

The Necessarily Efficient Point Method for Interval Molp Problems

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(Received September 22, 2008, accepted October 29, 2008)

Abstract. In the most real world situations, an objective function is not satisfied the decision maker's goals and reduce the efficiency of the models. Also the coefficients of decision variables are not exactly known. One way to illustrate the uncertainty is intervals. In this paper we consider multiobjective linear programming with interval coefficients and solve it with respect to necessarily efficient points.

Keywords: Interval multi objective programming, necessarily efficient points, interval equality

1. Introduction

We usually face some difficulties when a real world problem is formulated to a mathematical programming problem. One of the difficulties is caused by the uncertainty is knowledge, information and decision maker's preference. Another difficulty is that most of the problems are inherently characterized by multiple and conflicting aspects of evaluation.

Multiobjective linear programming with interval coefficients is one of the approaches to tackle the above difficulties in mathematical programming models. This paper is aimed at providing an approach, which is based on efficient points, to solve an uncertainty multi objective linear programming with equal constraints and inequality constraints, respectively.

Consider, with out loss of generality, the following MOLP with interval coefficients

$$\begin{aligned} & \text{Max } CX \\ & \text{s.t } AX = b \\ & X \geq 0, C \in \Phi \end{aligned} \tag{1.1}$$

Where Φ is a set of $p \times n$ matrices which its elements $C_{kj} \in [C_{kj}^L, C_{kj}^U]$ for $k=1, \dots, p$ and $j=1, \dots, n$. A is a $m \times n$ matrix, b is an $m \times 1$ vector and X is a $1 \times n$ vector.

We define necessarily efficient solution in this manner:

A solution is necessarily efficient to problem (1.1) if and only if it is efficient for any $C \in \Phi$. The necessarily efficient set (N_E) is obtained by

$$N_E = \bigcap_{C \in \Phi} X_E(C)$$

Where is the efficient solution set for each $C \in \Phi$.

1.1. Finding necessarily efficient points

To find the entire set of necessarily efficient solution to (1.1), we use the following algorithms but first, we state some necessary definitions.

Let $R^{(g, w_1, \dots, w_g)} = C_B^{(g, w_1, \dots, w_g)} B^{-1} N - C_U^N$, where the columns of the interval matrix, $C_B^{(g, w_1, \dots, w_g)}$, are defined as

$$C_B^{(g, w_1, \dots, w_g)} = \begin{cases} C_{B,j}^U & j \leq g, w_j = U \\ C_{B,j}^L & j \leq g, w_j = L \\ [C_{B,j}^L, C_{B,j}^U] & g \leq j \leq m. \end{cases}$$

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g is the tree level.

Let $R^{L(g, w_1, \dots, w_g)}$ and $R^{U(g, w_1, \dots, w_g)}$ be composed of the lower and upper bounds of each elements belonging to the interval matrix $R^{(g, w_1, \dots, w_g)}$, respectively. The operator "sc" is defined as:

$$sc(R^{L(g, w_1, \dots, w_g)}) = \{R^{L(g+1, w_1, \dots, w_g, L)}, R^{L(g+1, w_1, \dots, w_g, U)}\} \quad \text{and}$$

$$sc(R^{U(g, w_1, \dots, w_g)}) = \{R^{U(g+1, w_1, \dots, w_g, L)}, R^{U(g+1, w_1, \dots, w_g, U)}\}$$

1.2. The necessary efficiency tests

We listed below some necessarily efficiency tests:

The Bitran necessarily efficiency test [1]

Step1. let $s^L = \{R^{L(0)}\}$.

Step2. Select one element $R^{L(g, w_1, \dots, w_g)}$ from s^L and check whether it is efficient.

(a) If it is efficient then remove the element from s^L .

(b) Otherwise, add $sc(R^{L(g, w_1, \dots, w_g)})$ to s^L . If $R^{(g, w_1, \dots, w_g)} = R^{(m, w_1, \dots, w_m)}$, then R is not necessarily efficient.

Step3. If the set s^L is empty, then R is necessarily efficient.

Step4: Return to step 2.

If the solution being analyzed is not necessarily efficient, we use the following algorithm

To test the necessary efficiency.

The Ida necessarily efficiency test [4,5]

Step1. let $s^U = \{R^{U(0)}\}$.

Step2. Select one element $R^{U(g, w_1, \dots, w_g)}$ from s^U and check whether it is efficient.

