(2,1)-Total Labelling of Cactus Graphs

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Abstract. A (2,1)-total labelling of a graph $G = (V, E)$ is an assignment of integers to each vertex and edge such that: (i) any two adjacent vertices of $G$ receive distinct integers, (ii) any two adjacent edges of $G$ receive distinct integers, and (iii) a vertex and its incident edge receive integers that differ by at least 2. The span of a (2,1)-total labelling is the maximum difference between two labels. The minimum span of a (2,1)-total labelling of $G$ is called the (2,1)-total number and denoted by $\lambda_t(G)$.

A cactus graph is a connected graph in which every block is either an edge or a cycle. In this paper, we label the vertices and edges of a cactus graph by (2,1)-total labelling and have shown that, for a cactus graph, where $2 \leq \Delta \leq \Delta + 2$ for a cactus graph, where $\Delta$ is the degree of the graph $G$.

Keywords: Graph labelling; (2,1)-total labelling; cactus graph

1. Introduction

Motivated by frequency channel assignment problem Griggs and Yeh [5] introduced the $L(2,1)$-labelling of graphs. The notation was subsequently generalized to the $L(p,q)$-labelling problem of graphs. Let $p$ and $q$ be two non-negative integers. An $L(p,q)$-labelling of a graph $G$ is a function $c$ from its vertex set $V(G)$ to the set $\{0,1,\ldots,k\}$ such that $|c(x) - c(y)| \geq p$ if $x$ and $y$ are adjacent and $|c(x) - c(y)| \geq q$ if $x$ and $y$ are at distance 2. The $L(p,q)$-labelling number $\lambda_{p,q}(G)$ of $G$ is the smallest $k$ such that $G$ has an $L(p,q)$-labelling $c$ with $\max\{c(v) \mid v \in V(G)\} = k$.

The $L(p,q)$-labelling of graphs has been studied rather extensively in recent years [2, 8, 12, 16, 17, 18]. Whittlesey at el. [19] investigated the $L(2,1)$-labelling of incidence graphs. The incidence graph of a graph $G$ is the graph obtained from $G$ by replacing each edge by a path of length 2. The $L(2,1)$-labelling of the incident graph $G$ is equivalent to each element of $V(G) \cup E(G)$ such that:

(i) any two adjacent vertices of $G$ receive distinct integers,
(ii) any two adjacent edges of $G$ receive distinct integers, and
(iii) a vertex and an edge incident receive integers that differ by at least 2.

This labelling is called $(2,1)$-total labelling of graphs which introduced by Havet and Yu [6] and generalized to the $(d,1)$-total labelling, where $d \geq 1$ be an integer. A $k$-$(d,1)$-total labelling of a graph $G$ is a function $c$ from $V(G) \cup E(G)$ to the set $\{0,1,\ldots,k\}$ such that $c(u) \neq c(v)$ if $u$ and $v$ are adjacent and $|c(u) - c(e)| \geq d$ if a vertex $u$ is incident to an edge $e$. The $(d,1)$-total number, denoted by $\lambda_{d}(G)$, is the least integer $k$ such that $G$ has a $k$-$(d,1)$-total labelling. When $d = 1$, the $(1,1)$-total labelling is well known as total colouring of graphs.
Let $\Delta(G)$ (or simply $\Delta$) denote the maximum degree of a graph $G$.

Havet and Yu [6] proposed the following conjecture.

**Conjecture 1** $\lambda_d^t(G) \leq \min\{\Delta + 2d - 1, 2\Delta + d - 1\}$.

### 2. Some general bounds of $(d,1)$-total labelling

It is shown in [6] that for any graph $G$,

(i) $\lambda_d^t(G) \leq 2\Delta + d - 1$;

(ii) $\lambda_d^t(G) \leq 2\Delta - 2\log(\Delta + 2) + 2\log(16d - 8) + d - 1$; and

(iii) $\lambda_d^t(G) \leq 2\Delta - 1$ if $\Delta \geq 5$ is odd.

Again in [6] it was shown that

(i) $\lambda_d^t(G) \geq \Delta + d - 1$;

(ii) $\lambda_d^t(G) \geq \Delta + d$ if $G$ is $\Delta$-regular;

(iii) $\lambda_d^t(G) \geq \Delta + d$ if $d \geq \Delta$; and

(iv) $\lambda_d^t(G) \leq \chi(G) + \chi'(G) + d - 2$, where $\chi(G)$ and $\chi'(G)$ are known as chromatic number and chromatic index of $G$ respectively.

Let $\text{Mad}(G)$ is the maximum average degree of $G$, $\text{Mad}(G) = \max\{|E(H)|/|V(G)|, H \subseteq G\}$.

Montassier and Raspaud [15] proved that if $G$ be a connected graph with maximum degree $\Delta$, $d \geq 2$, then $\lambda_d^t(G) \geq \Delta - 2d - 2$ in the following cases:

(i) $\Delta \geq 2d + 1$ and $\text{Mad}(G) \leq \frac{5}{2}$;

(ii) $\Delta \geq 2d + 2$ and $\text{Mad}(G) \leq 3$;

(iii) $\Delta \geq 2d + 3$ and $\text{Mad}(G) \leq \frac{10}{3}$.

For a complete graph $K_n$, the result for $(d,1)$-total labelling is given in [6]. If $n$ is odd then $\lambda_d^t(K_n) = \min\{n + 2d - 2, 2n + d - 2\}$; if $n$ is even then $\lambda_d^t(K_n) = \min\{n + 2d - 2, 2n + d - 2\}$, $n \leq d + 5$, $\lambda_d^t(K_n) = n + 2d - 1$, $n > 6d^2 - 10d + 4$ and $\lambda_d^t(K_n) \in \{n + 2d - 2, 2n + d - 1\}$ otherwise. Then they focused in $(2,1)$-total labelling and shown that if $\Delta \geq 2$, then $\lambda_d^t(K_n) \leq 2\Delta + 2$ and therefore the $(d,1)$-total labelling conjecture is true when $p = 2$ and $\Delta = 3$. In fact, the bound for this special case is tight as $\lambda_d^t(K_3) = 7$ [6].

In [13], Molloy and Reed proved that the total chromatic number of any graph with maximum degree $\Delta$ is at most $\Delta$ plus an absolute constant. Moreover, in [14], they gave a similar proof of this result for sparse graphs.

