ij} = g_{ji} \geq 0 (j \neq i)$, and $g_{ii} = -\sum_{j=1, j\neq i}^{N} g_{ij}, i, j = 1, 2, ...N,$ then, all eigenvalues of the matrix $\overline{G} = G - diag(k_1, \cdots, k_l, 0, \cdots, 0)$ are negative, where $k_i > 0, i = 1, 2, \cdots, l$ are positive constants.

Lemma 3 (see[18]) If the matrix $A = (a_{ij})_{N\times N}$ satisfies $a_{ij} = a_{ji} \geq 0 (j \neq i)$ and $a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}$, $i, j = 1, 2, ...N.$ Then for any two vectors $x = (x_1, x_2, ..., x_N)^T$ and $y = (y_1, y_2, ..., y_N)^T$, we have $x^T A y = -\sum_{j>i} a_{ij}(x_j - x_i)(y_j - y_i)$.

Lemma 4. Let Q and R be two symmetric matrices, and matrix S has suitable dimension. Then

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} < 0$$

if and only if both $R < 0$ and $Q - SR^{-1}S^T < 0.$

3. Synchronizaton analysis

In this section, we will make the drive-response coupled complex dynamical networks achieve the synchronization, that is, we realize the inner synchronization inside each network and outer synchronization between them simultaneously.

We assume the coupling matrix A is irreducible, and let $\xi = (\xi_1, \xi_2, \cdots, \xi_N)^T$ be the left eigenvector of A corresponding to the eigenvalue 0 with $\xi_i > 0$. Then we define $U = \Xi - \xi\xi^T$ and $\Xi = diag(\xi_1, \xi_2, ..., \xi_N)$. We also let $\overline{A} = A - diag(k_1, \cdots, k_l, 0, \cdots, 0)$ and $(\Xi\overline{A})^s = \frac{\Xi\overline{A} + \overline{A}^T\Xi}{2}$. Since the coupling matrix A is irreducible, then ΞA is irreducible, that is, $\Xi A + A^T \Xi$ is also irreducible. From Lemma 2, we know $(\Xi\overline{A})^s$ is negative definite.

For convenience of writing, we denote $\tilde{x}_h(t) = (\tilde{x}_{1h}(t), \tilde{x}_{2h}(t), \cdots, \tilde{x}_{Nh}(t))^T$, $\tilde{e}_h(t) = (\tilde{e}_{1h}(t), \tilde{e}_{2h}(t), \cdots,$ $\tilde{e}_{Nh}(t))^T$. $h = 1, 2, \cdots, n.$

The drive network can be written in compact form as:

$$\dot{X} = F(X) + (c_1 A \otimes \Gamma_1) X(t) + (c_2 B \otimes \Gamma_2) X(t - \tau) \tag{3}$$

where $X(t) = (x_1(t), x_2(t), ..., x_N(t))^T$, $F(X) = (f(x_1), f(x_2), ..., f(x_N))^T$, $\otimes$ is the Kronecter product of two matrices.

Without loss of the generality, assume that the first l nodes $1 \leq i \leq l$ are selected and pinned with the linear feedback controllers, which are described by

$$u_i = \begin{cases} -c_1 k_i \Gamma_1 (y_i - x_i), 1 \le i \le l, \\ 0, \qquad 1 + l \le i \le N. \end{cases} \tag{4}$$

where k_i $(1 \le i \le l)$ are any positive constants.

Let $e_i(t) = y_i(t) - x_i(t)$, then the following error dynamical network can be obtained:

$$\begin{cases} \dot{e}_i(t) = f(y_i(t)) - f(x_i(t)) + c_1 \sum_{j=1}^{N} a_{ij} \Gamma_1 e_j(t) + c_2 \sum_{j=1}^{N} b_{ij} \Gamma_2 e_j(t-\tau) - c_1 k_i \Gamma_1 e_i(t), 1 \le i \le l, \\ \dot{e}_i(t) = f(y_i(t)) - f(x_i(t)) + c_1 \sum_{j=1}^{N} a_{ij} \Gamma_1 e_j(t) + c_2 \sum_{j=1}^{N} b_{ij} \Gamma_2 e_j(t-\tau), \qquad 1 + l \le i \le N. \end{cases} \tag{5}$$

Then we have the following result:

Theorem 1. Suppose A1 holds. If there exists a semi-positive definite matrix M and a positive definite matrix Q and constants $c_1, c_2, \delta_h, \gamma_1^h, \gamma_2^h (h = 1, 2, \cdots, n)$ such that

$$Z_1 = \begin{pmatrix} \delta_h U + (c_1 \gamma_1^h \Xi A)^s + M & \dfrac{c_2 \gamma_2^h UB}{2} \\ \dfrac{c_2 \gamma_2^h (UB)^T}{2} & -M \end{pmatrix} \le 0 \tag{6}$$

and

$$\delta_h \Xi + (c_1 \gamma_1^h \Xi \bar{A})^s + Q + \frac{(c_2 \gamma_2^h)^2 \Xi B Q^{-1} B^T \Xi}{4} < 0 \tag{7}$$

then, we can achieve the synchronization, that is, the inner synchronization inside each network and outer synchronization between them simultaneously.