(a) If it is not efficient, then R is not necessarily efficient.

(b) Otherwise, add $sc(R^{U(g, w_1, \dots, w_g)})$ to s^U . If $R^{(g, w_1, \dots, w_g)} = R^{(m, w_1, \dots, w_m)}$, Do not add any thing to s^U .

Step3. If the set s^U is empty, then R is necessarily efficient.

Step4. Return to step 2.

The Chernikova's efficiency test [7]

Step1: Compute R .

Step2: Analyze columns and rows of R and proceed as follows:

(a) If there are any columns in R such that $R_j \geq 0$, then eliminate these columns.

(b) If there are any rows in R such that $R_i \leq 0$, then eliminate these rows.

Step3: Analyze columns and rows of R and proceed as follow:

(a) If there is a column in R such that $R_j \leq 0$, then R is not efficient.

(b) If there is a row in r such that $R_i > 0$, then R is efficient.

(c) If there is a row in R such that $R_i \geq 0$ and a row $i' \neq i$ such that $R_{ij} > 0$ ($R_{ij} = 0$), then R is efficient.

Step4: Calculate the summation of the columns(R_{Σ}) and rows (R_{Σ}) of R .

(a) If $R_{\Sigma \leq 0}$, then R is not efficient.

(b) If $R_{\Sigma} > 0$, then R is efficient.

2. Interval MOLP with equal constraints

Definition 1: We define the set of $S = \{X | AX = b, A \in [A], b \in [b]\}$ as the solution of interval system of equations $[A]X = [b]$. [3]

That is, s is the set of all solutions of $AX = b$ for all $A \in [A], b \in [b]$. This set is not an interval vector.

Because s is generally so complicated in shape, it is usually impractical to use it. Instead, it is common practice to seek the interval vector $[X]$ containing s that has the narrowest possible interval components.

This interval vector is called hull and we solve the system when we find the hull_[X].

Theorem 1: For an interval equality constraint

$$\sum_{j=1}^n [A_j^L, A_j^U] x_j = [b^L, b^U]$$

The following pair of inequality constraints

$$\sum_{j=1}^n A_j' x_j \geq b^L \text{ and } \sum_{j=1}^n \tilde{A}_j x_j \geq b^U$$

where $A_j' = \begin{cases} A_j^U & x_j \geq 0 \\ A_j^L & x_j \leq 0 \end{cases}$ and $\tilde{A}_j = \begin{cases} A_j^L & x_j \geq 0 \\ A_j^U & x_j \leq 0 \end{cases}$, define a convex region of possibility in which every point could satisfy some legal version of the original interval equality by an appropriate choice of fixed values for the interval coefficients and vice versa.

Now, we consider the following MOLP with interval coefficients and equal constraints:

$$\begin{aligned} & \text{Max } CX \\ & \text{s.t. } [A]X = [b] \\ & X \geq 0, C \in \Phi \end{aligned} \tag{2.1}$$

where $[A]$ is a $m \times n$ interval matrix, $[b]$ is a $m \times 1$ interval vector and Φ is a set of all $p \times n$ matrices with $C_{kj} \in [C_{kj}^L, C_{kj}^U]$.

By using the theorem 1, problem (2.1) is equivalent to the following problem:

$$\begin{aligned} & \text{Max } CX \\ & \text{s.t. } A^U X \geq b^L \\ & A^L X \leq b^U \\ & X \geq 0, C \in \Phi \end{aligned} \tag{2.2}$$

Note that the problem (2.2) is a multiobjective programming with interval coefficients and crisp constraints. So we can find its necessarily efficient solution by using the algorithm we mentioned in the section 1.

3. Interval MOLP with inequality constraints

Consider the following interval MOLP with inequality constraints :

$$\begin{aligned} & \text{Max } CX \\ & \text{s.t. } [A]X \leq [b] \\ & X \geq 0, C \in \Phi \end{aligned} \tag{3.1}$$

where Φ is a set of all $p \times n$ matrices with $C_{kj} \in [C_{kj}^L, C_{kj}^U]$, $[A]$ is a $m \times n$ interval matrix and $[b]$ is a $m \times 1$ interval vector .

To find the necessarily efficient solution for the problem (3.1), we try to convert it to a problem similar to (1.1) and find the necessarily efficient points as the method which explain in section 1. For this purpose, we suggest the following method:

Consider the following constraint with interval coefficients:

$$\sum_{j=1}^n [A_j] x_j \leq [b] \tag{3.2}$$

where $[b] = [b^L, b^U]$, $[A_j] = [A_j^L, A_j^U]$, $x_j \geq 0, j = 1, \dots, n$.