In [7], it was shown that for any tree $T$, $\Delta + 1 \leq \lambda_d^t(T) \leq \Delta + 2$, where $\Delta$ is the maximum degree among all the vertices of the tree.

The $(d,1)$-total labelling for a few special graphs have been studied in literature, e.g., complete graphs [6], complete bipartite graphs [11], planar graphs [1], outer planar graphs [3], products of graphs [4], graphs with a given maximum average degree [15], etc. A more generalization of total colouring of graphs so called $[r,s,t]$-colouring, was defined and investigated in [9].

It is shown in [10] that for any cactus graphs, $\Delta + 1 \leq \lambda_{2,1} \leq \Delta + 3$. Now in this paper, we label the vertices and edges of a cactus graphs $G$ by $(2,1)$-total labelling and it is shown that $\Delta + 1 \leq \lambda_d^t \leq \Delta + 2$. 

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Lemma 1 [6] If $H$ is a subgraph of $G$, then $\lambda_2^1(H) \leq \lambda_2^1(G)$.

3. The (2,1)-total labelling of induce sub-graphs of cactus graphs

Let $G = (V, E)$ be a given graph and $U$ is a subset of $V$. The induced subgraph by $U$, denoted by $G[U]$, is the graph given by $G[U] = (U, E')$, where $E' = \{(u, v) : u, v \in U \text{ and } (u, v) \in E\}$.

![Figure 1: Some induce subgraphs of cactus graph.](image)

The star graph $K_{1,\Delta}$ is a subgraph of $K_{n,m}$. For any star graph $K_{1,\Delta}$ one can verify the following result.

Lemma 2 For any star graph $K_{1,\Delta}$, $\lambda_2^1(K_{1,\Delta}) = \Delta + 2$.

3.1. (2,1)-total labelling of cycles

3.1.1 (2,1)-total labelling of one cycle

Lemma 3 For any cycle $C_n$ of length $n$, $\lambda_2^1(C_n) = 4 = \Delta + 2$.

Proof. Let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of the cycle $C_n$. We classify $C_n$ into two groups, viz., $C_{2k}$, $C_{2k+1}$. Then the (2,1)-total labelling of vertices and edges of the cycle are as follows.

Case 1. Let $n = 2k$ (see Figure 2(a)).

$$c(v_0) = 0, \quad c(v_{2i+1}) = 1, \quad c(v_{2i}, v_{2i+1}) = 3, \quad \text{for} \quad i = 0, 1, 2, \ldots, k-1; \quad c(v_{2i+1}, v_{2i+2}) = 4, \quad \text{for} \quad i = 0, 1, 2, \ldots, k-2 \text{ and } c(v_{2k-1}, v_0) = 4.$$  

![Figure 2: Illustration of Lemma 3](image)

Case 2. Let $n = 3$ (see Figure 2(b)).

$$c(v_0) = 0, \quad c(v_1) = 2, \quad c(v_2) = 4, \quad c(v_0, v_1) = 4, \quad c(v_1, v_2) = 0 \text{ and } c(v_2, v_0) = 2.$$  

Case 3. Let $n = 2k + 1$ (see Figure 2(c)).

We label the vertices as $c(v_{2i}) = 0$, for $i = 0, 1, 2, \ldots, k-1$; $c(v_{2i+1}) = 1$, for $i = 0, 1, 2, \ldots, k-2$; $c(v_{2k+1}) = 2$ and $c(v_{2k}) = 4$. And we label the edges as

$$c(v_{2i-1}, v_2i) = 3, \quad c(v_{2i}, v_{2i+1}) = 4, \quad \text{for} \quad i = 0, 1, 2, \ldots, k-1; \quad c(v_{2k-1}, v_{2k}) = 0 \text{ and } c(v_{2k}, v_0) = 2.$$  

From all above cases, we conclude that, $\lambda_2^1(C_n) = 4 = \Delta + 2$. □
3.1.2 (2,1)-total labelling of two cycles

Lemma 4 If a graph $G = (C_n \cup C_m)$ contains two cycles having a common cutvertex with degree 4, then,

$$\lambda'_2(G) = \begin{cases} 
6, & \text{when length of each cycle is even;} \\
5, & \text{otherwise.}
\end{cases}$$

Proof. Let $G$ contains two cycles $C_n$ and $C_m$ of lengths $n$ and $m$ respectively. Again let $v_0$ be the cutvertex and $v_0, v_1, \ldots, v_{n-1}$ and $v_0', v_1', \ldots, v_{m-1}'$ be the vertices of $C_n$ and $C_m$ respectively. Now we label the vertices and edges of the graph as follows.

**Case 1.** For $n = 3, m = 3$ (shown in Figure 3(a)).

At first we label the cutvertex $v_0$ by 0. Then we label the vertices and edges of first $C_3$ (i.e., $C_n$) as same as given in case 2 of previous lemma. And then we label other vertices and edges as $c(v_i') = 1$, $c(v_1') = 2$, $c(v_0, v_1) = 3$, $c(v_1', v_2') = 4$ and $c(v_2', v_0) = 5$.

**Case 2.** For $n = 3, m = 2k + i$, $i = 0, 1$.

We label the edges and vertices of $C_3$ as same as in the above case. Then we label the second cycle as follows.

When $m$ is even, i.e., $m = 2k$ (shown in Figure 3(b)), then

$$c(v_{2i}') = 0, \quad c(v_{2i+1}') = 1, \quad c(v_{2i}', v_{2i+1}') = 3, \quad \text{for } i = 0, 1, 2, \ldots, k - 1; \quad c(v_{2i+1}', v_{2i+2}') = 4, \quad \text{for } i = 0, 1, 2, \ldots, k - 2; \quad c(v_{2k}', v_0') = 3 \quad \text{and} \quad c(v_0, v_1') = 5.$$  

When $m$ is odd, i.e., $m = 2k + 1$ (shown in Figure 3(c)), then

we label the vertices $v_i', i = 1, 2, \ldots, 2k - 1$ and the edges $(v_i', v_{i+1}')$, $i = 1, 2, \ldots, 2k - 2$, $(v_0, v_1')$, $(v_0, v_{2k}')$ as same as in the above except the label of the vertex $v_{2k}'$ and the edge $(v_{2k-1}', v_{2k}')$. We label that vertex and that edge as $c(v_{2k}') = 2$ and $c(v_{2k-1}', v_{2k}') = 4$.