Proof. According to Ref. [19], we define the synchronization state of the driving network as $s(t) = \sum_{i=1}^{N} \xi_i x_i(t),$ then, we have the inner synchronziation error $\delta x_i(t) = x_i(t) - s(t).$

We construct a Lyapunov function candidate as

$$V = V_1 + V_2 \tag{8}$$

where $V_1 = \frac{1}{2} \sum_{i=1}^{N} \xi_i (\delta x_i(t))^T P \delta x_i(t) + \sum_{h=1}^{n} p_h \int_{t-\tau}^{t} \tilde{x}_h^T(\theta) M \tilde{x}_h(\theta) d\theta = \frac{1}{2} X^T (U \otimes P) X$

$+\sum_{h=1}^{n} p_h \int_{t-\tau}^{t} \tilde{x}_h^T(\theta) M \tilde{x}_h(\theta) d\theta$ and $V_2 = \frac{1}{2} \sum_{i=1}^{N} \xi_i e_i^T(t) P e_i(t) + \sum_{h=1}^{n} p_h \int_{t-\tau}^{t} \tilde{e}_h^T(\theta) Q \tilde{e}_h(\theta) d\theta.$

By Assumption 1 and Lemma 3, the derivative of V_1 along the trajectories of (3) is

$$\begin{aligned} \dot{V}_1 &= X^T (U \otimes P) \dot{X} + \sum_{h=1}^{n} p_h \tilde{x}_h^T(t) M \tilde{x}_h(t) - \sum_{h=1}^{n} p_h \tilde{x}_h^T(t-\tau) M \tilde{x}_h(t-\tau) \\ &= X^T (U \otimes P)[F(X) + (c_1 A \otimes \Gamma_1) X(t) + (c_2 B \otimes \Gamma_2) X(t-\tau)] \\ &\quad + \sum_{h=1}^{n} p_h \tilde{x}_h^T(t) M \tilde{x}_h(t) - \sum_{h=1}^{n} p_h \tilde{x}_h^T(t-\tau) M \tilde{x}_h(t-\tau) \\ &= X^T (U \otimes P)[F(x) - (I_N \otimes \Delta) X(t)] + X^T (U \otimes P)[(I_N \otimes \Delta) + (c_1 A \otimes \Gamma_1)] X(t) \\ &\quad + X^T (U \otimes P)(c_2 B \otimes \Gamma_2) X(t-\tau) + \sum_{h=1}^{n} p_h \tilde{x}_h^T(t) M \tilde{x}_h(t) - \sum_{h=1}^{n} p_h \tilde{x}_h^T(t-\tau) M \tilde{x}_h(t-\tau) \\ &= -\sum_{j>i} U_{ij} (x_j - x_i) P (f(x_j) - f(x_i) - \Delta(x_j - x_i)) + X^T (U \otimes P)[(I_N \otimes \Delta) + (c_1 A \otimes \Gamma_1)] X(t) \end{aligned}$$

$$+X^T(U\otimes P)(c_2B\otimes\Gamma_2)X(t-\tau)+\sum_{h=1}^{n}p_h\tilde{x}_h^{\ T}(t)M\tilde{x}_h(t)-\sum_{h=1}^{n}p_h\tilde{x}_h^{\ T}(t-\tau)M\tilde{x}_h(t-\tau)$$

$$\le -\alpha X^T(U\otimes I_N)X+X^T(UI_N\otimes P\Delta)X+X^T(c_1UA\otimes P\Gamma_1)X(t)$$

$$+X^T(c_2UB\otimes P\Gamma_2)X(t-\tau)+\sum_{h=1}^{n}p_h\tilde{x}_h^{\ T}(t)M\tilde{x}_h(t)-\sum_{h=1}^{n}p_h\tilde{x}_h^{\ T}(t-\tau)M\tilde{x}_h(t-\tau)$$