Finding the feasible region is a essential problem for interval programming problem. Several definitions of feasible region have been proposed by researchers [1,2,6,7]. Here we use the definition given by Ishibuchi and Tanaaka which is based on the concept of degree of inequality holding true for two intervals[6].

Definition 2: For an interval $A=[A^L, A^U]$ and a real number x , the degree for inequality $A \leq x$ holding true is given as follows:

$$g(A \leq x) = \max\{0, \min\{1, \frac{x - A^L}{A^U - A^L}\}\}$$

Definition 3: For two intervals $A=[A^L, A^U]$ and $B=[B^L, B^U]$, the degree for inequality $A \leq B$ holding true is given as follows:

$$g(A \leq B) = g(A - B \leq 0) = \max\{0, \min\{1, \frac{B^U - A^L}{A^U - B^L + B^U - A^L}\}\}$$

According to definition 3, the feasible region of interval constraint (3.2) can be determined by the following theorem:

Theorem 2: For a given degree of inequality holding true q , the interval inequality constraint (3.2) can be transformed to

$$\sum_{j=1}^n (qA_j^U x_j + (1-q)A_j^L x_j) \leq (1-q)b^U + qb^L$$

For more detailed discussion on the feasible of interval constraint, see [6]. Therefore by using theorem 2, for a given degree of inequality holding true q , the problem (3.1) convert to:

$$\begin{aligned} & \text{Max } CX \\ & \text{s.t } \sum_{j=1}^n (qA_j^U x_j + (1-q)A_j^L x_j) \leq (1-q)b^U + qb^L \\ & X \geq 0, C \in \Phi \end{aligned} \quad (3.3)$$

Numerical example

Consider the following MOLP with inequality constraints in which the interval matrices C , A and b as follow:

$$C = \begin{pmatrix} [1,2] & [2,3] & [-2,-1] & [3,4] & [2,3] & [0,1] & [1,2] \\ [-1,0] & [1,2] & [1,2] & [2,3] & [3,4] & [1,2] & [0,1] \\ [3,4] & [0,1] & [1,2] & [1,2] & [0,1] & [-2,-1] & [-2,-1] \end{pmatrix}$$

$$[A] = \begin{pmatrix} [0,10] & [0,2,2] & [0,10] & [0,10] & [0,2,2] & [0,10] & [0,2,2] \\ [-3,7] & [-2,8] & [-1,9] & [0,10] & [0,2,2] & [-1,9] & [0,10] \\ [-2,8] & [-1,9] & [0,10] & [-1,9] & [0,2,2] & [-1,9] & [-3,7] \\ [-1,9] & [0,10] & [0,2,2] & [-2,8] & [0,10] & [-3,7] & [-2,8] \end{pmatrix}, \quad b = \begin{pmatrix} [7,17] \\ [7,17] \\ [7,17] \\ [7,17] \end{pmatrix}$$

If $q=0.1$ then by using theorem 2, the matrices in the interval constraints $[A]x=[b]$ transform to following matrices A and b :

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 & 2 & 1 & 2 \\ -2 & -1 & 0 & 1 & 2 & 0 & 1 \\ -1 & 0 & 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & 2 & -1 & 1 & -2 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 16 \\ 16 \\ 16 \\ 16 \end{pmatrix}$$

Now, we want to use the necessary efficiency test for the extreme point

$$(0, 0, \frac{32}{3}, \frac{16}{3}, 0, 0, 0, \frac{32}{3}, \frac{16}{3}, 0)^T$$

So, we obtain

$$B^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{2}{3} & 0 & 0 & -\frac{1}{3} \\ -\frac{1}{3} & 1 & 0 & \frac{1}{3} \\ -\frac{1}{3} & 0 & 1 & -\frac{1}{3} \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 2 & 2 & 1 & 2 & 1 & 0 \\ -2 & -1 & 2 & 0 & -2 & 0 & 0 \\ -1 & 0 & 2 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & -2 & -1 & 0 & 1 \end{pmatrix}, \quad C_N^U = \begin{pmatrix} 2 & 3 & 3 & 1 & 2 & 0 & 0 \\ 0 & 2 & 4 & 2 & 1 & 0 & 0 \\ 4 & 1 & 1 & -1 & -1 & 0 & 0 \end{pmatrix}$$