**Case 3.** For $n = 2k + i, m = 2k + i$, $i = 0, 1$.

When $n = 2k$ (even), $m = 2k$ (even) (shown in Figure 3(d)), then we label the vertices and edges of $C_n$ as same as in case 1 of Lemma 3. Now we label all the vertices of the cycle $C_n$ as the labelling of the vertices of the cycle $C_n$. Now we label the edges of $C_m$ as follows.

\[ \text{JIC email for contribution: editor@jic.org.uk} \]
\[ c(v_0, v'_1) = 5, \quad c(v'_{2k-1}, v_0) = 6 \quad \text{and} \quad c(v'_i, v'_{i+1}) = 3, \quad \text{for} \quad i = 0, 1, 2, \ldots, k-1, \]
\[ c(v''_{2i}, v''_{2i+1}) = 4, \quad \text{for} \quad i = 0, 1, 2, \ldots, k-2. \]

When \( n = 2k + 1 \) (odd), \( m = 2k \) (even) (shown in Figure 3(e)), then we label the vertices and edges of \( C_n \) as same as in case 3 of previous lemma. Then we label another cycle as same as in the above subcase except the label of the edges \( (v_0, v'_1) \) and \( (v'_{2k-1}, v_0) \) and we label those edges as
\[ c(v_0, v'_1) = 3 \quad \text{and} \quad c(v'_{2k-1}, v_0) = 5. \]

When \( n = 2k + 1 \) (odd), \( m = 2k + 1 \) (odd) (Figure 3(f)), then the labelling procedure of the \( C_n \) as same as given in case 3 of Lemma 3. And then we label the cycle \( C_m \) as same as given in case 2 (for \( n = 3, m = 2k + 1 \)).

Here the degree of the cutvertex \( v_0 \) is 4. Then from all the above cases, it follows that
\[
\lambda_2^*(G) = \begin{cases} 
6, & \text{both cycles are of even length}; \\
5, & \text{otherwise}. 
\end{cases}
\]

### 3.1.3 (2,1)-total labelling of three cycles

**Lemma 5** Let \( G \) be a graph contains three cycles and they have a common cutvertex \( v_0 \) with degree \( \Delta = 6 \), then
\[
\lambda_2^*(G) = \begin{cases} 
\Delta + 2, & \text{when three cycles are of even lengths}; \\
\Delta + 1, & \text{otherwise}. 
\end{cases}
\]

**Proof.** Let \( C_n, C_m \) and \( C_l \) be three cycles and \( v_0, v_1, \ldots, v_{n-1}; v_0, v'_1, \ldots, v'_{m-1}; v_0, v''_1, \ldots, v''_{l-1} \) be the vertices of them. They joined with a common cutvertex \( v_0 \) with degree \( \Delta (= 6) \). The labelling procedure of two cycles are given in previous lemma. Now according to the previous lemma we have to label the vertices and edges of the remaining cycle \( C_l \). When we label \( C_l \), there are three cases arise, viz., \( l = 3, l = 2k \) (even) and \( l = 2k + 1 \) (odd). Here the label of the cutvertex is 0. Then we label the third cycle as follows.

![Figure 4: Illustration of some cases of Lemma 5](image)

![Figure 3: (continuation)](image)
Case 1. When \( l = 3 \), then we relabel \((v_0, v'_0), (v''_1, v'_1), (v''_2, v'_2)\) and \((v''_3, v'_3)\) by 6, 1, 4, 2 and 7 respectively.

Case 2. When \( l = 2k \) (even), then we label the vertices of \( C_{2k} \) as
\[
\begin{align*}
c(v''_{2i}) &= 0, \quad \text{for } i = 1, 2, \ldots, k - 1; \\
c(v''_{2i+1}) &= 1, \quad \text{for } i = 0, 1, \ldots, k - 2; \\
\text{and } c(v''_{2k}) &= 2.
\end{align*}
\]
And the edges as
\[
\begin{align*}
c(v''_{2i}, v''_{2i+1}) &= 3, \quad \text{for } i = 1, 2, \ldots, k - 1; \\
c(v''_{2i+1}v''_{2i+2}) &= 4, \quad \text{for } i = 0, 1, \ldots, k - 2; \\
c(v'_0, v'_1) &= 6 \text{ and } c(v''_{2k-1}, v_0) = 7.
\end{align*}
\]
If the cycle \( C_j \) attach with two cycles of even lengths then the label of two edges incident on \( v_0 \) of \( C_j \) are different. And the labels are
\[
c(v_0, v'_0) = 7 \text{ and } c(v''_{2k-1}, v_0) = 8 \text{ respectively.}
\]
Case 3. When \( l = 2k + 1 \) (odd), then the labels of the vertices and edges of \( C_l \) are same as the labelling of the cycle \( C_m \) given in case 2 (for \( n = 3 \) and \( m = 2k + 1 \)) of lemma 4 except the labels of two edges \((v_0, v'_1)\) and \((v''_{2k-1}, v_0)\). And we relabel these two edges as
\[
c(v'_0, v'_1) = 6 \text{ and } c(v''_{2k-1}, v_0) = 7 \text{ respectively.}
\]
Here we see that the values of \( \lambda^2_c \) are 7 and 8.

Therefore we conclude that,
\[
\lambda^2_c(G) = \begin{cases} 
\Delta + 2, & \text{when three cycles are of even lengths;} \\
\Delta + 1, & \text{otherwise.}
\end{cases}
\]

3.1.4 \((2,1)\)-total labelling of finite number of cycles

We can extend the lemmas 4 and Lemma 5 for the finite number of cycles when they are joined at a common cutvertex.

Lemma 6 If a graph \( G \) contains finite number of cycles of finite lengths and if they are joined with a common cutvertex with degree \( \Delta \), then,
\[
\lambda^2_c(G) = \begin{cases} 
\Delta + 2, & \text{when all cycles are of even lengths;} \\
\Delta + 1, & \text{otherwise.}
\end{cases}
\]

Proof. Let us consider a graph \( G \) contains \( n \) number of cycles of length 3 (triangles). The \( n \) triangles joined with a common cutvertex say \( v_0 \) with degree \( \Delta = 2n \), then we have to prove that \( \lambda^2_c(G) = \Delta + 1 \). Let \( T_0, T_1, \ldots, T_{n-1} \) be the \( n \) number of triangles and \( v_0 \) be the cutvertex (see Figure 5). Then \( G \) is equivalent to \( \bigcup_{j} T_i \). Again let \( v_{ij}, \quad i = 1, 2 \) and \( j = 0, 1, \ldots, n - 1 \), be the vertices of \( G \). We label the vertices \( v_{ij}, v_{j+1} \) and \( (v_{ij}, v_{j+1}) \), for \( j = 1, 2, \ldots, n - 1 \), using the same procedure of labelling of \( v'_1, v'_2 \) and the edge \((v'_1, v'_2)\) of \( C_3 \) in case 1 of Lemma 3. Then we label the remaining two edges as
\[
c(v'_0, v'_1) = \begin{cases} 
2j + 2, & \text{if } i = 1; \\
2j + 3, & \text{if } i = 2, \quad \text{for } j = 0, 1, \ldots, n - 1.
\end{cases}
\]
Then the \((2,1)\)-total number of \( G \) is \( 2n + 1 \) which is exactly equal to \( \Delta + 1 \).

