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A numerical approach to solving an inverse parabolic problem using finite differential method

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Abstract. The present work is motivated by desire to obtain numerical approach combining the use of the finite difference method with the solution of ordinary differential equation(ODE) is proposed for the determination of unknown coefficient in an inverse heat conduction problem(IHCP). The least-squares method is adopted to modify the solution. Results show that an excellent estimation can be obtained within a couple of minutes CPU time at Pentium IV-2.4 GHz PC .

Keywords: inverse head conduction problem, finite difference method, least-squares method.

1. Introduction

The problem of determining unknown coefficient in parabolic partial differential equations has been treated by many authors [3-17]. Usually these problems involve the determination of a single unknown parameter from over specified boundary data. In some applications, however, it is desirable to be able to determine more than one parameter from the given boundary data [3-14].

The mathematical model of a one-dimensional inverse parabolic problem with initial and boundary condition is as follows:

$$U_t(x,t) = U_{xx}(x,t) + p(t)U(x,t), \quad 0 \prec x \prec 1, \quad 0 \prec t \prec T, \tag{1}$$

$$U(x,0) = r(x) \qquad 0 \le x \le 1, \tag{2}$$

$$U_x(0,t) = \zeta(t) \qquad 0 \le t \le T, \tag{3}$$

$$U_x(1,t) = q(t) \qquad 0 \le t \le T, \tag{4}$$

where $r(x), \zeta(t)$ and $q(t)$ are all continuously differentiable, while the temperature $U(x,t)$ and the coefficient $p(t)$ are unknown which remain to be determined. For an unknown coefficient $p(t)$ we must therefore provide additional information to provide a unique solution to the inverse problem (1)-(4).

The theoretical discussion is well addressed in [3,4], we will focus our study on the numerical method. The famous work of [5] was one of the first to call broad attention to the solution of this and other similar parabolic inverse problems. The existence and uniqueness and continuous dependence of the solution to some kind of these inverse problems are discussed in [5,6].

The numerical method suggested in this paper are based on time derivative of temperature in the heat conduction equation has been replaced by the first backward finite difference formula. As a result, an ordinary differential equation appears instead of the heat conduction equation.

2. Overview of the numerical method

Our method begins with the utilizing of the following transformation,

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$$U(x,t) = V(x,t)K(t) \tag{5}$$

where

$$K(t) = \exp(\int_0^t p(\tau)d\tau). \tag{6}$$

Therefore, we have

$$V(x,t) = \frac{U(x,t)}{K(t)} \tag{7}$$

and

$$p(t) = \frac{K'(t)}{K(t)} \tag{8}$$

Where

$$K'(t) = \frac{dK(t)}{K(t)} .$$

Transformation (6) allow us to eliminate the unknown term $p(t)$ from equation (1) and to obtain a new inverse parabolic problem with two unknown boundary conditions. Now using transformations (5)and (6), we can write (1)-(4) as follows,

$$V_t(x,t) = V_{xx}(x,t), \qquad 0 \prec x \prec 1 \ , \qquad 0 \prec t \prec T, \tag{9}$$

$$V(x,0) = r(x), \qquad 0 \le x \le 1 \ , \tag{10}$$

$$V_x(0,t) = \frac{\zeta(t)}{K(t)}, \qquad 0 \le t \le T \ , \tag{11}$$

$$V_x(1,t) = \frac{q(t)}{K(t)}, \qquad 0 \le t \le T, \tag{12}$$

The problem (9)-(12) may be analyzed as a direct problem, for the portion of the body from $x = 0$ to $x = 1$ with known boundary conditions, i.e. $p(t)$ is known function. There is a unique solution to the direct problem (9)-(12) and may be found by the following theorem:

Theorem 1. For piecewise continuous functions $r(x), \frac{\zeta(t)}{K(t)}$ and $\frac{q(t)}{K(t)}$ in their domains, the solution function $V(x,t)$ is given by

$$V(x,t) = \sum_{n=0}^{\infty} b_n \exp(-n^2\pi^2 t)\cos(n\pi x)$$

$$-2\int_0^t \theta(x-1,t-\tau)\frac{\zeta(\tau)}{K(\tau)}d\tau + 2\int_0^t \theta(x-1,t-\tau)\frac{q(\tau)}{K(\tau)}d\tau,$$

Where

$$b_n = 2\int_0^1 r(x)\cos n\pi x, \qquad n = 1,2,....$$

$$b_0 = \int_0^1 r(x)dx$$

$\theta(x,t) = \sum_{m=-\infty}^{\infty} K(x+2m,t)$, $t \succ 0$ and the fundamental solution $K(x,t)$ is defined by

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} , \quad t \succ 0,$$

is a unique solution of problem (1)-(4)

Proof. [1].