Thus,

$$\begin{aligned}
 \mathbf{R}^{(0)} = \mathbf{C}_B^{(0)} \mathbf{B}^{-1} \mathbf{N} - \mathbf{C}_N^U &= \begin{pmatrix} [-1,-2] & [3,4] & 0 & 0 \\ [1,2] & [2,3] & 0 & 0 \\ [1,2] & [1,2] & 0 & 0 \end{pmatrix} \mathbf{B}^{-1} \mathbf{N} - \mathbf{C}_N^U \\
 &= \begin{pmatrix} [-\frac{2}{3}, \frac{1}{3}] & [-2,0] & [-2,0] & [\frac{10}{3}, 5] & [\frac{7}{3}, \frac{13}{3}] & [\frac{4}{3}, \frac{7}{3}] & [-2, -\frac{4}{3}] \\ [\frac{5}{3}, \frac{8}{3}] & [1,3] & [-1,1] & [0, \frac{5}{3}] & [\frac{8}{3}, \frac{14}{3}] & [\frac{5}{3}, \frac{8}{3}] & [-\frac{2}{3}, 0] \\ [-3, -2] & [1,3] & [1,3] & [\frac{5}{3}, \frac{10}{3}] & [3,5] & [1,2] & [-\frac{1}{3}, \frac{1}{3}] \end{pmatrix}
 \end{aligned}$$

Hence

$$\mathbf{R}^{U(0)} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 5 & \frac{13}{3} & \frac{7}{3} & -\frac{4}{3} \\ \frac{8}{3} & 3 & 1 & \frac{5}{3} & \frac{14}{3} & \frac{8}{3} & 0 \\ -2 & 3 & 3 & \frac{10}{3} & 5 & 2 & \frac{1}{3} \end{pmatrix}, \quad \mathbf{R}^{L(0)} = \begin{pmatrix} -\frac{2}{3} & -2 & -2 & \frac{10}{3} & \frac{7}{3} & \frac{4}{3} & -2 \\ \frac{5}{3} & 1 & -1 & 0 & \frac{8}{3} & \frac{5}{3} & -\frac{2}{3} \\ -3 & 1 & 1 & \frac{5}{3} & 3 & 1 & -\frac{1}{3} \end{pmatrix}.$$

Since $\mathbf{R}_2^{U(0)} \geq 0$ and there is an element $\mathbf{R}_{37}^{U(0)} (\mathbf{R}_{27}^{U(0)} = 0)$, then $\mathbf{R}^{U(0)}$ is efficient and $\mathbf{R}^{L(0)}$ is not efficient. Therefore, by the two efficiency tests, it may be necessary to examine the efficiency of both $\mathbf{R}^{(1,L)}$ and $\mathbf{R}^{(1,U)}$. Thus,

$$\begin{aligned}
 \mathbf{R}^{(1,U)} = \mathbf{C}_B^{(1,U)} \mathbf{B}^{-1} \mathbf{N} - \mathbf{C}_N^U &= \begin{pmatrix} [-1,-1] & [3,4] & 0 & 0 \\ [2,2] & [2,3] & 0 & 0 \\ [2,2] & [1,2] & 0 & 0 \end{pmatrix} \mathbf{B}^{-1} \mathbf{N} - \mathbf{C}_N^U \\
 &= \begin{pmatrix} [-\frac{1}{3}, \frac{1}{3}] & [-1,0] & [-1,0] & [\frac{10}{3}, \frac{14}{3}] & [\frac{8}{3}, \frac{13}{3}] & [\frac{8}{3}, \frac{7}{3}] & [-\frac{5}{3}, -\frac{4}{3}] \\ [2, \frac{8}{3}] & [2,3] & [0,1] & [0, \frac{4}{3}] & [3, \frac{14}{3}] & [2, \frac{8}{3}] & [-\frac{1}{3}, 0] \\ [-\frac{8}{3}, -2] & [2,3] & [2,3] & [\frac{5}{3}, 3] & [\frac{10}{3}, 5] & [\frac{4}{3}, 2] & [0, \frac{1}{3}] \end{pmatrix}, \\
 \mathbf{R}^{(1,L)} = \mathbf{C}_B^{(1,L)} \mathbf{B}^{-1} \mathbf{N} - \mathbf{C}_N^U &= \begin{pmatrix} [-2,-2] & [3,4] & 0 & 0 \\ [1,1] & [2,3] & 0 & 0 \\ [1,1] & [1,2] & 0 & 0 \end{pmatrix} \mathbf{B}^{-1} \mathbf{N} - \mathbf{C}_N^U \\
 &= \begin{pmatrix} [-\frac{2}{3}, 0] & [-2,1] & [-2,1] & [\frac{11}{3}, 5] & [\frac{7}{3}, 4] & [\frac{4}{3}, 2] & [-2, -\frac{5}{3}] \\ [\frac{5}{3}, \frac{7}{3}] & [1,2] & [-1,0] & [1, \frac{5}{3}] & [\frac{8}{3}, \frac{13}{3}] & [\frac{5}{3}, \frac{7}{3}] & [-\frac{2}{3}, -\frac{1}{3}] \\ [-3, -\frac{7}{3}] & [1,2] & [1,2] & [2, \frac{10}{3}] & [3, \frac{14}{3}] & [1, \frac{5}{3}] & [-\frac{1}{3}, 0] \end{pmatrix},
 \end{aligned}$$