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Now we consider the graph $G$ which contains $n$ number of cycles of length 3 and $m$ number of cycles of length 4. They joined with a cutvertex with degree $\Delta = 2(n + m)$. Then the $\lambda'_2$-value for that graph is $\Delta + 1$.

Let $T_0, T_1, \ldots, T_{n-1}$ be the $n$ number of cycles of length 3 and $R_0, R_1, \ldots, R_{m-1}$ be the $n$ number of cycles of length 4 (shown in Figure 6). They are joined with a common cutvertex say $v_0$. Let $v'_i$, $i = 1, 2$ and $j = 0, 1, \ldots, n-1$, be the vertices of all $T_i$'s and $v_0$, $v'_k$, $k = 1, 2$ and $p = 0, 1, \ldots, m-1$, be the vertices of all $R_p$'s. Now the labelling of vertices of all $R_p$'s are same as the labelling of vertices of even number of cycles. Then we label the edges as follows:

$c(v'_{i,p}, v'_{2,p}) = 4$, $c(v'_{2,p}, v'_{3,p}) = 3$, for $p = 0, 1, \ldots, m-1$ and then we label the edges $(v_0, v'_k)$, for $k = 1, 3$ and $p = 0, 1, \ldots, m-1$ as follows:

$c(v_0, v'_k) = \begin{cases} 2n + 2(p + 1), & \text{if } k = 1; \\ 2n + 2(p + 1) + 1, & \text{if } k = 2, \text{ for } p = 0, 1, \ldots, m-1. \end{cases}$

We have $c(v_0, v'_{3,n-1}) = 2n + 2m + 1 = \Delta + 1$.

Lastly we prove that if a graph contains $n$ number of cycles of length 4 and all the cycles joined with a cutvertex then the value of $\lambda'_2$ is $\Delta + 2$.

Let us denote the $n$ number of cycles of length 4 by $R_0, R_1, \ldots, R_{n-1}$ (see Figure 7), joined with a
common cutvertex say $v_0$. Again let $v_0, v_j$, $j = 1, 2, 3$ and $i = 0, 1, \ldots, n-1$ be the vertices of $R_i$’s. We label all the vertices of each cycle as same as the label of the vertices of even cycle. And $c(v_i, v_j) = 4$, $c(v_{2i}, v_{3j}) = 3$, for $i = 0, 1, \ldots, n-1$. Then we label the edges which are incident to the cutvertex $v_0$ as

$$c(v_0, v_j) = \begin{cases} 2(i+1) + 1, & \text{if } j = 1; \\ 2(i+1) + 2, & \text{if } j = 2, \text{ for } i = 0, 1, \ldots, n-1. \end{cases}$$

We have $c(v_0, v_{2n-1}) = 2(n-1+1) + 2 = 2n + 2 = \Delta + 2$.

By using the above results, the general form can be proved by mathematical induction. That is, if a graph $G$ contains finite number of cycles of finite lengths, then

$$\lambda_2'(G) = \begin{cases} \Delta + 2, & \text{when all cycles are of even lengths}; \\ \Delta + 1, & \text{otherwise}. \end{cases}$$

Lemma 7 If a graph $G$ contains finite number of cycles of any length and finite number of edges joined with a common cutvertex of degree $\Delta$, then $\lambda_2'(G) = \Delta + 1$.

Proof. At first we prove that if a graph $G$ contains $n$ number of cycles of length 3, $m$ number of cycles of length 4, $p$ number of edges and they are joined with a common cutvertex with degree $\Delta (= 2n + 2m + p)$, then the value of $\lambda_2'(G)$ will be $\Delta + 1$. Let $v''_i$, $i = 0, 1, \ldots, p-1$ be the other end vertices of each edge. We label all $v''_i$’s as $c(v''_i) = 1$, for $i = 0, 1, \ldots, p-1$. Then according to the previous lemma we label the edges $(v_0, v'_i)$, for $i = 0, 1, \ldots, p-1$ as

$$c(v_0, v'_i) = 2n + 2m + p - 1 + 1 = 2(n + m) + p + 1 = \Delta + 1.$$

Again let us consider that the graph $G$ contains $n$ number of cycles of length 4 and $p$ number of edges joined with a cutvertex with degree $\Delta = 2n + p$. Then we have to prove that $\lambda_2'(G) = \Delta + 1$.

Now we label the vertex $v_0$ and the edge $(v_0, v'_0)$ by 4 and 2 respectively. Then according to the previous lemma we label the edges as $c(v_0, v'_j) = 2n + 2 + j$, for $j = 0, 1, \ldots, p-1$.

Then we have $c(v_0, v'_{p-1}) = 2n + 2 + p - 1 = 2n + p + 1 = \Delta + 1$.

By the above results, generally we conclude that if a graph contains finite number of cycles of any length and finite number of edges, then $\lambda_2'(G) = \Delta + 1$.

Lemma 8 Let $G$ be a graph, contains a cycle of any length and finite number of edges and they have a common cutvertex $v_0$. If $\Delta$ be the degree of the cutvertex, then $\lambda_2'(G) = \Delta + 2$, if the cycle is of even length and $\Delta + 1$, otherwise.

Proof. We consider that $G$ contains an cycle $C_n$ of length $n$ and $p$ number of edges. Let $v_0, v_1, \ldots, v_{n-1}$ are the vertices of $C_n$ and $v'_0, v'_1, \ldots, v'_{p-1}$ are the end vertices of all edges, joined with the cutvertex. Let $\Delta$ be the degree of $G$, then $\Delta = 2 + p$. Then we label the vertices and edges of $G$ as follows.