$$\le \alpha\sum_{j>i}U_{ij}(x_j-x_i)^T(x_j-x_i)+X^T(UI_N\otimes P\Delta)X+X^T(c_1UA\otimes P\Gamma_1)X(t)$$

$$+X^T(c_2UB\otimes P\Gamma_2)X(t-\tau)+\sum_{h=1}^{n}p_h\tilde{x}_h^{\ T}(t)M\tilde{x}_h(t)-\sum_{h=1}^{n}p_h\tilde{x}_h^{\ T}(t-\tau)M\tilde{x}_h(t-\tau)$$

$$\le -\frac{\alpha}{\max(p_h)}X^T(U\otimes P)X+\sum_{h=1}^{n}p_h\tilde{x}_h^{\ T}(t)(\delta_hU)\tilde{x}_h(t)+\sum_{h=1}^{n}p_h\tilde{x}_h^{\ T}(t)(c_1\gamma_1^h\Xi A)\tilde{x}_h(t)$$

$$+\sum_{h=1}^{n}p_h\tilde{x}_h^{\ T}(t)(c_2\gamma_2^hUB)\tilde{x}_h(t-\tau)+\sum_{h=1}^{n}p_h\tilde{x}_h^{\ T}(t)M\tilde{x}_h(t)-\sum_{h=1}^{n}p_h\tilde{x}_h^{\ T}(t-\tau)M\tilde{x}_h(t-\tau)$$

$$=-\frac{\alpha}{\max(p_h)}X^T(U\otimes P)X+\sum_{h=1}^{n}p_h[\tilde{x}_h^{\ T}(t),\tilde{x}_h^T(t-\tau)]Z_1\begin{pmatrix}\tilde{x}_h(t)\\ \tilde{x}_h(t-\tau)\end{pmatrix}$$

From the condition (6) of Theorem 1, we obtain

$$\dot{V}_1\le -\frac{\alpha}{\max(p_h)}X^T(U\otimes P)X \tag{9}$$

Evaluating the time derivative of V_2 along the trajectory of (5), one obtains

$$\dot{V}_2=\sum_{i=1}^{N}\xi_ie_i^T(t)P[f(y_i(t))-f(x_i(t))-\Delta e_i(t)]+\sum_{i=1}^{N}\xi_ie_i^T(t)P[\Delta e_i(t)+c_1\sum_{j=1}^{N}a_{ij}\Gamma_1e_j(t)$$

$$-c_1\sum_{i=1}^{l}k_i\xi_ie_i^T(t)P\Gamma_1e_i(t)+c_2\sum_{j=1}^{N}b_{ij}\Gamma_2e_j(t-\tau)]+\sum_{h=1}^{n}p_h\tilde{e}_h^{\ T}(t)Q\tilde{e}_h(t)$$

$$-\sum_{h=1}^{n}p_h\tilde{e}_h^{\ T}(t-\tau)Q\tilde{e}_h(t-\tau)$$

$$\le -\alpha\sum_{i=1}^{N}\xi_ie_i^T(t)e_i(t)+\sum_{h=1}^{n}p_h\tilde{e}_h^{\ T}(t)(\delta_h\Xi)\tilde{e}_h(t)+\sum_{h=1}^{n}p_h\tilde{e}_h^{\ T}(t)(c_1\gamma_1^h\Xi\overline{A})^s\tilde{e}_h(t)$$

$$-\sum_{h=1}^{n}p_h\tilde{e}_h^{\ T}(t)(c_2\gamma_2^h\Xi B)\tilde{e}_h(t-\tau)+\sum_{h=1}^{n}p_h\tilde{e}_h^{\ T}(t)Q\tilde{e}_h(t)-\sum_{h=1}^{n}p_h\tilde{e}_h^{\ T}(t-\tau)Q\tilde{e}_h(t-\tau)$$

$$=-\alpha\sum_{i=1}^{N}\xi_ie_i^T(t)e_i(t)+\sum_{h=1}^{n}p_h[\tilde{e}_h(t),\tilde{e}_h^T(t-\tau)]Z_2\begin{pmatrix}\tilde{e}_h(t)\\ \tilde{e}_h(t-\tau)\end{pmatrix}$$

where $Z_2=\begin{pmatrix}\delta_h\Xi+(c_1\gamma_1^h\Xi\overline{A})^s+Q & \dfrac{c_2\gamma_2^h\Xi B}{2}\\ \dfrac{c_2\gamma_2^hB^T\Xi}{2} & -Q\end{pmatrix}.$