Hence

$$\begin{aligned}
 \mathbf{R}^{U(1,U)} &= \begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{14}{3} & \frac{13}{3} & \frac{7}{3} & -\frac{4}{3} \\ \frac{8}{3} & 3 & 1 & \frac{4}{3} & \frac{14}{3} & \frac{8}{3} & 0 \\ -2 & 3 & 3 & 3 & 5 & 2 & \frac{1}{3} \end{pmatrix}, \quad \mathbf{R}^{L(1,U)} = \begin{pmatrix} -\frac{1}{3} & -1 & -1 & \frac{10}{3} & \frac{8}{3} & \frac{8}{3} & -\frac{5}{3} \\ 2 & 2 & 0 & 0 & 3 & 2 & -\frac{1}{3} \\ -\frac{8}{3} & 2 & 2 & \frac{5}{3} & \frac{10}{3} & \frac{4}{3} & 0 \end{pmatrix}. \\
 \mathbf{R}^{U(1,L)} &= \begin{pmatrix} 0 & 1 & 1 & 5 & 4 & 2 & -\frac{5}{3} \\ \frac{7}{3} & 2 & 0 & \frac{5}{3} & \frac{13}{3} & \frac{7}{3} & -\frac{1}{3} \\ -\frac{7}{3} & 2 & 2 & \frac{10}{3} & \frac{14}{3} & \frac{5}{3} & 0 \end{pmatrix}, \quad \mathbf{R}^{L(1,L)} = \begin{pmatrix} -\frac{2}{3} & -2 & -2 & \frac{11}{3} & \frac{7}{3} & \frac{4}{3} & -2 \\ \frac{5}{3} & 1 & -1 & \frac{1}{3} & \frac{8}{3} & \frac{5}{3} & -\frac{2}{3} \\ -3 & 1 & 1 & 2 & 3 & 1 & -\frac{1}{3} \end{pmatrix}.
 \end{aligned}$$

Since $\mathbf{R}_2^{U(1,U)} \geq 0$ and there is an element $\mathbf{R}_{37}^{U(1,U)} (\mathbf{R}_{27}^{U(1,U)} = 0)$, then $\mathbf{R}^{U(1,U)}$ is efficient. However, $\mathbf{R}^{L(1,U)}$ is not efficient since $\mathbf{R}_7^{L(1,U)} \leq 0$. On the other hand, $\mathbf{R}^{L(1,L)}$ is not efficient since $\mathbf{R}_7^{L(1,L)} \leq 0$. Finally, $\mathbf{R}^{(2,L,L)}$ and $\mathbf{R}^{(2,L,U)}$ are not efficient because $\mathbf{R}_7^{U(1,L)} \leq 0$. Therefore, by the two efficiency tests we conclude that \mathbf{R} is not necessarily efficient (there is an element in S^U , $\mathbf{R}^{U(1,U)}$, that is not efficient) and the solution $(0, 0, \frac{32}{3}, \frac{16}{3}, 0, 0, 0, 0, \frac{32}{3}, \frac{16}{3}, 0)^T$ is not necessarily efficient when $q = 0.1$.