![Figure 8: Illustration of Lemma 8](image-url)

Case 1. Let $n = 2k$ (even).
Here $c(v_0) = 0$, then we label all the endvertices of the edges as $c(v'_i) = 1$, for $i = 0, 1, \ldots, p-1$.
Now we label the edges $(v_0, v'_j)$ as $c(v_0, v'_j) = 5 + j$ for $j = 0, 1, \ldots, p-1$.
Now $c(v_0, v'_{p-1}) = p + 4 = \Delta + 2$.

Case 2. Let $n = 3$ and $n = 2k + 1$ (odd).
Here we label the first edge $(v_0, v'_0)$ by 3. Then the labelling procedure of all endvertices are same as given in the above case. And we label the remaining edges as follows

Now $c(v_0, v'_{p-1}) = 3 + k = \Delta + 1$.

From the above two cases we see that $\lambda^1_n(G) = \Delta + 2$, if the cycle is of even length and $\Delta + 1$, otherwise.

3.2. (2,1)-labelling of sun
Let us consider the sun $S_{2n}$ of $2n$ vertices. This graph is obtained by adding an edge to each vertex of a cycle $C_n$. So $C_n$ is a subgraph of $S_{2n}$. The result for any sun $S_{2n}$ is given below.

Lemma 9 For any sun $S_{2n}$, $\lambda^1_n(S_{2n}) = 5 = \Delta + 2$.

Proof. Let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of $C_n$ and $v_i$ is adjacent to $v_{i+1}$ and $v_{i-1}$. To complete $S_{2n}$, we add an edge $(v_i, v'_i)$ to the vertex $v_i$, i.e., $v'_i$’s are the pendant vertices. To label this graph we consider the following three cases.

Case 1. Let $n = 3$ (shown in Figure 9(a)).
We label the cycle $C_3$ according to the Case 2 of Lemma 3. Then we label other vertices and edges as follows:
$c(v'_0) = 1, c(v'_1) = 5, c(v'_2) = 0, c(v_0, v'_0) = 3, c(v_1, v'_1) = 1$ and $c(v_2, v'_2) = 5$.

Case 2. Let $n = 2k$ (even) (see Figure 9(b)).
We label the cycle $C_n$ as per Case 1 of Lemma 3. And we label other vertices and edges of $S_{2n}$ as follows:
$c(v'_2) = 1, c(v'_{2i+1}) = 0$ for $i = 0, 1, \ldots, k-1$ and $c(v_i, v'_i) = 5$ for $i = 0, 1, \ldots, n - 1$.

Case 3. Let $n = 2k + 1$ (odd) (see Figure 9(c)).
Here the labelling procedure of the cycle $C_{2k+1}$ is same as the Case 3 of Lemma 3. Now the labelling of other vertices and edges are as follows:

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\[ c(v'_2) = 1, \quad c(v'_{2i+1}) = 0 \quad \text{for} \quad i = 0, 1, \ldots, k - 1, \quad c(v'_{n-1}) = 5, \quad c(v_i, v'_i) = 5 \quad \text{for} \quad i = 1, 2, \ldots, n - 1, \]
\[ c(v_0, v'_0) = 3 \quad \text{and} \quad c(v_{n-1}, v'_{n-1}) = 1. \]

Here we see that the (2,1)-total number for that graph is 5.

Hence \( \lambda_2'(S_{2n}) = 5 = \Delta + 2. \]

**Lemma 10** Let \( G \) be a graph obtained from \( S_{2n} \) by adding an edge to each of the pendent vertex of \( S_{2n} \), then

\[ \lambda_2'(S_{2n}) = \Delta + 2 = 5. \]

**Proof.** Follows from Figure 10.

![Figure 10: Illustration of Lemma 10](image)

**Lemma 11** Let a graph \( G \) contains two cycles of any length and they are joined by an edge. If \( \Delta(= 3) \) be the degree of \( G \), then,

\[ \lambda_2'(G) = 5 = \Delta + 2. \]

**Proof.** Let the graph \( G \) contains two cycles \( C_n \) and \( C_m \) with vertices \( v_0, v_1, \ldots, v_{n-1} \) and \( v'_0, v'_1, \ldots, v'_{m-1} \) respectively. And the cycles are joined by an edge \( (v_0, v'_0) \). The degree of the graph is \( \Delta(= 3) \) Now we label the vertices and edges of the graph as follows.

![Figure 11: The graph G](image)

**Case 1.** Let \( n = 3, \ m = 3 \).

First we label the vertices and edges of \( C_3 \) as same as given in case 2 of Lemma 3. Now we label the edge \( (v_0, v'_0) \) by 3 and then we label the other cycles as follows.

\[ c(v'_0) = 1, \quad c(v'_1) = 0, \quad c(v'_2) = 2, \]
\[ c(v'_0, v'_0) = 4, \quad c(v'_1, v'_1) = 3, \quad c(v'_2, v'_2) = 5. \]

**Case 2.** Let \( n = 3, \ m = 2k + i, \ i = 0, 1, \)

We label the vertices and edges of \( C_3 \) as same as given in the above case. Then we label the edge

\[ JIC \text{email for contribution: editor@jic.org.uk} \]
When \( m \) is even, i.e., \( m = 2k \), then
\[
c'(v'_1) = 1, \quad c'(v'_{2i+1}) = 0, \quad \text{for} \quad i = 0, 1, \ldots, k - 1,
\]
\[
c'(v'_{2i}, v'_{2i+1}) = 3, \quad \text{for} \quad i = 0, 1, \ldots, k - 1,
\]
\[
c'(v'_{2i+1}, v'_{2i+2}) = 4, \quad \text{for} \quad i = 0, 1, \ldots, k - 2
\]
and \( c'(v'_{n-1}, v'_n) = 4 \).

When \( m \) is odd, i.e., \( m = 2k + 1 \), then we label the vertices and edges of \( C_n \) as same as given in the above subcase except the label of the vertex \( v'_{m-1} \), i.e., \( v'_{2k} \) and the edge \((v'_{m-2}, v'_{m-1})\), i.e., \((v'_{2k-1}, v'_{2k})\). We label the vertex and the edge as follows.
\[
c'(v'_{2k}) = 2 \quad \text{and} \quad c'(v'_{2k-1}, v'_{2k}) = 5.
\]

**Case 3.** Let \( n = 2k + i \), \( m = 2k + i \), \( i = 0, 1 \).

When \( n = 2k \) and \( m = 2k \), then we label the cycle \( C_n \) as same as given in Case 1 of Lemma 3. Then we label the edges \((v_0, v'_0)\) by 5 and the cycle \( C_m \) as same as in the subcase (when \( m \) is even) in Case 2 of this lemma.