For unknown is unknown function $\dfrac{\zeta(t)}{K(t)} = \phi(t)$ and $\dfrac{q(t)}{K(t)} = \psi(t)$,i.e. $p(t)$ is unknown function, we must therefore provide additional information

$$V(x_i, t_k) = V_{i,k} \ , \quad i = 1, 2, ..., M \quad , \qquad k = 1, 2, ..., N \tag{13}$$

to provide a solution to the inverse problem (9)-(12). For this purpose, we estimate the unknown functions $\phi(t)$ and $\psi(t)$, by using of additional information of discrete temperature measurements. Thus the temperature history at some locations are measured in a slab. It is assumed that M thermocouples are used to record the temperature information at selected locations. Temperature histories taken from the thermocouples at successive specific dimensionless time t_k are denoted by $V_{i,k}$ for $i = 1,2,..., M$ and $k = 1, 2, ..., N$, where N denotes the number of the discrete measurement times.

The application of the numerical method to find the solution of problem (9)-(13) can be described as follows.

To remove time dependent terms in problem (1)-(4) we replace the time derivative of temperature in the equation (1) with the first backward finite difference formula .We arrive after some evaluation to the following problem

$$\frac{d^2 V_k(x)}{dx^2} - \lambda^2 V_k(x) = -\lambda^2 V_{k-1}(x) \ , \qquad 0 \prec x \prec 1 \ , \ k = 1, 2, ..., N \tag{14}$$

$$\frac{dV_k(x)}{dx} = \phi_k \ , \quad x = 0 \ , \ k = 0, 1, ..., N \tag{15}$$

$$\frac{dV_k(x)}{dx} = \psi_k \ , \quad x = 1, \ k = 0, 1, ..., N \tag{16}$$

Where $V_k(x) = V(x, k\Delta t)$, $0 \prec x \prec 1$ with Δt being a dimensionless time step, $\lambda^2 = \dfrac{1}{\Delta t}$.

The problem (14)-(16) is a two-point boundary-value problem and the prescribed conditions (15) and (16) are called the boundary conditions. Equations of this sort arise quite frequently in scientific and engineering applications and often pose a difficult challenge to the numerical analyst. The equation (14) is a second order linear ordinary differential equation that may have no solution, a unique solution, or infinitely many solutions.

Theorem 2. Let $V_{k-1}(x)$ be continuous on [0,1]Then the boundary value problem (14)-(16), has a unique solution for any constants ϕ_k and ψ_k .

Proof. [2]

The analytical unique solution of problem (6)-(8) is as follows

$$V_k(x) = f(x)\phi_k + g(x)\psi_k + W_k(x) \tag{17}$$

where

$$f(x) = \frac{-\cosh \lambda(1-x)}{\lambda \sinh \lambda} \tag{18}$$

$$g(x) = \frac{\cosh \lambda x}{\lambda \sinh \lambda} \tag{19}$$

and

$$\begin{aligned} W_k(x) &= \lambda \cosh \lambda x \coth \lambda \int_0^1 V_{k-1}(t) \cosh \lambda t dt \\ &- \lambda \cosh \lambda x \int_0^1 V_{k-1}(t) \sinh \lambda t dt + W^*(x) \end{aligned} \tag{20}$$

with

$$W^*(x) = \lambda \cosh \lambda x \int_0^x V_{k-1}(t) \sinh \lambda t dt$$
$$-\lambda \sinh \lambda x \int_0^x V_{k-1}(t) \cosh \lambda t dt \qquad (21)$$