4. Multiobjective linear programming with fuzzy coefficients.

In conventional mathematical programming, coefficients of problems are usually determined by the experts and crisp values. but in reality, in an imprecise and uncertain environment, it is an uncertain assumption that the knowledge and representation of an expert are so precise. Hence in order to develop good operation research methodology fuzzy , interval and stochastic approaches are frequently used to describe and treat imprecise and uncertain elements present in a real decision problem. In fuzzy programming problems the constraints and goals are viewed as fuzzy sets and assumed that their membership functions are known. But, in reality, to a decision maker (DM) it is not always easy to specify the membership function.

At least in some of the cases, use an interval coefficients may serve the purpose better. though by using α - cuts, fuzzy numbers can be degenerated into interval numbers.

Moreover , most real word problems are inherently characterized by multiple, conflicting and incommensurate aspects of evaluation .these axes of evaluation are generally operationalized by objective functions to be optimized in framework of multiple objective linear programming models.

In this section, we focus on multiobjective linear programming with fuzzy coefficient. By introducing α -cuts and *Ramik and Raminek* ranking, we degenerate the problem to a multiobjective linear programming with interval objective functions and crisp constraints then define necessarily efficient points and find these points for new problems.

The multiobjective linear programming with fuzzy coefficients can be formulated as follow:

$$\begin{aligned} \max \quad & \sum_{j=1}^n \tilde{c}_{kj} x_j \quad k = 1, \dots, p \\ \text{s.t} \quad & \sum_{j=1}^n \tilde{A}_{ij} x_j \leq \tilde{b}_i \quad i = 1, \dots, m \\ & x_j \geq 0 \end{aligned} \quad (4.1)$$

Where \tilde{c}_{kj} , \tilde{A}_{ij} and \tilde{b}_i are fuzzy numbers.

To find the necessarily efficiency points for problem (4.1), we transform it to a problem with interval objective function and crisp constraints. so we use α -cuts to degenerate coefficients of objective function to interval ,as indicated follow, and *Ramik* and *Raminek* ranking to obtain crisp constraints from our fuzzy constraints.

4.1. Some definitions

The α -level set (α -cut) of a fuzzy set \tilde{M} is defined as an ordinary set \tilde{M}_α for which the degree of its membership function exceeds the level α :

$$\tilde{M}_\alpha = \{ x \mid \mu_{\tilde{M}} \geq \alpha \}, \alpha \in [0, 1]$$

Actually, an α -level set is an ordinary set whose elements belong to the corresponding fuzzy set to a certain degree α . A fuzzy number is a convex normalized fuzzy set of the real line R whose membership function is piecewise continuous.

From the definition of a fuzzy number \tilde{M} , it is significant to note that the α -level set \tilde{M}_α of a fuzzy number \tilde{M} can be represented by the closed interval which depends on interval value of α . Namely,

$$\tilde{M}_\alpha = \{ x \mid \mu_{\tilde{M}} \geq \alpha \} = [M_\alpha^L, M_\alpha^U]$$

Where M_α^L or M_α^U represents the left or right extreme point of the α -level set \tilde{M}_α respectively.

Especially if $M = (m, \gamma, \beta)$ is a triangular fuzzy number then $\tilde{M}_\alpha = [\gamma(\alpha - 1) + m, m - \beta(\alpha - 1)]$ and if $M = (m^L, m^R, \gamma, \beta)$ is a trapezoidal fuzzy number then $\tilde{M}_\alpha = [\gamma(\alpha - 1) + m^L, m^R - \beta(\alpha - 1)]$.

4.2. Ranking fuzzy numbers

Dubios and prade proposed a method of ranking fuzzy numbers as follow:

Definition 2. Let \tilde{M} and \tilde{N} be fuzzy numbers, then we have

$$\tilde{M} \geq \tilde{N} \leftrightarrow \tilde{M} \vee \tilde{N} = \tilde{M} .$$

By using the definition Ramik and raminek suggested the following lemma:

Lemma 2. If \tilde{M} and \tilde{N} be a fuzzy numbers, $\tilde{M} \vee \tilde{N} = \tilde{M}$ if and only if for every $h \in [0,1]$ we have

$$\begin{aligned} \inf \{ s : \mu_{\tilde{M}}(s) \geq h \} &\geq \inf \{ t : \mu_{\tilde{N}}(t) \geq h \} \\ \sup \{ s : \mu_{\tilde{M}}(s) \geq h \} &\geq \sup \{ t : \mu_{\tilde{N}}(t) \geq h \} \end{aligned}$$