When \( n = 2k \) and \( m = 2k + 1 \), then we label the edges and vertices of \( C_m \) as same as given in the subcase (when \( m \) is odd) of the above case.

When \( n = 2k + 1 \) and \( m = 2k + 1 \), then we label the vertices and edges of \( C_n \) as same as given in Case 2.

Finally, we get \( \lambda'_2(G) = 5 = \Delta + 2 \). \( \square \)

**Corollary 1** Let a graph \( G \) contains two cycles of any lengths and they are joined by two edges. If \( \Delta \) be the degree of the graph \( G \), then
\[
\lambda'_2(G) = \Delta + 2.
\]

**Lemma 12** Let a graph \( G \) contains a cycle of any length and each vertex of the cycle contain another cycle of any length, then
\[
\lambda'_2(G) = 6 = \Delta + 2.
\]

**Proof.** At first we take the main cycle are of two types, viz., \( C_{2k} \), i.e., even and \( C_{2k+1} \), i.e., odd. Let \( v_0, v_1, \ldots, v_{n-1} \) be the vertices of \( C_n \).
Case 1. Let $n = 2k$ (even).

When each vertex of $C_n$ contains the cycles of length 3 (shown in Figure 12(a)).

Let $v_0, v'_0, v''_0; v_1, v'_1, v''_1; \ldots; v_{n-1}, v'_{n-1}, v''_{n-1}$ are the vertices of the cycles of length 3. Now the labelling of the cycle $C_n$ is same as the labelling procedure of the cycle of even length. Then we label the other vertices and edges as follows:

$$
\begin{align*}
  c(v'_0) = 1, \quad c(v''_0) &= 0 \quad \text{for } i = 0, 1, \ldots, k - 1 \quad \text{and} \quad c(v''_i) = 2 \quad \text{for } i = 0, 1, \ldots, n - 1. \\
  c(v'_i, v''_i) = 5, \quad c(v''_i, v''_i) &= 6 \quad \text{for } i = 0, 1, \ldots, n - 1.
\end{align*}
$$

When each vertex of $C_n$ contains the cycles of length 4 (see Figure 12(b)).

Let $v_0, v'_0, v''_0, v'''_0; v_1, v'_1, v''_1, v'''_1; \ldots; v_{n-1}, v'_{n-1}, v''_{n-1}, v'''_{n-1}$ be the vertices of all the cycles of length 4. We label the cycles as follows:

$$
\begin{align*}
  c(v'_0) = 1, \quad c(v''_0) &= 0 \quad \text{and} \quad c(v'''_0) = 1 \quad \text{for } i = 0, 1, \ldots, k - 1; \\
  c(v'_i, v''_i) = 0, \quad c(v''_i) &= 1 \quad \text{and} \quad c(v'''_i) = 0 \quad \text{for } i = 0, 1, \ldots, k - 1; \\
  c(v''_i, v''_i) = 5, \quad c(v''_i, v''_i) &= 4 \quad \text{for } i = 0, 1, \ldots, n - 1.
\end{align*}
$$

Case 2. Let $n = 2k + 1$ (odd).

When $n = 3$ and all cycles are of length 3 (see Figure 12(c)).

The labelling procedure of the cycle $C_n$ is same as given in case 2 of Lemma 3. Now we label the other vertices and edges as follows:

$$
\begin{align*}
  c(v'_0) = 1, \quad c(v''_0) &= 2, \quad c(v''_0, v'_0) = 3, \quad c(v''_0, v''_0) = 4, \quad c(v''_0, v''_0) = 5; \\
  c(v'_i) = 0, \quad c(v''_i) &= 1, \quad c(v'_i, v''_i) = 5, \quad c(v''_i, v''_i) = 4, \quad c(v''_i, v''_i) = 6; \\
  c(v''_i, v''_i) = 3, \quad c(v''_i, v''_i) &= 2, \quad c(v''_i, v''_i) = 1, \quad c(v''_i, v''_i) = 0, \quad c(v''_i, v''_i) = 6.
\end{align*}
$$

When each vertex of $C_n$ contains the cycles of length 3 (shown in Figure 12(d)).

The labelling procedure for the vertices $v'_i, v''_i$ and the edges $(v'_i, v'_i), (v''_i, v''_i), (v''_i, v''_i)$ for $i = 1, 2, \ldots, 2k - 2$ are same as the labelling of the graph which contains a cycle of even length and each vertex of the cycle contain cycles of length 3 given in case 1. And the labelling of $v'_i, v''_i, (v'_i, v'_i), (v''_i, v''_i), (v''_i, v''_i)$ for $i = 0, 2k - 2, 2k$ as same as the labelling of the above graph for $i = 0, 1, 2$ respectively.

When all the cycles are of length 4 except the main cycle (shown in Figure 12(e)).
We label the vertices and edges \( v_i', v_i'', v_i''', (v_i, v_i') \), \((v_i', v_i'')\), \((v_i'', v_i''')\) and \((v_i''', v_i)\) for \(i = 1, 2, \ldots, 2k - 1\) as same as the labelling procedure of the graph which contains a cycle of even length and each vertex contains another cycle of length 4 except the label of the vertex \(v_{2k-1}\). We label this vertex as \(c(v_{2k-1}) = 2\). For \(i = 0, 2k\), we label the remaining vertices and edges of the graph as follows:

\[
\begin{align*}
&c(v_0') = 1, c(v_0'') = 0, c(v_0''') = 1, c(v_0', v_0'') = 3, c(v_0'', v_0'') = 4, c(v_0''', v_0) = 5; \\
&c(v_{2k}') = 3, c(v_{2k}'') = 2, c(v_{2k}''') = 1, c(v_{2k}', v_{2k}'') = 1, c(v_{2k}''', v_{2k}''') = 0, c(v_{2k}'''', v_{2k}') = 4, c(v_{2k}''''', v_{2k}') = 6.
\end{align*}
\]

Here we see that the minimum label number is 6 which is exactly equal to \(\Delta + 2\).

Finally, we conclude that if a graph \(G\) contains a cycle of any length and each vertex of the cycle contains another cycle of any length then,

\[
\lambda_2(G) = \Delta + 2.
\]

An edge is nothing but \(P_2\), so \(\lambda_2(G) = 3\).