Now, one may write equation (17) for M points, therefore we obtain the following algebraic system of equations,

$$V_k^{mea} = A\Theta_k + W_k \qquad (22)$$

where $V_k^{mea}(x_i), i = 1(1)M$, , represent the measured temperature taken from the M thermocouples at the location x_i for $i = 1, 2, ..., M$, at the successive specific dimensionless time t_k and

$$V_k^{mea} = (V_k^{mea}(x_1)...V_k^{mea}(x_M))^T \ , \ A = \begin{pmatrix} f(x_1) & \dots & f(x_M) \\ g(x_1) & \dots & g(x_M) \end{pmatrix}^T \ ,$$
$$\Theta_k = (\phi_k \ \ \psi_k)^T \ , \ W_k = (W_k(x_1) \dots W_k(x_M))^T \qquad (23)$$

Remark. When $M = 2$ the system of equations (22) has a unique solution.

Proof.

We must prove that the

$$\det(A) \neq 0$$

Now, if the determinant of A is equal to zero then

$$f(x_1)g(x_2) - f(x_2)g(x_1) = 0,$$
$$(\sinh \lambda)\sinh \lambda(x_2 - x_1) = 0,$$

and finally

$$\sinh \lambda(x_2 - x_1) = 0,$$

therefore $x_2 = x_1$. This is impossible, since $x_2 \neq x_1$. Thus the assumption that the $\det(A) = 0$ leads to a contradiction.

However, when $M \succ 2$, it becomes an over determined system that can be solved in a sense of minimization of a criterion function (linear least-squares method). Let

$$S_k = \sum_{i=1}^{M} (V_k^{mea}(x_i) - V_k(x_i))^2 \ , \qquad (24)$$

describe this function. To obtain the minimum value of S_k with respect to ϕ_k and ψ_k , differentiation of S_k with respect to ϕ_k and ψ_k , will be performed .Thus to minimize S_k one has to solve the following system

$$\begin{cases} \dfrac{\partial S_k}{\partial \phi_k} = 0 \\ \\ \dfrac{\partial S_k}{\partial \psi_k} = 0 \end{cases} \qquad (25)$$

Thus the system (25), corresponding to the values of ϕ_k and ψ_k , can be expressed as

$$\Theta_k = (A^T A)^{-1} A^T (V_k^{mea} - W_k) \qquad (26)$$

and $(A^T A)^{-1} A^T$ is the reverse matrix of the inverse problem. Note that the matrix $A^T A$ is invertible, Since

$$\det(A^T A) = (\sum_{i=1}^{M} f^2(x_i))(\sum_{i=1}^{M} g^2(x_i)) - (\sum_{i=1}^{M} f(x_i)g(x_i))^2 \ ,$$

now, by using the Cauchy-Schwartz inequality we obtain $\det(A^T A) \geq 0$, with the equality only if $\frac{f(x_1)}{g(x_1)} = ... = \frac{f(x_M)}{g(x_M)}$, while Remark shows that $\frac{f(x_1)}{g(x_1)} \neq \frac{f(x_M)}{g(x_M)}$, , therefore

$$\det(A^T A) > 0$$

Now from (17) and (26), we obtain

$$V_k(x) = \rho(x)\Theta_k + W_k(x) \tag{27}$$

where

$$\rho(x) = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}^T .$$

Equation (26) is assumed to measure all the discredited points in the problem. The realistic experimental approach is to measure only the few point in the problem. We can construct the parts of the matrices $(A^T A)^{-1} A^T$, $U_k^{mea} - V_k$ and Θ_k corresponding to the measuring positions in order to estimate the boundary heat flux of problem. According to the above derivation, it is possible to identify whether the solution is unique or not. The method by which to identify the properties of the solution is based on the theory of linear algebra, which will be shown in the following descriptions.

If the rank of the reverse matrix is equal to the number of elements of the coefficient vector, then the perpendicular distance from θ_k to the column space of A needs to be checked. If the distance vanishes then the solution is unique.