From Lemma 4, we know if $\delta_h\Xi+(c_1\gamma_1^h\Xi\overline{A})^s+Q+\dfrac{(c_2\gamma_2^h)^2\Xi BQ^{-1}B^T\Xi}{4}<0,$ then $Z_2<0.$

Thus, we obtain

$$\dot{V}_2\le -\alpha\sum_{i=1}^{N}\xi_ie_i^T(t)e_i(t) \tag{10}$$

Then, we have

$$\dot{V} = \dot{V}_1 + \dot{V}_2 \le -\frac{\alpha}{\max(p_h)} X^T (U \otimes P) X - \alpha \sum_{i=1}^{N} \xi_i e_i^T(t) e_i(t) \tag{11}$$

Therefore, the inner synchronization of the drive network (1) and the response network (2), respectively, and the outer synchronization of the two networks are achieved simultaneously.

4. Numerical simulations

In this section, to verify and demonstrate the effectiveness of the proposed method, we consider numerical examples, that is, the Lü chaotic system as the node dynamic system of the networks.

The Lü chaotic system is described by

$$\dot{x} = \begin{pmatrix} a(x_2 - x_1) \\ bx_2 + x_1 x_3 \\ x_1 x_2 - cx_3 \end{pmatrix}, \tag{12}$$

when the parameters $a = 36, b = 20, c = 3,$ the Lü system is chaotic.

In the numerical simulations, for simplicity, we assume $c_1 = c_2 = 1,\ \tau = 3,\ \Gamma_1 = diag(5,5,5),$ $\Gamma_2 = diag(0.2, 0.2, 0.2)$, $l = 1, k_1 = 15, k_i = 0,\ \ i = 2,..., N.$ Consider a complex network with nontime-delay and time delayed coupling consisting of the Lü chaotic system 6 nodes to verify the correctness of theorem 1. Choosing the coupling configuration matrices

$$A = \begin{pmatrix} 5 & 1 & 3 & 1 & 0 & 0 \\ 0 & -8 & 0 & 0 & 6 & 2 \\ 0 & 0 & -4 & 2 & 0 & 2 \\ 9 & 0 & 1 & -10 & 0 & 0 \\ 0 & 0 & 0 & 5 & -5 & 0 \\ 0 & 3 & 0 & 2 & 0 & -5 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 5 & -5 \end{pmatrix} \tag{13}$$

The left eigenvector of A associated with eigenvalue 0 is $\xi = (0.2785, 0.0847, 0.2476, 0.1547, 0.1016,$ $0.1329)^T$, $\Xi = diag(0.2785, 0.0847, 0.2476, 0.1547, 0.1016, 0.1329),$ $U = \Xi - \xi\xi^T$. Using the LMI Toolbox in MATLAB, we obtain

$$M = \begin{pmatrix} 0.3209 & -1.0138 & -3.5088 & -6.1719 & 0.0228 & 0.0361 \\ * & 5.0130 & 0.0229 & 0.0075 & -1.8895 & -2.1125 \\ * & * & 7.3480 & -2.4001 & 0.0261 & -1.8224 \\ * & * & * & 11.4758 & -1.8788 & -0.9676 \\ * & * & * & * & 3.7415 & 0.0119 \\ * & * & * & * & * & 4.9074 \end{pmatrix}$$

and

$$Q = \begin{pmatrix} 4.2175 & -0.8400 & -2.5697 & -4.9207 & -0.0557 & -0.0544 \\ * & 4.8104 & -0.0153 & -0.0339 & -1.8060 & -2.0172 \\ * & * & 6.9288 & -2.3631 & -0.0019 & -1.7546 \\ * & * & * & 10.7332 & -1.7854 & -0.9347 \\ * & * & * & * & 3.6016 & -0.0060 \\ * & * & * & * & * & 4.7013 \end{pmatrix} \tag{14}$$

Then, the conditions (6) and (7) of Theorem 1 are satisfied. Therefore, according to Theorem 1, two coupled neworks can achieve the inner and outer synchronization, that is, the inner synchronization inside each network and outer synchronization between them simultaneously. And the simulation results are shown in Figs.1 and 2. Fig. 1 (a)-(c) shows the state variables of the drive-response networks. Fig. 1 (d) exhibits the inner synchronization error of the drive network and that of the response network, respectively. The outer synchronization errors are shown respectively in Fig.2. The numerical results show that the scheme for the drive-response complex network is effective in the Theorem 1.