3.3. \((2,1)\)-labelling of paths

Lemma 13 For any path \(P_n\) of length \(n\),

\[
\lambda_2^2(P_n) = 4 = \Delta + 2.
\]

Proof. Let \(v_0, v_1, \ldots, v_{n-2}, v_{n-1}\) be the vertices of the path \(P_n\) of length \(n\) (shown in Figure 13). We classify the path into two cases, viz., even and odd.

![Figure 13: (2,1)-total labelling of path \(P_n\)](image)

Case 1. When \(n = 2k\), i.e., the path is even.

We label the vertices and edges of \(P_n\) according to the following rules.

\[
\begin{align*}
&c(v_i) = 0, \text{ for } i = 0, 1, \ldots, k - 1; \\
&c(v_{2i+1}) = 1, \text{ for } i = 0, 1, \ldots, k - 1; \\
&c(v_{2i}, v_{2i+2}) = 3, \text{ for } i = 0, 1, \ldots, k - 1; \\
&\text{and } c(v_{2i+1}, v_{2i+2}) = 4, \text{ for } i = 0, 1, \ldots, k - 1.
\end{align*}
\]

Case 2. When \(n = 2k + 1\), i.e., the path is odd.

The labelling of the vertices and edges of the path is same as in the above case, only the label of the last vertex \(v_{2k}\) and last edge \((v_{2k-1}, v_{2k})\) are different. We label that vertex and edge as follows:

\[
\begin{align*}
&c(v_{2k}) = 1 \text{ and } c(v_{2k-1}, v_{2k}) = 3.
\end{align*}
\]

From all above cases we see that \(\lambda_2^2(G) = 4 = \Delta + 2\).

3.4. \((2,1)\)-total labelling of caterpillar graph

Now, we label another important subclass of cactus graphs called caterpillar graph.

Definition 1 A caterpillar \(C\) is a tree where all vertices of degree \(\geq 3\) lie on a path, called the backbone of \(C\). The hairlength of a caterpillar graph \(C\) is the maximum distance of a non-backbone vertex to the backbone.

Lemma 14 For any caterpillar graph \(G\), \(\lambda_2^2(G) = \Delta + 2\), where \(\Delta\) is the degree of the caterpillar graph.
Proof. Let \( P_n \) be the backbone of length \( n \) of the caterpillar graph \( G \) and \( v_0, v_1, \ldots, v_{n-2}, v_{n-1} \) be the vertices of \( P_n \). We label the vertices and edges of the path by using the previous lemma. Let \( v_k \) be a vertex on the path \( P_n \) with degree \( k \). Then \( k-2 \) different paths (other than backbone) are originated from \( v_k \) of variable lengths. We denote such paths by \( P_{ij}^k \), where \( i (= 0, 1, \ldots, k-2) \) represents the \( i \)th path originated from the vertex \( k \) and \( j \) is the length of the path. Let us take the first path \( P_{11}^k \) and \( v_k, v_1^k, v_2^k, \ldots, v_{m-1}^k \) be the vertices of it. We label all the vertices of \( P_{11}^k \) by 0 or 1 and label all the edges adjacent to \( v_k \) by 5, 6, 7, \ldots, \( k+2 \) because the label of the edges incident on the vertex \( v_k \) of the path \( P_n \) are either 3 and 4 respectively. We label the first edge of \( P_{11}^k \) by 5 and other edges of \( P_{11}^k \) by using the labelling procedure given in the previous lemma. All the labels are allowed to label the vertices of the remaining portion of the path \( P_{m}^{k1} \). Now we take the second path \( P_{12}^k \). Here also the labelling procedure for the path is same as given in Lemma 13 except the label of the edge incident on the vertex \( v_k \). We label the edge by 6 and so on. Lastly, we label the first edge of the \( (k-2) \)th path incident on the vertex \( v_k \) by \( k+2 \). Here \( \Delta = k \), so the value of \( \lambda_i^k \) is \( \Delta + 2 \). Similar method apply to all paths joined with the vertices of the path \( P_n \).

\[
\begin{align*}
\lambda_3^7(G) &= 6 = \Delta + 2 \\
\lambda_5^7(G) &= 6 = \Delta + 2 \\
\lambda_7^7(G) &= 10 = \Delta + 2 \\
\lambda_5^7(G) &= 10 = \Delta + 2
\end{align*}
\]

Figure 14: Labelling of caterpillar graphs

Therefore, we conclude that, for any caterpillar graph, \( \lambda_i^k(G) = \Delta + 2 \).

The proof of lemma 14 is illustrated in Figure 14.

4. (2,1)-total labelling of lobster

Another subclass of cactus graphs is the lobster graph. The definition of lobster graph is given below.

Definition 2 A lobster is a tree having a path (of maximum length) from which every vertex has distance at most \( k \), where \( k \) is an integer.

The maximum distance of the vertex from the path is called the diameter of the lobster graph. For the above definition \( k \) is the diameter. There are many types of lobsters given in literature like diameter 2, diameter 4, diameter 5, etc. Figure 16 shows a lobster of diameter 4.

Lemma 15 For any lobster \( G \), \( \lambda_3^k(G) = \Delta + 2 \), where \( \Delta \) is the degree of the lobster.

Proof. Assume that \( P_n \) be a path of length \( n \) of the lobster graph \( G \) and \( v_0, v_1, \ldots, v_{n-1} \) be the vertices of it. Let us consider a vertex \( v_k \) on \( P_n \) from which \( p \) number of trees be originated. Let \( T_1, T_2, \ldots, T_p \) be such
trees. Without lose of generality let the label of the vertex \( v_k \) be 0. Again, let \( \Delta_i \), \( i = 1, 2, \ldots, p \) be the degrees of these trees. We know that \( \lambda_i^1(T_i) \) is \( \Delta_i + 2 \) (if \( \Delta_i \geq 4 \)) [7].

Figure 15: Illustration of Lemma 15

Now we label the edge of the tree \( T_i \) \( i = 1, 2, \ldots, p \) originated from \( v_k \) by \( i + 4 \). Let \( v_{k-1} \) and \( v_{k+1} \) be two adjacent vertices of \( v_k \) on \( P_n \). We label these vertices \( v_{k-1} \) and \( v_{k+1} \) by 1 (or 0) because the label of \( v_k \) can be assigned to 0 (resp. 1). And we label the edges \((v_k, v_{k-1})\) and \((v_k, v_{k+1})\) by 3 and 4 respectively. So we see that there are no extra labels are required to label the edges incident on \( v_k \) of the path \( P_n \). So, the value of \( \lambda_i^1 \) of the lobster is \( \Delta + 2 \), where \( \Delta = \max \{ \Delta_1, \Delta_2, \ldots, \Delta_p \} \).