Especially if $\tilde{M} = (m^L, m^R, \alpha, \beta)_{LR}$ and $\tilde{N} = (n^L, n^R, \gamma, \delta)_{LR'}$ be trapezoidal fuzzy numbers, the above relation is true if and only if

$$\begin{aligned} m^l - L^*(h) \alpha &\geq n^l - L^*(h) \beta \quad \forall h \in [0,1] \\ m^r + R^*(h) \alpha &\geq n^r + R^*(h) \beta \quad \forall h \in [0,1] \end{aligned}$$

Where

$$\begin{aligned} L^*(h) &= \sup\{ z : L(z) \geq h \} \\ L'^*(h) &= \sup\{ z : L'(z) \geq h \} \\ R^*(h) &= \sup\{ z : T(z) \geq h \} \\ R'^*(h) &= \sup\{ z : R'(z) \geq h \} . \end{aligned}$$

From the definition of Ramik and Raminek, if $\tilde{M} = (m, \alpha, \beta)_{LR}$ and $\tilde{N} = (n, \gamma, \delta)_{LR'}$, be triangular fuzzy numbers ,then we have

$$\tilde{M} \leq \tilde{N} \leftrightarrow \begin{cases} m \leq n \\ m - \alpha \leq n - \gamma \\ m + \beta \leq n + \delta \end{cases}$$

If $\tilde{M} = (m^l, m^r, \alpha, \beta)_{LR}$ and $\tilde{N} = (n^l, n^r, \gamma, \delta)_{LR'}$ be trapezoidal fuzzy numbers, similarly we can compare them as follow

$$\tilde{M} \leq \tilde{N} \leftrightarrow \begin{cases} m^l \leq n^l \\ m^r \leq n^r \\ m^l - \alpha \leq n^l - \gamma \\ m^r + \beta \leq n^r + \delta \end{cases}$$

By using above definition and with respect to the kind of fuzzy numbers of the problem, we can transform the fuzzy constraints to crisp ones. In this way, if $\tilde{c}_{kj} = (c_{kj}, \alpha_{kj}, \beta_{kj})$, $\tilde{A}_{ij} = (a_{ij}, \gamma_{ij}, \delta_{ij})$ and $\tilde{b}_i = (b_i, \eta_i, \nu_i)$ be triangular fuzzy numbers, problem (4.1) is equivalent to

$$\begin{aligned} \text{Max} \quad & \sum_{j=1}^n [\alpha_{kj}(\alpha - 1) + c_{kj}, c_{kj} - \beta_{kj}(\alpha - 1)] x_j \quad k = 1, \dots, p \\ \text{s.t} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad i = 1, \dots, m \\ & \sum_{j=1}^n (a_{ij} - \gamma_{ij}) x_j \leq b_i - \eta_i \quad i = 1, \dots, m \end{aligned} \tag{4.2}$$

$$\sum_{j=1}^n (a_{ij} + \beta_{ij})x_j \leq b_i + v_i \quad i = 1, \dots, m$$

$$x_j \geq 0.$$

Similarly, if $\tilde{c}_{kj} = (c_{kj}^L, c_{kj}^R, \alpha_{kj}, \beta_{kj})$, $\tilde{A}_{ij} = (a_{ij}^L, a_{ij}^R, \gamma_{ij}, \delta_{ij})$ and $\tilde{b}_i = (b_i^L, b_i^R, \eta_i, v_i)$ be trapezoidal fuzzy numbers then problem (4.1) is equivalent to

$$\text{Max } \sum_{j=1}^n [\alpha_{kj}(\alpha - 1) + c_{kj}^L, c_{kj}^R - \beta_{kj}(\alpha - 1)]x_j \quad k = 1, \dots, p$$

$$\text{s.t. } \sum_{j=1}^n a_{ij}^L x_j \leq b_i^L \quad i = 1, \dots, m$$

$$\sum_{j=1}^n a_{ij}^R x_j \leq b_i^R \quad i = 1, \dots, m \quad (4.3)$$

$$\sum_{j=1}^n (a_{ij}^L - \gamma_{ij})x_j \leq b_i^L - \eta_i \quad i = 1, \dots, m$$

$$\sum_{j=1}^n (a_{ij}^R + \beta_{ij})x_j \leq b_i^R + v_i \quad i = 1, \dots, m$$

$$x_j \geq 0.$$

5. Conclusion

In this paper, a MOLP with interval coefficients is focused. we proposed a procedure to transform the problem to a Molp with interval objective coefficients and crisp constraints. Then we try to finding the necessarily efficient points for solving primary problem.

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