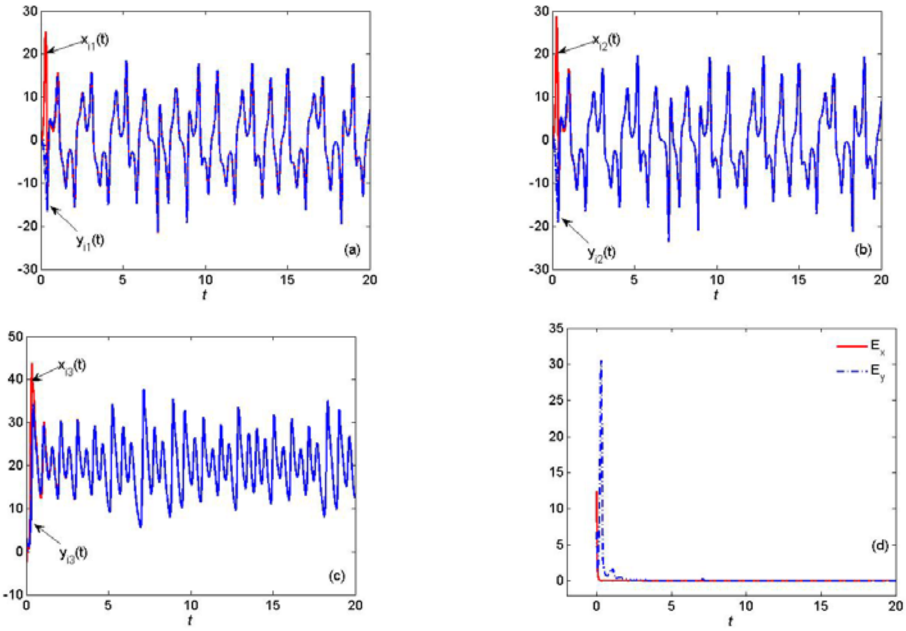

Fig.1. The state of the variables (a) x_{i1} (red line) and y_{i1} (blue dot line) (b) x_{i2} (red line) and y_{i2} (blue dot line) (c) x_{i3} (red line) and y_{i3} (blue dot line); (d) the inner synchronization error of the drive network E_x and that of the response network E_y, respectively.

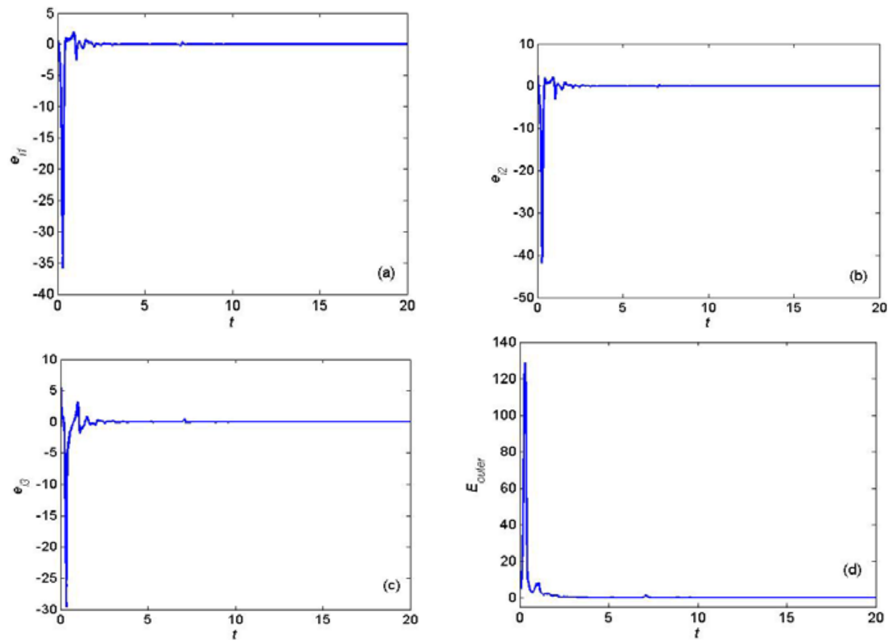

Fig.2. Outer synchronization errors of the drive- response coupled networks: $(a)\, e_{i1}$ (b) e_{i2} (c) e_{i3} (d) E_{outer}

5. Conclusion

In this paper, the pinning controllers have been proposed to study the inner synchronization within each network and outer synchronization between two coupled complex networks with delayed and non-delayed coupling. With the Lyapunov stability theory and adaptive control theory, some criteria are derived. Analytical results show that two networks can realize the synchronization: the outer synchronization between the drive-response networks, and the inner synchronization in the drive network and the response newtwork, respectively. A numerical example has been given to show the usefulness of the theoretical result.

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