Figure 16 is an example of 4-diameter lobster and the proof of Lemma 15 is illustrated here.

![Figure 16: (2,1)-total labelling of 4-diameter lobster](image)

**Lemma 16** Let \( G_1 \) and \( G_2 \) be two cactus graphs. If \( \Delta_i + 1 \leq \lambda_i^1(G_i) \leq \Delta_i + 2 \) and \( \Delta_2 + 1 \leq \lambda_2^1(G_2) \leq \Delta_2 + 2 \), then \( \Delta + 1 \leq \lambda_i^1(G) \leq \Delta + 2 \), \( G \) is the union of two graphs \( G_1 \) and \( G_2 \), they have only one common vertex \( v \) and \( \max \{ \Delta_1, \Delta_2 \} \leq \Delta \leq \Delta_1 + \Delta_2 \).

**Proof.** Let \( G_1 \) and \( G_2 \) be two cactus graphs and \( \Delta_1, \Delta_2 \) be the degrees of them. Now if we merge two cactus graphs \( G_1 \) and \( G_2 \) with the vertex \( v \) then we get a new cactus graph \( G \) \( (= G_1 \cup G_2) \). Let \( \Delta \) be the degree of new cactus graph \( G \) and it can be shown that \( \max \{ \Delta_1, \Delta_2 \} \leq \Delta \leq \Delta_1 + \Delta_2 \). For the graph \( G_1 \), \( \Delta_1 + 1 \leq \lambda_i^1(G_1) \leq \Delta_1 + 2 \) and \( G_2, \Delta_2 + 1 \leq \lambda_2^1(G_2) \leq \Delta_2 + 2 \). Now we have to prove that the lower and upper bounds of \( \lambda_i^1 \) will preserve for the new cactus graph \( G \). Let \( u \) and \( v \) be two vertices of that graphs and \( u_0, u_1; v_0, v_1 \) be the adjacent vertices of \( u \) and \( v \) respectively. Let \( x \) be the label of \( u \), then the label of \( u_0 \) and \( u_1 \) may be \( x + 1 \) and \( x + 1 \) or \( x + 4 \). And the label of the edges \((u, u_0)\) and \((u, u_1)\) may be \( x + 3 \)
and \( x+4 \) or \( x+1 \) respectively. Similarly, if \( y \) be the label of \( v \), then the label of \( v_0 \) and \( v_1 \) may be \( y+1 \) and \( y+1 \) or \( y+4 \). And the label of the edges \((v,v_0)\) and \((v,v_1)\) may be \( y+3 \) and \( y+4 \) or \( y+1 \) respectively.

Assume that the label of \( u \) be fixed and let it be 0, i.e., \( x = 0 \), and the label \( y \) of \( v \) lies between 0 to \( \Delta_2 + 2 \). That is, the label difference between \( x \) and \( y \) is one of the integer 0,1,…,\( \Delta_2 + 2 \).

![Figure 17](image1.png)

Let the label of the vertices \( u \) and \( v \) be same, i.e., \( x = y \) (Figure 17). If we join two cactus graphs at \( v \), then the label of \( v \) remains unchanged and the labels of adjacent vertices \( v_0 \) and \( v_1 \) will change to \( x+1 \) and \( x+1 \) or \( x+2 \). And the labels of the edges \((v,v_0)\) and \((v,v_1)\) will change to \( x+5 \) and \( x+6 \) or \( x+4 \) and \( x+5 \). If we increase the label numbers by 1 of all the vertices and edges of \( G_2 \) except \( v \) then there are at least one vertex or edge in which we adjust the labelling to preserve the lower and upper bounds of \( \lambda'_2 \).

When the label difference between \( x \) and \( y \) is 1, i.e., \( y = x+1 \) (see Figure 18), then without loss of generality we assume that the label numbers of adjacent vertices of \( u \) are \( x+1 \) and \( x+1 \) or \( x+4 \). And the label of the edges \((u,u_0)\) and \((u,u_1)\) are \( x+3 \) and \( x+4 \) or \( x+1 \). Now the label numbers of adjacent vertices of \( v \) are \( x \) or \( x+2 \) and \( x \) or \( x+2 \) or \( x+3 \) respectively. And for the edges \((v,v_0)\) and \((v,v_1)\), \( x+3 \) or \( x+4 \) and \( x+4 \) or \( x \) respectively. Now if we increase the label numbers by 1 of all the vertices and edges of \( G_2 \) except \( v \) then we get at least one vertex or edge in which we adjust the labelling to preserve the lower and upper bounds of \( \lambda'_2 \), i.e. the \( \lambda'_2 \)-value of new cactus graph can't be less than \( \Delta+1 \) and greater than \( \Delta+2 \).

![Figure 18](image2.png)

Similarly, for the label differences 2,3,…,\( \Delta_2 + 2 \), the lower and upper bounds of \( \lambda'_2 \) for the new cactus graph will preserve.

\[\square\]
Figure 19: (2,1)-total labelling of cactus graphs

The (2,1)-labelling of all subgraphs of cactus graphs and their combinations are discussed in the previous lemmas. From these results we conclude that the $\lambda_1^t$-value of any cactus graph can not be more than $\Delta + 2$ and less than $\Delta + 1$. Hence we have the following theorem.

**Theorem 1** If $\Delta$ is the degree of a cactus graph $G$, then

$$\Delta + 1 \leq \lambda_1^t(G) \leq \Delta + 2.$$ 

The graph of Figure 19 is an example of a cactus graph, contains all possible subgraphs and its (2,1)-total labelling.

5. **Conclusion**

The bounds of (2,1)-total labelling of a cactus graph and various subclass viz., cycle, sun, star, tree, caterpillar and lobster are investigated. The bounds of $\lambda_1^t(G)$ for these graphs are $\lambda_1^t(C_n) = 4$ and for sun, star, caterpillar and lobster it is $\Delta + 2$. For the cactus graph the bound for $\lambda_1^t$ is $\Delta + 1 \leq \lambda_2^t(G) \leq \Delta + 2$, where $\Delta$ is the maximum degree of the cactus graph $G$.

6. **References**