Finally, for the evaluating $U(x,t)$ and $p(t)$ we use (7), (8) and (17), therefore

$$K(t_k)\frac{\zeta(t_k)}{V_k(0)}$$

$$U(x,t_k) = V_k(x)K(t_k),$$

and

$$p(t_k) = \frac{K(t_{k-1}) - K(t_{K+1})}{2k \times K(t_k)}$$

3. Numerical results and discussion

In this section, we are going to demonstrate numerically, some of the results for the temperature $U(x,t)$ and the coefficient $p(t)$ in the inverse problem (1)-(4). Mathematically, IHCPs belong the class of ill-posed problems, i.e. small errors in the measured data can lead to large deviations in the estimated quantities. The physical reason for the ill-posed ness of the estimation problem is that variations in the surface conditions of the solid body are damped towards the interior because of the diffusive nature of heat conduction. As a consequence, large-amplitude changes at the surface have to be inferred from small-amplitude changes in the measurements data. Errors and noise in the data can therefore be mistaken as significant variations of the surface state by the estimation procedure. The purpose of this section is to illustrate the applicability of the present method described in Section 2 for solving IHCP. All the computations are performed on the PC.

Example. In this example let us consider following one-dimensional inverse parabolic problem

$$U_t = U_{xx} + p(t)U, \qquad 0 \prec x \prec 1, \quad 0 \prec t \prec T \tag{28}$$

$$U(x,0) = \sin(x), \qquad 0 \le x \le 1, \tag{29}$$

$$U_x(0,t) = \exp(-\frac{t^2}{2} - t), \qquad 0 \le t \le T, \tag{30}$$

$$U_x(1,t) = \exp(-\frac{t^2}{2} - t)\cos(1), \qquad 0 \le t \le T, \tag{31}$$

with the exact solution

$$U(x,t) = \exp(-\frac{t^2}{2} - t)\sin(x),$$

and

$$p(t) = -t$$

To give a clear overview of the present method, the above example will be considered.

Tables 1 and 2, are shown the values of $U(x,t)$ and $p(t)$ in $x = ih$ and $t = jk$ when $k = 0.025$ $h = 0.05$.

Table1. The analytical and numerical results for the temperature $U(x,t)$, in $x = ih$ and $t = jk$ where $k = 0.025, h = 0.05$.

x	Numerical $U(x,k)$	Exact $U(x,k)$	Numerical $U(x,3k)$	Exact $U(x,3k)$	Numerical $U(x,5k)$	Exact $U(x,5k)$
0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1/20	0.04873	0.04875	0.04624	0.04638	0.04376	0.04411
1/5	0.19370	0.19376	0.18380	0.18431	0.17396	0.17533
1/2	0.46744	0.46759	0.44353	0.44478	0.41980	0.42309
3/4	0.66460	0.66481	0.63061	0.63238	0.59686	0.60154
9/10	0.76375	0.76399	0.72469	0.72673	0.68590	0.69128
1	0.82044	0.82069	0.77848	0.78067	0.73682	0.74260

Table 2.The analytical and numerical results for the values $p(t)$ in $t = jk$, , where $k = 0.025$ and $j = 1(1)5$.

t	Numerical $p(t)$	Exact $p(t)$
k	-0.02499	0.02500
2k	-0.04998	-0.05000
3k	-0.07498	-0.07500
4k	-0.09997	-0.10000
5k	-0.12496	-0.12500

Figures 1 and 2 show the comparison of $U(x,t)$ and $p(t)$ between the exact results and the present numerical results.

Fig.1, U(x,3k), x=0(1/20)1,k=0.025

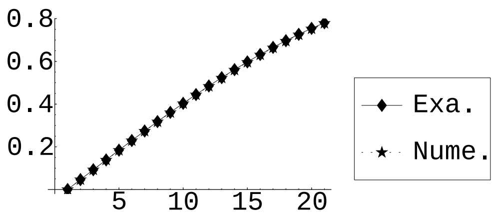
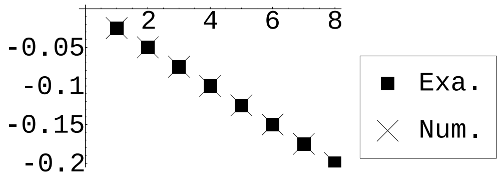


Fig.2, P(jk),j=1(1)5,k=0.025



4. Conclusion

A numerical method to estimate the temperature $U(x,t)$ and the coefficient $p(t)$ is proposed for an IHCP and the following results are obtained.

1. The present study, successfully applies the numerical method involving the finite difference method in conjunction with the least-squares scheme to an IHCP.

2. From the illustrated example it can be seen that the proposed numerical method is efficient and accurate to estimate the temperature $U(x,t)$ and the coefficient $p(t)$.

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